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On pseudo-convergent sequences

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MATHEMATICS

ON PSEUDO-CONVERGENT SEQUENCES

BY

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Introduction

In the theory of non-Archimedian valuated fields A. OSTROWSKI [1] [1]) introduced the concept of pseudo-convergent sequence. He proved the theorem:

If $\{a_i\}$ is a pseudo-convergent sequence of elements of a non-Archimedian valuated field K, and if $f(x)$ is a polynomial with coefficients in K then the sequence $\{f(a_i)\}$ is also pseudo-convergent. In his proof [1, pp. 371–374] he uses valuated algebraic extensions of K, for which, as is well-known [3, § 78], also complete extension of K is used.

F. LOONSTRA [2] conjectured the possibility of avoiding these extensions. His proof, however, is incorrect, because his lemma stated as "Satz IV" [2]) is false, as is shown by the following counter-example.

Take for K the field of rational numbers with the 2-adic valuation and let $\overline{K}$ be its completion i.e. the field of 2-adic numbers.

The polynomial $x^2 + 7$ has a zero in $\overline{K}$, but not in K, (for the underlying theory see [3, § 79]).

Let $a = \sum_{j=0}^{\infty} 2^{\nu_j}$ (with $\nu_{j+1} > \nu_j$) be the 2-adic expansion of such a zero. Now put $a_i = \sum_{j=0}^{i} 2^{\nu_j}$, then $\{a_i\}$ is pseudo-convergent, since

$$|a_{i+1} - a_i| = 2^{-\nu_{i+1}} < 2^{-\nu_i} = |a_i - a_{i-1}|.$$

Besides $\{a_i - \alpha\}$ is pseudo-convergent of the first kind for all $\alpha \in K$. For, if $\alpha = \sum_{j=0}^{\infty} 2^{\mu_j}$ (with $\mu_{j+1} > \mu_j$) and if j_0 is the smallest index, such that $\mu_{j_0} \neq \nu_{j_0}$ (such a j_0 always exists because $a \neq \alpha$), then we have

$$|a_i - \alpha| = 2^{-\min(\mu_{j_0}, \nu_{j_0})}, \text{ for } i \geqslant j_0.$$

Now we take for $f(x)$ the polynomial $x^2 + 7$.

The sequence $\{a_i^2 + 7\}$ is convergent with the limit 0, hence $|a_i^2 + 7| \to 0$ and since $|a_i^2 + 7| \neq 0$ we come upon a contradiction with "Satz IV".

1) The numbers in the square brackets refer to the bibliography at the end of the paper.

2) "Satz IV. Sei $\{a_i\}$ eine pseudokonvergente Folge. Es gebe weiter in K kein Element α derart, dass die pseudokonvergente Folge $\{a_i - \alpha\}$ von der 2. Art ist. Sei $f(x)$ ein Polynom mit Koeffizienten aus K. Dann ist $|f(a_i)|$ konstant von einem gewissen i_0 an."

In this paper we prove Ostrowski's theorem without the use of extensions of the field K, thus establishing Loonstra's conjecture.

Preliminaries and notation

Let K be a non-Archimedian valuated field. We shall denote the value of an element a of K by $|a|$. A sequence $\{a_i\}$ of elements of K is called pseudo-convergent if

$$\text{either } a_i = a_{i+1} \text{ for all } i \geqslant i_0,$$
$$\text{or } |a_{i+1} - a_i| < |a_i - a_{i-1}| \text{ for all } i \geqslant i_1.$$

It follows immediately that

$$|a_{i+j} - a_i| = |a_{i+1} - a_i| \text{ for all } j \geqslant 1; \text{ [4, p. 39].}$$

An other direct consequence is that

$$\text{either } |a_i| = |a_{i+1}| \text{ for all } i \geqslant i_2$$
$$\text{or } |a_{i+1}| < |a_i| \text{ for all } i \geqslant i_3 \quad \text{[1, p. 369] and [4, p. 39].}$$

In the first case $\{a_i\}$ is called pseudo-convergent of the first kind and in the second case of the second kind.

Proof of the theorem stated in the introduction

We shall prove the somewhat stronger theorem:

If $f(x) = \sum_{l=0}^{n} d_l x^l$ is a polynomial of degree n with coefficients in a non-Archimedian valuated field K and if $\{a_i\}$ is a pseudo-convergent sequence in K, then

$$|f(a_{i+1}) - f(a_i)| = c\,|a_{i+1} - a_i|^k,$$

with integral k and $n \geqslant k \geqslant 1$ and real constant $c \geqslant 0$, for all $i \geqslant i_5$. The constants k and c do not depend on i and $c = 0$ only if $n = 0$.

Remark: If the theorem is valid then clearly $\{f(a_i)\}$ is pseudo-convergent. If it is so of the first kind then $|f(a_i)| =$ constant, and if it is of the second kind then

$$|f(a_i)| = |f(a_{i+1}) - f(a_i)| = c\beta_i^k \text{ with } \beta_i = |a_{i+1} - a_i|.$$

Hence for sufficiently large i we may write in both cases

$$|f(a_i)| = c\beta_i^k \text{ with } k \geqslant 0 \text{ and } c \geqslant 0.$$

Proof: The theorem is evident if the sequence is such that $a_{i+1} = a_i$ for all $i \geqslant i_0$. So we may suppose $0 < \beta_{i+1} < \beta_i$.

We shall apply mathematical induction with respect to n.

Basis of induction: $n = 0$, hence $f(x) = d_0$ and

$$|f(a_{i+1}) - f(a_i)| = 0 \text{ for all } i.$$

Suppose the theorem to be true for all polynomials of a degree less than n, with $n > 0$.

We shall evaluate the value of $f(a_i) - f(a_{i+j})$ with the use of the identity

$$\text{(A)} \qquad f(a_i) - f(a_{i+j}) = \sum_{l=1}^{n} f_l(a_{i+j})\,(a_i - a_{i+j})^l$$

in which

$$f_l(a_{i+j}) = \sum_{m=0}^{n-l} d_{l+m} \binom{l+m}{l} x^m$$

is a polynomial of a degree less than n and hence by hypotheses and the remark above

$$|f_l(a_{i+j})| = c_l\,\beta_{i+j}^{k(l)}$$

with $k(l) \geqslant 0$ and $c_l \geqslant 0$ and $i + j \geqslant i_6$.

We shall denote the values of the terms of the right hand side of (A) by

$$\gamma_l(i, j) = c_l\,\beta_{i+j}^{k(l)}\,\beta_i^l.$$

We shall prove that one of them, say $\gamma_\lambda(i, j)$, dominates the others for all $i \geqslant i_7$ and all $j \geqslant j_0(i)$.

We shall treat the cases i: $\lim\limits_{i\to\infty} \beta_i = \beta = 0$ and ii: $\lim\limits_{i\to\infty} \beta_i = \beta > 0$ separately.

Case i; $\beta = 0$.

Divide the indices l in three classes V_1, V_2 and V_3 such that

$l \in V_1$ if and only if $c_l = 0$
$l \in V_2$ if and only if $c_l \neq 0$ and $k(l) \neq 0$ and
$l \in V_3$ if and only if $c_l \neq 0$ and $k(l) = 0$.

The set V_3 is not empty because $f_n(x)$ is a non-zero constant.

Let λ be the smallest index of V_3.

First choose i so large that $c_\lambda \beta_i^\lambda > c_l \beta_i^l$ for all $l \in V_3$ and $\neq \lambda$, which is possible because $\beta_i \to 0$. Now fix i and choose j so large, $\geqslant j_0(i)$, that $\gamma_l(i, j) < c_\lambda \beta_i^\lambda$ for all $l \in V_2$, which is possible because also $\lim\limits_{j\to\infty} \beta_{i+j} = 0$ and $c_\lambda \beta_i^\lambda > 0$.

Case ii; $\beta > 0$.

Here we have

$$\Lambda = \max_{l=1,\ldots,n} (\lim_{i\to\infty} \gamma_l(i, j)) = \max_{l=1,\ldots,n} (c_l\,\beta^{k(l)+l}) \geqslant c_n\,\beta^n > 0.$$

Further let V_4 be the set of indices such that $c_l\,\beta^{k(l)+l} = \Lambda$ for all $l \in V_4$ and $c_l\,\beta^{k(l)+l} < \Lambda$ for all $l \notin V_4$ and let λ be the greatest index of V_4 (evidently V_4 is not empty).

Now we first choose i so large that $\Lambda > c_l\,\beta_i^{k(l)+l}$ for all $l \notin V_4$ and hence

$$\gamma_\lambda(i, j) > \Lambda > c_l\,\beta_i^{k(l)+l} \geqslant \gamma_l(i, j).$$

In order to make clear that we can find a $j_0(i)$ such that, for all $j \geqslant j_0(i$ we have $\gamma_\lambda(i, j) > \gamma_l(i, j)$ for all $l \neq \lambda$, $l \in V_4$, we put $\beta_i = (1 + \eta_i)\beta$ and require

$$c_\lambda\,\beta^{k(\lambda)+\lambda}(1 + \eta_{i+j})^{k(\lambda)}(1 + \eta_i)^\lambda > c_l\,\beta^{k(l)+l}(1 + \eta_{i+j})^{k(l)}(1 + \eta_i)^l$$

or equivalently

$$(1+\eta_i)^{\lambda-l} > (1+\eta_{i+j})^{k(l)-k(\lambda)}.$$

Because $(1+\eta_i)^{\lambda-l} > 1$ (since $\lambda - l > 0$) and $\lim_{j\to\infty} (1+\eta_{i+j})^{k(l)-k(\lambda)} = 1$, we can choose j so large that the requirement is fulfilled.

In both cases we get

$$|f(a_i) - f(a_{i+j})| = c_\lambda \beta_{i+j}^{k(\lambda)} \beta_i^\lambda$$

for some fixed $\lambda \geqslant 1$ and some constant $c_\lambda > 0$, for all sufficiently large i and all $j \geqslant j_0(i)$. Since the same holds for $i+1$ and $j-1$, provided $j \geqslant \max(j_0(i), j_0(i+1)+1)$, we have

$$\text{(B)} \quad \begin{cases} |f(a_{i+1}) - f(a_i)| = |f(a_{i+1}) - f(a_{i+1+j-1}) - (f(a_i) - f(a_{i+j}))| = \\ = \max(c_\lambda \beta_{i+j}^{k(\lambda)} \beta_i^\lambda, c_\lambda \beta_{i+j}^{k(\lambda)} \beta_{i+1}^\lambda) = c_\lambda \beta_{i+j}^{k(\lambda)} \beta_i^\lambda. \end{cases}$$

The integer $k(\lambda)$ is necessarily zero. This follows at once from the fact that the left hand side of (B) is independent of j and hence $\beta_{i+j}^{k(\lambda)} = \beta_{i+j+1}^{k(\lambda)}$ while on the other hand $\beta_{i+j} > \beta_{i+j+1}$.

So we get $|f(a_{i+1}) - f(a_i)| = c\, \beta_i^k$ with $c = c_\lambda > 0$ and $n \geqslant k = \lambda \geqslant 1$, which proves the theorem.

Finally I would like to express my thanks to Dr W. Peremans, collaborator at the Mathematical Centre, for his valuable criticism and his assistance in the construction of the counter-example stated in the introduction.

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