Partial ordering of quantifiers and of clopen equivalence relations

by

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1. **Introduction**

A quantifier in a boolean algebra $A$ is a closure operator $C$ in $A$ having the additional property that $C(-Ca) = -Ca$ for every $a \in A$ ("every open element is closed"). It is natural to introduce a partial ordering in the set $Q(A)$ of all quantifiers in $A$ in the following way: $C_1 \preceq C_2$ iff $C_1 a \preceq C_2 a$ for every $a \in A$.

Another natural definition would be: $C_1 \preceq C_2$ iff $C_1 \circ C_2 = C_2$ (here $\circ$ denotes composition: $C_1 \circ C_2 a = C_1(C_2 a)$). It is shown in section 4 that both definitions are equivalent, and that furthermore they are equivalent to the following one: $C_1 \preceq C_2$ iff the subalgebra $C_1(A)$ of $A$ contains the subalgebra $C_2(A)$ of $A$.

As is well-known, every boolean algebra $A$ can be considered as the algebra of clopen subsets of a zero-dimensional compact Hausdorff space $X$. A quantifier $C$ in $A$ can be interpreted as an open- and -closed equivalence relation $\equiv$ in $X$, or as a continuous decomposition $D$ of $X$. These facts are exposed in section 3. As the set of all equivalence relations in a space $X$ is partially ordered ($R_1 \preceq R_2$ if $xR_1 y$ implies $xR_2 y$), in this way again a partial ordering is induced in $Q(A)$. This partial ordering also turns out to be identical to the previous ones (see proposition 9 in section 4).

An interesting problem is the following. The class of all equivalence relations in a set $X$ is a lattice under the partial ordering mentioned above. The subclass $Cl(X)$ of all clopen equivalence relations is not a sublattice, not even an (upper or lower) sub-semilattice, as is shown by examples in section 2. In these examples, $X$ can be taken as a zero-dimensional compact Hausdorff space. However, it is still possible, of course, that $Cl(X)$ is lattice-ordered by $\preceq$. To the best of my knowledge, this is an open question.

Because of the fact that the partially ordered set $Q(A)$ is order-isomorphic to the partially ordered set $Cl(X)$, the problem just stated is equivalent to the following one: is the set $Q(A)$ of all quantifiers in a boolean algebra $A$ always a lattice under the partial ordering described above? It would already be of interest if it could be shown that $Q(A)$ always is an upper semilattice (which I suppose is true); I did not succeed yet in proving this, however,
In the last section, the fact that two quantifiers $c_1$ and $c_2$ in a boolean algebra $A$ commute: $c_1 \circ c_2 = c_2 \circ c_1$, is shown to have some connection with the partial ordering of $Q(A)$. In fact, if $c_1 c_2 = c_2 c_1$, then $c_1$ and $c_2$ have a l.u.b. in $Q(A)$, and this l.u.b. is $c_1 c_2$. However, as is shown by an example, $c_1$ and $c_2$ may have a l.u.b. even if they do not commute.
2. Open and closed equivalence relations

Let $X$ be a set, and $R$ an equivalence relation in $X$. If $A \subseteq X$, then $R[A] = \{ x \in X : xRy \text{ for some } y \in A \}$. Instead of $R[\{ x \}]$ we write $R[x]$. The decomposition

$$X = \bigcup_{x \in X} R[x]$$

is denoted by $D_R$.

As is well-known (see e.g. [4]), the equivalence relations in a set $X$ are lattice-ordered by the relation $\leq$.

$$(2.1) \quad R_1 \leq R_2 \iff (\forall x,y \in X)(xR_1y \rightarrow xR_2y).$$

We have

$$(2.2) \quad x(R_1 \lor R_2)y \iff xR_1y \text{ and } xR_2y;$$

$$(2.3) \quad x(R_1 \land R_2)y \iff \text{there exists a finite chain } xR_1x_1, x_1R_2x_2, x_2R_1x_3, \ldots, x_{2n}R_1y.$$

It is clear that

$$(2.4) \quad A = (R_1 \lor R_2)[A] \iff A = R_1[A] = R_2[A].$$

If we define

$$(2.5) \quad S^0 = A; S^{2n+1}A = R_1[S^{2n}A]; S^{2n+2}A = R_2[S^{2n+1}A];$$

then

$$(2.6) \quad (R_1 \lor R_2)[A] = \bigcup_{n=0}^{\infty} S^nA.$$ 

Remark: It should be kept in mind that in general

$$(R_1 \land R_2)[A] \neq R_1[A] \cap R_2[A].$$

The equality holds, however, if $A$ consists of at most one point.

Now let $X$ be a topological space. An equivalence relation $R$ is called open (closed) if $R[A]$ is open (closed) whenever $A$ is open (closed). In a different terminology (cf. [6], [7]), $R$ is called open (closed) iff $D_R$ is a lower (upper) semicontinuous decomposition.
The relation $R$ is called **clopen** if it is both open and closed (i.e. iff $D_R$ is a **continuous decomposition**).

The decomposition space, obtained from $X$ by identifying points in the same $D_R$-set, is denoted by $X/R$, and is always supposed to be provided with the quotient topology.

Then the identification map $\pi_R : X \rightarrow X/R$ is always continuous; it is open (closed) iff $R$ is open (closed).

**Proposition 1:** If $R_1$ and $R_2$ are open, then $R_1 \lor R_2$ is open.

**Proof.** If $A$ is open, then every $S^nA$ is open, where the $S^n$ are defined as in (2.5); hence \( \bigcup_{n=0}^{\infty} S^nA = (R_1 \lor R_2)A \) is open.

It is not true, in general, that $R_1 \lor R_2$ is open if $R_1$ and $R_2$ are.

**Example 1.** Let $X$ be the unit square, i.e. the set of all pairs $(x,y)$, $0 \leq x, y \leq 1$, with the euclidean topology. Put

\[(x,y) R_1 (u,v) \iff x = u.\]

A second equivalence relation $R_2$ will be described by means of the corresponding decomposition $D_{R_2}$. The sets of $D_{R_2}$ are the straight-line segments joining the points $(a,0)$ and $(2a,0)$ ($0 \leq a \leq \frac{1}{3}$), and the segments joining the points $(a,0)$ and \( (\frac{a+1}{2},0) \) ($\frac{1}{3} < a \leq 1$).

Then both $R_1$ and $R_2$ are clopen. The equivalence classes of $R_1 \lor R_2$ are all singletons \( \{(x,y) : x \neq 0, 1, \} \), and the two segments \( \{(0,y) : 0 \leq y \leq 1\} \) and \( \{(1,y) : 0 \leq y \leq 1\} \). Hence $R_1 \lor R_2$ is not open.

**Proposition 2.** If $X$ is a normal Hausdorff space, then $R_1 \lor R_2$ is closed if $R_1, R_2$ are closed.
Proof. We use the fact that an equivalence relation \( R \) is closed iff each equivalence class \( R[x] \) has a neighbourhood base consisting of saturated sets (see e.g. pag. 62).

Let \( A_1 = R_1[x] \) (\( i=1, 2 \)), and let \( U \) be an open set containing \( A_1 \cap A_2 \).
The sets \( A_1, A_2 \) are closed, as \( X \) is a \( T_1 \)-space; the disjoint closed sets \( A_1 \setminus U \) and \( A_2 \setminus U \) have disjoint neighbourhoods \( V_1, V_2 \), as \( X \) is normal. The neighbourhood \( U \cup V_i \) of \( A_i \) contains an open set \( W_i \) such that \( A_i \subseteq W_i = R_i[W_i] \) (\( i=1, 2 \)); it follows that

\[
A_1 \cap A_2 \subseteq W_1 \cap W_2 = (R_1 \cap R_2)[W_1 \cap W_2] \subseteq U.
\]

Hence \( R_1 \cap R_2 \) is closed.

Remark. I do not know whether the assumption that \( X \) is a \( T_4 \)-space is superfluous or not.

It may happen that \( R_1, R_2 \) are closed while \( R_1 \cup R_2 \) is not.

Example 2. Let \( X \) be the discontinuum of Cantor. We will define two clopen equivalence relations \( R_1, R_2 \) in \( X \) such that \( R_1 \cup R_2 \) is not closed. These equivalence relations will be defined by means of the corresponding decompositions.

Represent \( X \) in the canonical way as a subset of the unit segment. For \( n=0, 1, 2, \ldots \) we define the subset \( A_n \) of \( X \) as follows:

\[
A_n = \left\{ x \in X : 1 - \frac{1}{3^n} \leq x \leq 1 - \frac{2}{3^n} \right\}.
\]

The decomposition \( D_{R_1} \) will consist of the sets \( A_0 ; A_{2n+1} \cup A_{2n} \) (\( n=1, 2, \ldots \)); and \( \{1\} \). The decomposition \( D_{R_2} \) will consist of the sets \( A_{2n} \cup A_{2n+1} \) (\( n=0, 1, \ldots \)) and the set \( \{1\} \). Both \( D_{R_1} \) and \( D_{R_2} \) are continuous, i.e. \( R_1 \) and \( R_2 \) are clopen. But \( R_1 \cup R_2 \) is not closed, for the set \( A_0 \) is closed (even clopen) while \( (R_1 \cup R_2)[A_0] = X \setminus \{1\} \) is not closed.

We are particularly interested in the case where \( X \) is a zero-dimensional compact Hausdorff space, and \( R_1, R_2 \) are both clopen equivalence relations.

Let \( Cl(X) \) be the set of all clopen equivalence relations in \( X \).
(In another terminology, \( Cl(X) \) is (in 1-1-correspondence to) the set of all continuous decompositions of \( X \).) The set \( Cl(X) \) is partially ordered by \( \preceq \).
It is shown by example 2 that, even in the case where \( X \) is zero-dimensional compact Hausdorff, \( \Cl(X) \) need not be a sublattice of the lattice of all equivalence relations in \( X \). However, the two equivalence relations in example 2 have a l.u.b. in \( \Cl(X) \), namely, the universal relation (the relation that holds between every two elements \( x, y \in X \)).

We will denote the l.u.b. and g.l.b. in \( \Cl(X) \) by \( \uparrow \) and \( \downarrow \).

Example 1 can be easily modified to show that \( R_1 \land R_2 \) need not be clopen, for \( R_1, R_2 \in \Cl(X) \), even if \( X \) is a compact Hausdorff space. One can e.g. take \( X \) to be the set of all rational points in the unit square, instead of the full unit square.

In example 1 (modified or not) \( R_1 \) and \( R_2 \) have a g.l.b. \( R_1 \downarrow R_2 \) in \( \Cl(X) \), namely the identity relation:

\[
x(R_1 \downarrow R_2) y \iff x = y.
\]

The following problem is still open:

**Problem.** If \( X \) is zero-dimensional compact Hausdorff, is it true that \( \Cl(X) \) is lattice-ordered by \( \leq ? \) i.e., for \( R_1, R_2 \in \Cl(X) \) do \( R_1 \uparrow R_2 \) and \( R_1 \downarrow R_2 \) always exist? If not, is \( \Cl(X) \) at least a semilattice (upper or lower)?

**Proposition 3.** Let \( X \) be a compact Hausdorff space, \( A \) a clopen subset of \( X \), and \( R_1, R_2 \) two clopen equivalence relations in \( X \). The set \( (R_1 \lor R_2)[A] \) is closed (and then also clopen) iff \( (R_1 \lor R_2)[A] = S^n A \), for some integer \( n \geq 0 \).

**Proof.** All \( S^n A \) are clopen, hence if \( (R_1 \lor R_2)[A] = S^n A \), then \( (R_1 \lor R_2)[A] \) is closed. Conversely, suppose \( (R_1 \lor R_2)[A] \) is closed.

Assume \( S^{n+1} A \neq S^n A \), for all \( n > 1 \). Then the clopen sets \( S^n A \setminus S^{n-1} A \) are non-void, for \( n = 0, 1, 2, \ldots \) (we put \( S^{-1} A = \emptyset \)). Let \( x_n \in S^n A \setminus S^{n-1} A \) (\( n=0,1,\ldots \)); the sequence \( \{ x_n \} \) has a limit point \( a \) in the compact set \( (R_1 \lor R_2)[A] \).

Let \( n \) be the smallest integer \( \geq 0 \) such that \( a \in S^n A \); then \( S^n A \setminus S^{n-1} A \) is a neighbourhood of \( a \) that contains only one \( x_n \). This is a contradiction; hence we conclude that \( S^{n+1} A = S^n A \), for some \( n > 1 \). It follows at once that \( (R_1 \lor R_2)[A] = S^n A \).
In the case of example 2, there are two pathological facts. In the first place we saw that \((R_1 \vee R_2)[A_0]\) is not closed, where \(A_0\) is a clopen set. In the second place we may remark that even for the one-point sets like \(\{0\}\), \((R_1 \vee R_2)[0]\) is not closed. Hence \(X/R_1 \vee R_2\) is not even a \(T_1\)-space.

Of course, if \(X/R\) is a Hausdorff space, and \(X\) is compact, then \(R\) is closed. For if \(\pi\) is the identification map, then \(\pi\) is continuous; if \(A\) is a closed subset of \(X\), then \(A\) is compact, hence \(\pi(A)\) is compact, and therefore closed. It follows that \(R[A] = \pi^{-1}(\pi(A))\) is closed.

Something can be said also if it is only assumed that \(X/R\) is a \(T_1\)-space.

**Proposition 4.** Let \(X\) be a zero-dimensional compact Hausdorff space, and let \(R\) be an equivalence relation in \(X\). If \(R[x]\) is closed, for every \(x \in X\), and if \(R[A]\) is closed, for every clopen subset \(A\) of \(X\), then \(R\) is closed.

**Proof.** Let \(x \in X\), and let \(U\) be a neighbourhood of \(R[x]\). We must show that there exists neighbourhood \(V\) of \(R[x]\) such that \(V = R[V]\) and \(V \subseteq U\).

As \(R[x]\) is closed and hence compact, \(U\) contains a clopen neighbourhood \(W\) of \(R[x]\):

\[
R[x] \subseteq W \subseteq U.
\]

Then \(R[X \setminus W]\) is closed by assumption, and disjoint with \(R[x]\); hence \(W = X \setminus R[X \setminus W]\) is an open neighbourhood of \(R[x]\). Of course \(W\) is contained in \(U\).

I was not yet able to answer the following question: is the assumption that \(X/R\) is a \(T_1\)-space really necessary in the case interesting us, i.e. the case \(R = R_1 \vee R_2\), \(R_1\) and \(R_2\) being clopen? (In that case \(R\) itself is open.). In other words:

Let \(R_1\) and \(R_2\) be clopen equivalence relations in a zero-dimensional compact Hausdorff space \(X\). Let \((R_1 \vee R_2)[A]\) be a clopen subset of \(X\) if \(A\) is any clopen subset of \(X\). Does it follow that \(R_1 \vee R_2\) is closed?
3. Quantifiers and equivalence relations

Definition 1. A quantifier in a boolean algebra \( A \) is a transformation \( C : A \to A \) with the following properties:

(i) \( CO = 0 \);
(ii) \( a \not\in Ca \), for all \( a \in A \);
(iii) \( C(a \land b) = Ca \landCb \), for all \( a, b \in A \).

The set of all quantifiers in \( A \) is denoted by \( Q(A) \).

Proposition 5. The quantifiers in a boolean algebra \( A \) are exactly the closure operators in \( A \) having the additional property that every open element is closed:

\[
C(-Ca) = -Ca, \text{ for all } a \in A.
\]

A proof can be found in Halmos [3].

Definition 2. A subalgebra \( B \) of a boolean algebra \( A \) is called \( A \)-complete if, for each \( a \in A \), the subset \( \{ b \in B : b \not\supset a \} \) of \( A \) has a g.l.b. in \( A \), and if this g.l.b. belongs again to \( B \).

Proposition 6. (cf. Halmos [3]): If \( C \in Q(A) \), then \( C(A) \) is an \( A \)-complete subalgebra of \( A \). The correspondence \( C \mapsto C(A) \) is a 1-1-correspondence, between the set of all quantifiers in \( A \) and the set of all \( A \)-complete subalgebras of \( A \).

If \( B \) is an \( A \)-complete subalgebra, the corresponding quantifier \( C \) is defined by

\[
Ca = \bigwedge \{ b \in B, a \not\in b \}.
\]

Examples of quantifiers are given by

Proposition 7. (Varsavsky [5]). Let \( X \) be any set, and let \( A \) be the boolean algebra of all subsets of \( X \). Let \( R \) be any equivalence relation in \( X \). The operation \( C_R : Y \to R[Y] (Y \subseteq X) \) is a quantifier in \( A \), and the correspondence \( R \mapsto C_R \) is a 1-1-correspondence between the set of all equivalence relations in \( X \) and the set \( Q(A) \).

If \( A \) is any boolean algebra, the zero-dimensional compact Hausdorff space of all ultrafilters in \( A \), provided with the hull-kernel
topology, will be denoted by $T(A)$.

If $a \in A$, then $\hat{a} = \{ U \in T(A) : a \in U \}$ is a clopen set in $T(A)$; if $F$ is a filter in $A$, then $\hat{F} = \{ U \in T(A) : F \subseteq U \}$ is closed in $T(A)$.

The mapping $a \mapsto \hat{a}$ is an isomorphism of $A$ onto the boolean algebra of all clopen subsets of $T(A)$. If $S$ is a closed set in $T(A)$, then $S = \bigwedge S$.

It is a result of Davis [2] (cf. also Varsavsky [5]), that there is a 1-1-correspondence between the set of all subalgebras of a boolean algebra $A$ and the set of all those equivalence relations $R$ in $T(A)$ such that $T(A)/R$ is a zero-dimensional compact Hausdorff space. This correspondence is the following.

If $B$ is a subalgebra of $A$, the corresponding equivalence relation $R$ is defined by

$$ U_1RU_2 \iff U_1 \cap B = U_2 \cap B. \tag{3.3} $$

If $R$ is an equivalence relation in $T(A)$ such that $T(A)/R$ is zero-dimensional compact Hausdorff, the corresponding subalgebra $B$ of $A$ is defined by

$$ B = \{ b \in A : R[\hat{b}] = \hat{b} \}. \tag{3.4} $$

Furthermore, if $B$ and $R$ correspond, then $T(B)$ and $T(A)/R$ are homeomorphic; and for $S \subseteq T(A)$ we have

$$ R[S] = S \iff S = \bigwedge \{ b : \hat{b} \in B \text{ and } S \subseteq \hat{b} \}. \tag{3.5} $$

Proposition 8. (Halmos [3]). Under the correspondence just described, the set of all $A$-complete subalgebras of $A$ corresponds with $Cl(T(A))$.

Let $C \in Q(A)$, $B = C(A)$, and let $R \in Cl(T(A))$ correspond with $B$; this relation will be denoted by $\mathcal{C}$. Then the following holds:

$$ \mathcal{C}[\hat{a}] = \bigwedge \{ b \in B : a \not\subseteq b \} = \hat{C}a. \tag{3.6} $$

Conversely: let $C \in Q(A)$, and let $R$ be any equivalence relation in $T(A)$ such that $R[\hat{a}] = \hat{C}a$, for all $a \in A$. Then it follows that $R$ is clopen, and hence that $R = \mathcal{C}$. 

-9-
4. The partial ordering of $Q(A)$

**Definition 3.** In $Q(A)$, we define a partial ordering $\preceq$ by

\[(4.1) \quad C_1 \preceq C_2 \iff C_1 a \preceq C_2 a \quad \text{for all } a \in A.\]

(It is evident that this is indeed a partial ordering).

**Proposition 9.** $C_1 \preceq C_2 \iff C_2 \circ C_1 = C_2 \iff C_1 \circ C_2 = C_2 \iff C_2 (A) \preceq C_1 (A) \iff C_1 \preceq C_2$.

**Proof.** Assume $C_1 \preceq C_2$. Take $a \in A$; $C_1 a \preceq C_2 a \iff C_2 C_1 a \preceq C_2 a = C_2 a \preceq C_1 C_2 a$; hence $C_2 C_1 = C_2$.

Assume $C_2 C_1 = C_2$. Take $a \in A$; $C_2 a = C_2 C_2 a = C_2 C_1 C_2 a$; hence $C_2 (C_1 C_2 a A - C_2 a) = C_2 C_2 C_2 a A - C_2 a = 0$. Thus $C_1 C_2 a \wedge C_2 a = 0$, or $C_1 C_2 a \preceq C_2 a$. As certainly $C_2 a \preceq C_1 C_2 a$, it follows that $C_1 C_2 = C_2$.

It is trivial that $C_1 C_2 = C_2 \iff C_2 (A) \preceq C_1 (A)$.

Assume $C_1 \preceq C_2$. Then, for $a \in A$: $\widehat{C_1 a} = \widehat{C_1} A = \widehat{C_2} a = C_2 a$; hence $C_1 a \preceq C_2 a$. This implies $C_1 \preceq C_2$.

Assume $C_1 \not\preceq C_2$. Then there exists a point $U \in T(A)$ such that $C_1 [U] \setminus C_2 [U] \neq \emptyset$; let $V \in C_1 [U] \setminus C_2 [U]$. As $\{U\}$ is closed and $C_2$ clopen, the set $C_2 [U]$ is closed; $V \notin C_2 [U]$, hence (as $T(A)$ is zero-dimensional) $V$ has a clopen neighbourhood $a$ that is disjoint with $C_2 [U]$. It follows that $U \in C_2 [a]$. As $U \in C_1 [V] \subseteq C_1 [a]$, we must have:

\[(4.2) \quad \widehat{C_1 a} = \widehat{C_1} [a] \not\preceq \widehat{C_2} [a] = \widehat{C_2 a};\]

i.e. $C_1 a \not\preceq C_2 a$. Thus $C_1 \not\preceq C_2$.

As it follows that $Q(A)$ and $C_1 (T(A))$ are order-isomorphic, it is natural to denote the g.l.b. and l.u.b. in $Q(A)$, as far as they exist, by $\downarrow$ and $\uparrow$, respectively.

The main problem of section 2 turns out to be equivalent to the following problem about quantifiers:

**Problem.** Is $Q(A)$ lattice-ordered by $\preceq$, for every boolean algebra $A$?
Remark. Suppose $C_1, C_2 \in Q(A)$ have a g.l.b. $C_1 \wedge C_2$ in the partial ordering $\preceq$ of $Q(A)$. Then $C_1 \wedge C_2 \preceq C_i$ $(i=1,2)$; hence

\[(4.3) \quad (C_1 \wedge C_2)a \preceq C_1a \wedge C_2a, \quad \text{for all } a \in A.\]

In general, however, $(C_1 \wedge C_2)a \neq C_1a \wedge C_2a$, as $a \rightarrow C_1a \wedge C_2a$ need not be a quantifier.

Example 3. Let $A$ be the boolean algebra of all subsets of the unit square $X = [0,1]^2$. If $a \in A$, then let $C_1$ be the union of all horizontal straight-line segments intersecting $a$, and let $C_2$ be the union of all vertical straight line segments intersecting $a$. Let $C = C_1a \cap C_2a$. Then $C$ is not a quantifier, as it does not satisfy the condition $C(a \cap Cb) = C(a \wedge Cb)$, for all $a,b \in A$.

\[\text{fig. 2}\]

If $a,b$ are chosen as in figure 2, then $b = Cb$; $a \cap Cb = \emptyset$, hence $C(a \cap Cb) = \emptyset$; but $C(a \cap Cb) \neq \emptyset$.

In this example it is immediately seen that $C_1 \downarrow C_2$ exists and is the identity quantifier:

\[(C_1 \downarrow C_2)a = a,\]

for all $a \in A$. Similarly, $C_1 \uparrow C_2$ exists and is equal to the unit quantifier:

\[(C_1 \uparrow C_2)a = X,\]

for every $a \neq \emptyset$.

As a consequence of proposition 3, we have:

Proposition 10. If $C_1 \vee C_2 = C_1 \uparrow C_2$, then for each $a \in A$ there exists a non-negative integer $n$ such that
Here \((C_1 \uparrow C_2)^n a\) stands for \(C_1 C_2 \ldots C_1 C_2 a\), a being preceded by \(2n\) quantifiers in total.

In particular it follows, that for every \(a \in A\) there exists an \(n > 0\) such that
\[
(C_1 \uparrow C_2)^n a = (C_1 C_2)^n a.
\]

If \(n\) is any non-negative integer with this property, then \((C_1 C_2)^n a = (C_1 \uparrow C_2)^n a\). We can also write:
\[
(C_1 \uparrow C_2)^n a = \bigvee_{k=0}^{\infty} (C_1 C_2)^k a.
\]

As long as the question following proposition 4 has not been answered, the following problem remains open:

Let \(C_1, C_2\) be quantifiers in a boolean algebra \(A\). Suppose that for every \(a \in A\) there exists a non-negative integer \(n\) such that (3.10) holds. Is it then true that the mapping \(C : A \rightarrow A\) defined by
\[
Ca = \bigvee_{k=0}^{\infty} (C_1 C_2)^k a
\]
is a quantifier?

If \(C\) is a quantifier, then it is immediate that \(C = C_1 \uparrow C_2\), and that \(C_1 \uparrow C_2 = C_1 \lor C_2\).
5. Commutativity of quantifiers

Proposition 11. Let $A$ be a boolean algebra, and let $C_1, C_2 \in Q(A)$. Then

\[ C_1 \ast C_2 = Q(A) \iff C_1(C_2(A)) \subseteq C_2(A) \iff C_1(C_2(A)) \subseteq C_1(A) \cap C_2(A) \iff C_1(C_2(A)) = C_1(A) \cap C_2(A) \iff C_1 \uparrow C_2 \text{ exists, and is equal to } C_1 \ast C_2.\]

Proof. Suppose $C_1 C_2 \in Q(A)$. Then, for any $a \in A$: $C_1 C_2 \ast_C 2 C_1 C_2 \ast_C 1 C_2 \ast_C 2 \ast_C 1 C_2 = C_1 C_2 a$; hence $C_1 C_2 = C_2 C_1 C_2$, implying that $C_1(C_2(A)) \subseteq C_2(A)$.

Suppose $C_1(C_2(A)) \subseteq C_2(A)$. Certainly $C_1 C_2 0 = 0$, and $a \in C_1 C_2 a$, for all $a \in A$. As $C_2 C_1 C_2 = C_1 C_2$ it also follows that

\[ C_1 C_2(a \land C_1 C_2 b) = C_2 C_1 C_2(a \land C_1 C_2 C_1 C_2 b) = C_1(C_2 a \land C_1 C_2 b) = C_1 C_2 a \land C_1 C_2 b; \text{ hence } C_1 C_2 \in Q(A).\]

It is trivial that $C_1(C_2(A)) = C_1(A) \cap C_2(A) \iff C_1(C_2(A)) \subseteq C_1(A) \cap C_2(A) \iff C_1(C_2(A)) \subseteq C_2(A)$.

Assume $C_1(C_2(A)) \subseteq C_1(A) \cap C_2(A)$; it then follows that $C_1(C_2(A)) = C_1(A) \cap C_2(A)$, for if $a \in C_1(A) \cap C_2(A)$, then $a = C_1 a = C_2 a = C_1 C_2 a = C_1(C_2 a)$.

Suppose $C_1 \uparrow C_2 \text{ exists and is equal to } C_1 C_2$. Then certainly $C_1 C_2 \in Q(A)$. Conversely, suppose $C_1 C_2 \in Q(A)$. Then it follows (see the first part of the proof) that $C_1 C_2 = C_2 C_1 C_2$, and hence (by prop.9) that $C_2 \subseteq C_1 C_2$. As certainly $C_1 C_2 \subseteq C_1 C_2$, we also have $C_1 \subseteq C_2 C_2$. Now let $C \subseteq Q(A)$ such that $C_1 \subseteq C$ and $C_2 \subseteq C$. Then $C = C_1 C = C_2 C = C_1 C_2 C$, hence $C_1 C_2 \subseteq C$. This shows that $C_1 \uparrow C_2 \text{ exists and is equal to } C_1 C_2$.

Proposition 12. Let $A$ be a boolean algebra, and let $C_1, C_2 \in Q(A)$. Then

\[ C_1 \ast C_2 = C_2 \ast C_1 \iff C_1 \ast C_2 \in Q(A) \text{ and } C_2 \ast C_1 \in Q(A).\]

Proof. If $C_1 C_2 \in Q(A)$ and $C_2 C_1 \in Q(A)$, then $C_1 C_2 = C_1 \uparrow C_2 = C_2 C_1$. Conversely, assume $C_1 C_2 = C_2 C_1$. Then $C_1 C_2 \in Q(A)$. For certainly $C_1 C_2 0 = 0$, and $a \in C_1 C_2 a$, for all $a \in A$; and

\[ C_1 C_2(a \land C_1 C_2 b) = C_1 C_2(a \land C_1 C_2 C_1 b) = C_1(C_2 a \land C_1 C_2 C_1 b) = C_1 C_2 a \land C_1 C_2 b = C_1 C_2 a \land C_1 C_2 b.\]
If \( C_1 C_2 = C_2 C_1 \), then \( C_1 \uparrow C_2 \) exists. The converse is not true, however: \( C_1 \uparrow C_2 \) may exist while \( C_1 C_2 \neq C_2 C_1 \). This is shown by example 4. Before exhibiting this example, we describe the general construction behind it.

Let \( A \) be the boolean algebra of all subsets of a non-void set \( X \). One can construct quantifiers in \( A \) as follows.

Let \( \sigma : X \to X \) be an involution (i.e. \( \sigma \circ \sigma = \text{identity map} \); for \( a \in X \), let \( C_a = a \cup \sigma a \) (where \( \sigma a = \{ x : \sigma x a \} \)). Then \( C \) is a quantifier in \( A \).

It is immediate that \( C \emptyset = \emptyset \) and that \( a \subseteq C_a \), for all \( a \in A \). We now show that \( C(a \cap Cb) = C_a \cap Cb \), for arbitrary \( a, b \in A \).

\[
C(a \cap Cb) = (a \cap (b \cup \sigma b)) \cup (a \cap (b \cup \sigma b)) = \]
\[
= ((a \cap b) \cup (a \cap \sigma b)) \cup (\sigma a \cap (b \cup \sigma b)) = \quad \text{(as } \sigma \text{ is } 1-1) \]
\[
= (a \cap b) \cup (a \cap \sigma b) \cup (\sigma a \cap (b \cup \sigma b)) = \]
\[
= ((a \cup \sigma a) \cap (b \cup \sigma b)) = \]
\[
= (a \cup \sigma a) \cap (b \cup \sigma b) = C_a \cap Cb.
\]

This construction is applied in

**Example 4.** Let \( X = \{ \alpha, \beta, \gamma, \delta \} \); define \( \sigma_1, \sigma_2 : X \to X \) as follows:

\[
\sigma_1 \alpha = \alpha, \quad \sigma_1 \beta = \gamma, \quad \sigma_1 \gamma = \beta, \quad \sigma_1 \delta = \delta; \]
\[
\sigma_2 \alpha = \gamma, \quad \sigma_2 \beta = \alpha, \quad \sigma_2 \gamma = \delta, \quad \sigma_2 \delta = \beta.
\]

Let \( A \) be the boolean algebra of all subsets of \( X \); if \( a \in A \), let \( C_a = a \cup \sigma_1 a \) (i=1,2).

As \( A \) is finite, \( Q(A) \) is finite, hence \( C_1 \uparrow C_2 \) exists. However, \( C_1 C_2 \neq C_2 C_1 \), for

\[
C_1 C_2 \{ \alpha \} = C_1 \{ \alpha, \gamma \} = \{ \alpha, \beta, \gamma \} ;
\]
\[
C_2 C_1 \{ \alpha \} = C_2 \{ \alpha \} = \{ \alpha, \gamma \} .
\]

(One verifies at once that \( C_1 \uparrow C_2 \) is the unit quantifier: \( (C_1 \uparrow C_2) a = X \), for every \( a \neq \emptyset \).)

Topologically the situation is very simple; the discrete space \( X \) "is" the Stone-space \( T(A) \) of \( A \). The decomposition corresponding to \( C_1 \) is given by

\[
X = \{ \alpha \} \cup \{ \beta, \gamma \} \cup \{ \delta \} ;
\]

the decomposition introduced by \( C_2 \) is given by

\[
X = \{ \alpha, \gamma \} \cup \{ \beta, \delta \} .
\]

There is only one decomposition of which both are refinements, namely the decomposition having \( X \) as its only component.
References


A fairly complete list of the literature concerning quantifiers can be found in