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Partial ordering of quantifiers and of clopen equivalence relations

by

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1. Introduction

A quantifier in a boolean algebra A is a closure operator C in A having the additional property that C(-Ca) = -Ca for every $a \in A$ ("every open element is closed"). It is natural to introduce a partial ordering in the set Q(A) of all quantifiers in A in the following way: $C_1 \leq C_2$ iff $C_1 a \leq C_2 a$ for every $a \in A$.

Another natural definition would be: $C_1 \leq C_2$ iff $C_1 \circ C_2 = C_2$ (here $\circ$ denotes composition: $C_1 \circ C_2 a = C_1(C_2 a)$). It is shown in section 4 that both definitions are equivalent, and that furthermore they are equivalent to the following one: $C_1 \leq C_2$ iff the subalgebra $C_1(A)$ of A contains the subalgebra $C_2(A)$ of A.

As is well-known, every boolean algebra A can be considered as the algebra of clopen subsets of a zero-dimensional compact Hausdorff space X. A quantifier C in A can be interpreted as an open- and - closed equivalence relation $\mathcal{C}$ in X, or as a continuous decomposition D of X. These facts are exposed in section 3. As the set of all equivalence relations in a space X is partially ordered ($R_1 \leq R_2$ if xR_1y implies xR_2y), in this way again a partial ordering is induced in Q(A). This partial ordering also turns out to be identical to the previous ones (see proposition 9 in section 4).

An interesting problem is the following. The class of all equivalence relations in a set X is a lattice under the partial ordering mentioned above. The subclass Cl(X) of all clopen equivalence relations is not a sublattice, not even an (upper or lower) sub-semilattice, as is shown by examples in section 2. In these examples, X can be taken as a zero-dimensional compact Hausdorff space. However, it is still possible, of course, that Cl(X) is lattice-ordered by $\leq$. To the best of my knowledge, this is an open question.

Because of the fact that the partially ordered set Q(A) is order-isomorphic to the partially ordered set $\mathcal{C}$l(X), the problem just stated is equivalent to the following one: is the set Q(A) of all quantifiers in a boolean algebra A always a lattice under the partial ordering described above? It would already be of interest if it could be shown that Q(A) always is an upper semilattice (which I suppose is true); I did not succeed yet in proving this, however, In the last section, the fact that two quantifiers C_1 and C_2 in a boolean algebra A commute: $C_1 \circ C_2 = C_2 \circ C_1$, is shown to have some connection with the partial ordering of Q(A). In fact, if $C_1C_2 = C_2C_1$, then C_1 and C_2 have a l.u.b. in Q(A), and this l.u.b. is C_1C_2. However, as is shown by an example, C_1 and C_2 may have a l.u.b. even if they do not commute.

2. Open and closed equivalence relations

Let X be a set, and R an equivalence relation in X. If $A \subset X$, then $R[A] = \{x \in X: xRy \text{ for some } y \in A\}$. Instead of $R[\{x\}]$ we write $R[x]$. The decomposition

$$X = \bigcup_{x \in X} R[x]$$

is denoted by D_R.

As is well-known (see e.g. [4]), the equivalence relations in a set X are lattice-ordered by the relation $\leq$,

$$(2.1) \qquad R_1 \leq R_2 \Leftrightarrow (\forall x, y \in X)(xR_1y \rightarrow xR_2y).$$

We have

$$(2.2) \qquad x(R_1 \wedge R_2)y \Leftrightarrow xR_1y \text{ and } xR_2y;$$

$$(2.3) \qquad x(R_1 \vee R_2)y \Leftrightarrow \text{there exists a finite chain } xR_1x_1,\ x_1R_2x_2,\ x_2R_1x_3, \ldots, x_{2n}R_1y.$$

It is clear that

$$(2.4) \qquad A = (R_1 \vee R_2)[A] \Leftrightarrow A = R_1[A] = R_2[A].$$

If we define

$$(2.5) \qquad S^o = A;\ S^{2n+1}A = R_1[S^{2n}.A];\ S^{2n+2}A = R_2[S^{2n+1}A];$$

then

$$(2.6) \qquad (R_1 \vee R_2)[A] = \bigcup_{n=0}^{\infty} S^nA.$$

Remark: It should be kept in mind that in general $(R_1 \wedge R_2)[A] \neq R_1[A] \cap R_2[A]$. The equality holds, however, if A consists of at most one point.

Now let X be a topological space. An equivalence relation R is called open (closed) if $R[A]$ is open (closed) whenever A is open (closed). In a different terminology (cf. [6], [7]), R is called open (closed) iff D_R is a lower (upper) semicontinuous decomposition).

The relation R is called <u>clopen</u> if it is both open and closed (i.e. iff D_R is a <u>continuous decomposition</u>).

The decomposition space, obtained from X by identifying points in the same D_R-set, is denoted by X/R, and is always supposed to be provided with the quotient topology.

Then the identification map $\pi_R : X \to X/R$ is always continuous; it is open (closed) iff R is open (closed).

<u>Proposition 1</u>: If R_1 and R_2 are open, then $R_1 \vee R_2$ is open.

<u>Proof</u>. If A is open, then every $S^n A$ is open, where the S^n are defined as in (2.5); hence $\bigcup_{n=0}^{\infty} S^n A = (R_1 \vee R_2)\, A$ is open.
It is not true, in general, that $R_1 \wedge R_2$ is open if R_1 and R_2 are.

<u>Example 1.</u> Let X be the unit square, i.e. the set of all pairs (x,y), $0 \leq x,y \leq 1$, with the euclidean topology. Put

$$(x,y)\ R_1\ (u,v) \Longleftrightarrow x = u.$$

A second equivalence relation R_2 will be described by means of the corresponding decomposition D_{R_2}. The sets of D_{R_2} are the straight-line segments joining the points (a,0) and (2a,0) $(0 \leq a \leq \frac{1}{3})$, and the segments joining the points (a,0) and $(\frac{a+1}{2}, 0)$ $(\frac{1}{3} \leq a \leq 1)$.

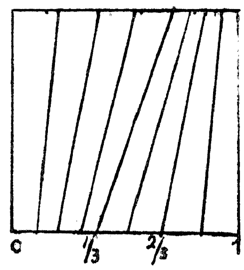

fig.1

Then both R_1 and R_2 are clopen. The equivalence classes of $R_1 \wedge R_2$ are all singletons $\{(x,y)\}$, $x \neq 0,1$, and the two segments $\{(0,y) : 0 \leq y \leq 1\}$ and $\{(1,y) : 0 \leq y \leq 1\}$. Hence $R_1 \wedge R_2$ is not open.

<u>Proposition 2.</u> If X is a normal Hausdorff space, then $R_1 \wedge R_2$ is closed if R_1, R_2 are closed.

Proof. We use the fact that an equivalence relation R is closed iff each equivalence class $R[x]$ has a neighbourhood base consisting of saturated sets (see e.g. [1] pag.62).

Let $A_i = R_i[x]$ (i=1,2), and let U be an open set containing $A_1 \cap A_2$. The sets A_1, A_2 are closed, as X is a T_1-space; the disjoint closed sets $A_1 \setminus U$ and $A_2 \setminus U$ have disjoint neighbourhoods V_1, V_2, as X is normal. The neighbourhood $U \cup V_i$ of A_i contains an open set W_i such that $A_i \subset W_i = R_i[W_i]$ (i=1,2); it follows that

$$A_1 \cap A_2 \subset W_1 \cap W_2 = (R_1 \wedge R_2)[W_1 \cap W_2] \subset U.$$

Hence $R_1 \wedge R_2$ is closed.

Remark. I do not know whether the assumption that X is a T_4-space is superfluous or not.

It may happen that R_1, R_2 are closed while $R_1 \vee R_2$ is not.

Example 2. Let X be the discontinuum of Cantor. We will define two clopen equivalence relations R_1, R_2 in X such that $R_1 \cup R_2$ is not closed. These equivalence relations will be defined by means of the corresponding decompositions.

Represent X in the canonical way as a subset of the unit segment. For n=0,1,2,... we define the subset A_n of X as follows:

$$A_n = \left\{ x \in X \ : \ 1 - \frac{1}{3^n} \leqslant x \leqslant 1 - \frac{2}{3^n} \right\} .$$

The decomposition D_{R_1} will consist of the sets A_o; $A_{2n-1} \cup A_{2n}$ (n=1,2,...); and $\{1\}$. The decomposition D_{R_2} will consist of the sets $A_{2n} \cup A_{2n+1}$ (n=0,1,...) and the set $\{1\}$. Both D_{R_1} and D_{R_2} are continuous, i.e. R_1 and R_2 are clopen. But $R_1 \cup R_2$ is not closed, for the set A_o is closed (even clopen) while $(R_1 \vee R_2)[A_o] = X \setminus \{1\}$ is not closed.

We are particularly interested in the case where X is a zero-dimensional compact Hausdorff space, and R_1, R_2 are both clopen equivalence relations.

Let Cl(X) be the set of all clopen equivalence relations in X. (In another terminology, Cl(X) is (in 1-1-correspondence to) the set of all continuous decompositions of X.). The set Cl(X) is partially ordered by $\leqslant$.

It is shown by example 2 that, even in the case where X is zero-dimensional compact Hausdorff, Cl(X) need not be a sublattice of the lattice of all equivalence relations in X. However, the two equivalence relations in example 2 have a l.u.b. in Cl(X), namely, the universal relation (the relation that holds between every two elements $x,y \in X$).

We will denote the l.u.b. and g.l.b. in Cl(X) by $\uparrow$ and $\downarrow$.

Example 1 can be easily modified to show that $R_1 \wedge R_2$ need not be clopen, for $R_1, R_2 \in Cl(X)$, even if X is a compact Hausdorff space. One can e.g. take X to be the set of all rational points in the unit square, instead of the full unit square.

In example 1 (modified or not) R_1 and R_2 have a g.l.b. $R_1 \downarrow R_2$ in Cl(X), namely the identity relation:

$$x(R_1 \downarrow R_2)y \Leftrightarrow x = y.$$

The following problem is still open:

<u>Problem</u>. If X is zero-dimensional compact Hausdorff, is it true that Cl(X) is lattice-ordered by $\leq$? I.e., for $R_1, R_2 \in Cl(X)$ do $R_1 \uparrow R_2$ and $R_1 \downarrow R_2$ always exist? If not, is Cl(X) at least a semilattice (upper or lower)?

<u>Proposition 3.</u> Let X be a compact Hausdorff space, A a clopen subset of X, and R_1, R_2 two clopen equivalence relations in X. The set $(R_1 \vee R_2)[A]$ is closed (and then also clopen) iff $(R_1 \vee R_2)[A] = S^nA$, for some integer $n \geq 0$.

<u>Proof.</u> All S^nA are clopen, hence if $(R_1 \vee R_2)[A] = S^nA$, then $(R_1 \vee R_2)[A]$ is closed. Conversely, suppose $(R_1 \vee R_2)[A]$ is closed.

Assume $S^{n+1}A \neq S^nA$, for all $n > 1$. Then the clopen sets $S^nA \setminus S^{n-1}A$ are non-void, for $n = 0,1,2,\ldots$ (we put $S^{-1}A = \emptyset$). Let $x_n \in S^nA \setminus S^{n-1}A$ $(n=0,1,\ldots)$; the sequence $\{x_n\}$ has a limit point a in the compact set $(R_1 \vee R_2)[A]$.

Let n be the smallest integer ≥ 0 such that $a \in S^nA$; then $S^nA \setminus S^{n-1}A$ is a neighbourhood of a that contains only one x_n. This is a contradiction; hence we conclude that $S^{n+1}A = S^nA$, for some $n > 1$. It follows at once that $(R_1 \vee R_2)[A] = S^nA$.

In the case of example 2, there are two pathological facts. In the first place we saw that $(R_1 \vee R_2)[A_o]$ is not closed, where A_o is a clopen set. In the second place we may remark that even for the one-point sets like $\{0\}$, $(R_1 \vee R_2)[0]$ is not closed. Hence $X/R_1 \vee R_2$ is not even a T_1-space.

Of course, if X/R is a Hausdorff space, and X is compact, then R is closed. For if π is the identification map, then π is continuous; if A is a closed subset of X, then A is compact, hence $\pi(A)$ is compact, and therefore closed. It follows that $R[A] = \pi^{-1}(\pi(A))$ is closed.

Something can be said also if it is only assumed that X/R is a T_1-space.

<u>Proposition 4.</u> Let X be a zero-dimensional compact Hausdorff space, and let R be an equivalence relation in X. If $R[x]$ is closed, for every $x \in X$, and if $R[A]$ is closed, for every clopen subset A of X, then R is closed.

<u>Proof.</u> Let $x \in X$, and let U be a neighbourhood of $R[x]$. We must show that there exists neighbourhood V of $R[x]$ such that $V = R[V]$ and $V \subset U$.

As $R[x]$ is closed and hence compact, U contains a clopen neighbourhood W of $R[x]$:

$$R[x] \subset W \subset U.$$

Then $R[X \setminus W]$ is closed by assumption, and disjoint with $R[x]$; hence $W = X \setminus R[X \setminus W]$ is an open neighbourhood of $R[x]$. Of course W is contained in U.

I was not yet able to answer the following question: is the assumption that X/R is a T_1-space really necessary in the case interesting us, i.e. the case $R = R_1 \vee R_2$, R_1 and R_2 being clopen? (In that case R itself is open.). In other words:

Let R_1 and R_2 be clopen equivalence relations in a zero-dimensional compact Hausdorff space X. Let $(R_1 \vee R_2)[A]$ be a clopen subset of X if A is any clopen subset of X. Does it follow that $R_1 \vee R_2$ is closed?

3. Quantifiers and equivalence relations

Definition 1. A quantifier in a boolean algebra A is a transformation $C : A \to A$ with the following properties:

(i) $C0 = 0$;

(ii) $a \leq Ca$, for all $a \in A$;

(iii) $C(a \wedge Cb) = Ca \wedge Cb$, for all $a, b \in A$.

The set of all quantifiers in A is denoted by Q(A).

Proposition 5. The quantifiers in a boolean algebra A are exactly the closure operators in A having the additional property that every open element is closed:

$$C(-Ca) = -Ca, \text{ for all } a \in A. \tag{3.1}$$

A proof can be found in Halmos [3] .

Definition 2. A subalgebra B of a boolean algebra A is called A-complete if, for each $a \in A$, the subset $\{b \in B : b \geq a\}$ of A has a g.l.b. in A, and if this g.l.b. belongs again to B.

Proposition 6. (cf. Halmos [3]): If $C \in Q(A)$, then C(A) is an A-complete subalgebra of A. The correspondence $C \to C(A)$ is a 1-1-correspondence, between the set of all quantifiers in A and the set of all A-complete subalgebras of A.

If B is an A-complete subalgebra, the corresponding quantifier C is defined by

$$Ca = \bigwedge\{b \in B,\ a \leq b\}. \tag{3.2}$$

Examples of quantifiers are given by

Proposition 7. (Varsavsky [5]). Let X be any set, and let A be the boolean algebra of all subsets of X. Let R be any equivalence relation in X. The operation $C_R : Y \to R[Y]$ $(Y \subset X)$ is a quantifier in A, and the correspondence $R \to C_R$ is a 1-1-correspondence between the set of all equivalence relations in X and the set Q(A).

If A is any boolean algebra, the zero-dimensional compact Hausdorff space of all ultrafilters in A, provided with the hull-kernel topology, will be denoted by T(A).

If $a \in A$, then $\hat{a} = \{U \in T(A) : a \in U\}$ is a clopen set in T(A); if F is a filter in A, then $\hat{F} = \{U \in T(A) : F \subset U\}$ is closed in T(A).

The mapping $a \rightarrow \hat{a}$ is an isomorphism of A onto the boolean algebra of all clopen subsets of T(A). If S is a closed set in T(A), then $S = \widehat{\cap S}$.

It is a result of Davis [2] (cf. also Varsavsky [5]), that there is a 1-1-correspondence between the set of all subalgebras of a boolean algebra A and the set of all those equivalence relations R in T(A) such that T(A) / R is a zero-dimensional compact Hausdorff space. This correspondence is the following.

If B is a subalgebra of A, the corresponding equivalence relation R is defined by

$$U_1 R U_2 \Longleftrightarrow U_1 \cap B = U_2 \cap B. \tag{3.3}$$

If R is an equivalence relation in T(A) such that T(A) / R is zero-dimensional compact Hausdorff, the corresponding subalgebra B of A is defined by

$$B = \{b \in A : R[\hat{b}] = \hat{b}\}. \tag{3.4}$$

Furthermore, if B and R correspond, then T(B) and T(A) / R are homeomorphic; and for $S \subset T(A)$ we have

$$R[S] = S \Leftrightarrow S = \cap\{\hat{b} : b \in B \text{ and } S \subset \hat{b}\}. \tag{3.5}$$

<u>Proposition 8.</u> (Halmos [3]). Under the correspondence just described, the set of all A-complete subalgebras of A corresponds with Cl(T(A)).

Let $C \in Q(A)$, $B = C(A)$, and let $R \in Cl(T(A))$ correspond with B; this relation will be denoted by $\mathcal{C}$. Then the following holds:

$$\mathcal{C}[\hat{a}] = \wedge\{b \in B : a \leqslant b\} = \widehat{Ca}. \tag{3.6}$$

Conversely: let $C \in Q(A)$, and let R be any equivalence relation in T(A) such that $R[\hat{a}] = \widehat{Ca}$, for all $a \in A$. Then it follows that R is clopen, and hence that $R = \mathcal{C}$.

4. The partial ordering of Q(A)

Definition 3. In Q(A), we define a partial ordering $\leqslant$ by

(4.1) $$C_1 \leqslant C_2 \Leftrightarrow C_1 a \leqslant C_2 a \quad , \text{ for all } a \in A.$$

(It is evident that this is indeed a partial ordering).

Proposition 9. $C_1 \leqslant C_2 \Leftrightarrow C_2 \circ C_1 = C_2 \Leftrightarrow C_1 \circ C_2 = C_2 \Leftrightarrow$
$$\Leftrightarrow C_2(A) \subset C_1(A) \Leftrightarrow \mathcal{C}_1 \leqslant \mathcal{C}_2.$$

Proof. Assume $C_1 \leqslant C_2$. Take $a \in A$; $C_1 a \leqslant C_2 a \Rightarrow C_2 C_1 a \leqslant C_2 C_2 a = C_2 a \leqslant C_2 C_1 a$; hence $C_2 C_1 = C_2$.

Assume $C_2 C_1 = C_2$. Take $a \in A$; $C_2 a = C_2 C_2 a = C_2 C_1 C_2 a$; hence $C_2(C_1 C_2 a \wedge -C_2 a) = C_2 C_1 C_2 a \wedge - C_2 a = 0$. Thus $C_1 C_2 a \wedge -C_2 a = 0$, or $C_1 C_2 a \leqslant C_2 a$. As certainly $C_2 a \leqslant C_1 C_2 a$, it follows that $C_1 C_2 = C_2$.

It is trivial that $C_1 C_2 = C_2 \Leftrightarrow C_2(A) \subset C_1(A)$.

Assume $\mathcal{C}_1 \leqslant \mathcal{C}_2$. Then, for $a \in A$: $\widehat{C_1 a} = \mathcal{C}_1[\hat{a}] \subset \mathcal{C}_2[\hat{a}] = \widehat{C_2 a}$; hence $C_1 a \leqslant C_2 a$. This implies $C_1 \leqslant C_2$.

Assume $\mathcal{C}_1 \not\leqslant \mathcal{C}_2$. Then there exists a point $U \in T(A)$ such that $\mathcal{C}_1[U] \setminus \mathcal{C}_2[U] \neq \emptyset$; let $V \in \mathcal{C}[U] \setminus \mathcal{C}_2[U]$. As $\{U\}$ is closed and $\mathcal{C}_2$ clopen, the set $\mathcal{C}_2[U]$ is closed; $V \notin \mathcal{C}_2[U]$, hence (as T(A) is zero-dimensional) V has a clopen neighbourhood $\hat{a}$ that is disjoint with $\mathcal{C}_2[U]$. It follows that $U \notin \mathcal{C}_2[\hat{a}]$. As $U \in \mathcal{C}_1[V] \subset \mathcal{C}_1[\hat{a}]$, we must have:

(4.2) $$\widehat{C_1 a} = \mathcal{C}_1[\hat{a}] \not\subset \mathcal{C}_2[\hat{a}] = \widehat{C_2 a};$$

i.e. $C_1 a \not\leqslant C_2 a$. Thus $C_1 \not\leqslant C_2$.

As it follows that Q(A) and Cl(T(A)) are order-isomorphic, it is natural to denote the g.l.b. and l.u.b. in Q(A), as far as they exist, by $\downarrow$ and $\uparrow$, respectively.

The main problem of section 2 turns out to be equivalent to the following problem about quantifiers:

Problem. Is Q(A) lattice-ordered by $\leqslant$, for every boolean algebra A?

Remark. Suppose $C_1, C_2 \in Q(A)$ have a g.l.b. $C_1 \wedge C_2$ in the partial ordering $\leqslant$ of $Q(A)$. Then $C_1 \wedge C_2 \leqslant C_i$ $(i=1,2)$; hence

$$(4.3) \qquad (C_1 \wedge C_2)a \leq C_1 a \wedge C_2 a, \qquad \text{for alle } a \in A.$$

In general, however, $(C_1 \wedge C_2)a \neq C_1 a \wedge C_2 a$, as $a \longrightarrow C_1 a \wedge C_2 a$ need not be a quantifier.

Example 3. Let A be the boolean algebra of all subsets of the unit square $X = [0,1]^2$. If $a \in A$, then let C_1 be the union of all horizontal straight-line segments intersecting a, and let $C_2 a$ be the union of all vertical straight line segments intersecting a. Let $Ca = C_1 a \cap C_2 a$. Then C is not a quantifier, as it does not satisfy the condition $C(a \cap Cb) =$ $= Ca \cap Cb$, for all $a, b \in A$.

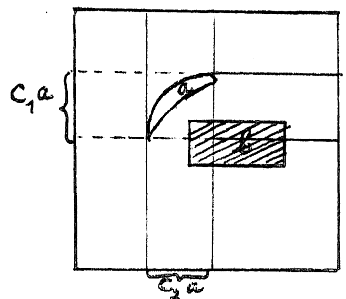

fig.2

If a,b are choosen as in figure 2, then $b = Cb$; $a \cap Cb = \emptyset$, hence $C(a \cap Cb) = \emptyset$; but $Ca \cap Cb \neq \emptyset$.

In this example it is immediately seen that $C_1 \downarrow C_2$ exists and is the identity quantifier:

$$(C_1 \downarrow C_2)a = a,$$

for all $a \in A$. Similarly, $C_1 \uparrow C_2$ exists and is equal to the unit quantifier:

$$(C_1 \uparrow C_2)a = X,$$

for every $a \neq \emptyset$.

As a consequence of proposition 3, we have:

Proposition 10. If $\mathcal{C}_1 \vee \mathcal{C}_2 = \mathcal{C}_1 \uparrow \mathcal{C}_2$, then for each $a \in A$ there exists a non-negative integer n such that

$$(C_1 \uparrow C_2)a = (C_1C_2)^n a.$$

Here $(C_1C_2)^n a$ stands for $C_1C_2C_1C_2\ldots C_1C_2a$, a being preceded by 2n quantifiers in total.

In particular it follows, that for every $a \in A$ there exists an $n \geq 0$ such that

$$C_2(C_1C_2)^n a = (C_1C_2)^n a \; ; \tag{4.4}$$

if n is any non-negative integer with this property, then $(C_1C_2)^n a = (C_1 \uparrow C_2)a$. We can also write:

$$(C_1 \uparrow C_2)a = \bigvee_{k=0}^{\infty} (C_1C_2)^k a. \tag{4.5}$$

As long as the question following proposition 4 has not been answered, the following problem remains open:

Let C_1, C_2 be quantifiers in a boolean algebra A. Suppose that for every $a \in A$ there exists a non-negative integer n such that (3.10) holds. Is it then true that the mapping $C : A \to A$ defined by

$$Ca = \bigvee_{k=0}^{\infty} (C_1C_2)^k a \tag{4.6}$$

is a quantifier?

If C is a quantifier, then it is immediate that $C = C_1 \uparrow C_2$, and that $\mathcal{C}_1 \uparrow \mathcal{C}_2 = \mathcal{C}_1 \vee \mathcal{C}_2$.

5. Commutativity of quantifiers

Proposition 11. Let A be a boolean algebra, and let $C_1, C_2 \in Q(A)$. Then

$$C_1 \circ C_2 \in Q(A) \Leftrightarrow C_1(C_2(A)) \subset C_2(A) \Leftrightarrow C_1(C_2(A)) \subset C_1(A) \cap C_2(A) \Leftrightarrow$$
$$\Leftrightarrow C_1(C_2(A)) = C_1(A) \cap C_2(A) \Leftrightarrow$$
$$\Leftrightarrow C_1 \uparrow C_2 \text{ exists, and is equal to } C_1 \circ C_2.$$

Proof. Suppose $C_1C_2 \in Q(A)$. Then, for any $a \in A$: $C_1C_2a \leq C_2C_1C_2a \leq C_1C_2C_1C_2a = C_1C_2a$; hence $C_1C_2 = C_2C_1C_2$, implying that $C_1(C_2(A)) \subset C_2(A)$.

Suppose $C_1(C_2(A)) \subset C_2(A)$. Certainly $C_1C_2 0 = 0$, and $a \leq C_1C_2a$, for all $a \in A$. As $C_2C_1C_2 = C_1C_2$ it also follows that

$$C_1C_2(a \wedge C_1C_2b) = C_1C_2(a \wedge C_2C_1C_2b) = C_1(C_2a \wedge C_2C_1C_2b) = C_1(C_2a \wedge C_1C_2b) = C_1C_2a \wedge C_1C_2b;$$ hence $C_1C_2 \in Q(A)$.

It is trivial that $C_1(C_2(A)) = C_1(A) \cap C_2(A) \Rightarrow C_1(C_2(A)) \subset C_1(A) \cap C_2(A) \Leftrightarrow C_1(C_2(A)) \subset C_2(A)$.

Assume $C_1(C_2(A)) \subset C_1(A) \cap C_2(A)$; it then follows that $C_1(C_2(A)) = C_1(A) \cap C_2(A)$, for if $a \in C_1(A) \cap C_2(A)$, then $a = C_1a = C_2a = C_1C_2a \in C_1(C_2(A))$.

Suppose $C_1 \uparrow C_2$ exists and is equal to C_1C_2. Then certainly $C_1C_2 \in Q(A)$. Conversely, suppose $C_1C_2 \in Q(A)$. Then it follows (see the first part of the proof) that $C_1C_2 = C_2C_1C_2$, and hence (by prop.9) that $C_2 \leq C_1C_2$. As certainly $C_1C_1C_2 = C_1C_2$, we also have $C_1 \leq C_1C_2$. Now let $C \in Q(A)$ such that $C_1 \leq C$ and $C_2 \leq C$. Then $C = C_1C = C_2C = C_1C_2C$, hence $C_1C_2 \leq C$. This shows that $C_1 \uparrow C_2$ exists and is equal to C_1C_2.

Proposition 12. Let A be a boolean algebra, and let $C_1, C_2 \in Q(A)$. Then

$$C_1 \circ C_2 = C_2 \circ C_1 \Leftrightarrow C_1 \circ C_2 \in Q(A) \text{ and } C_2 \circ C_1 \in Q(A).$$

Proof. If $C_1C_2 \in Q(A)$ and $C_2C_1 \in Q(A)$, then $C_1C_2 = C_1 \uparrow C_2 = C_2C_1$. Conversely, assume $C_1C_2 = C_2C_1$. Then $C_1C_2 \in Q(A)$. For certainly $C_1C_2 0 = 0$, and $a \leq C_1C_2a$, for all $a \in A$, and

$$C_1C_2(a \wedge C_1C_2b) = C_1C_2(a \wedge C_2C_1b) = C_1(C_2a \wedge C_2C_1b) = C_1(C_2a \wedge C_1C_2b) = C_1C_2a \wedge C_1C_2b.$$

If $C_1C_2 = C_2C_1$, then $C_1 \uparrow C_2$ exists. The converse is not true, however: $C_1 \uparrow C_2$ may exist while $C_1C_2 \neq C_2C_1$. This is shown by example 4. Before exhibiting this example, we describe the general construction behind it.

Let A be the boolean algebra of all subsets of a non-void set X. One can construct quantifiers in A as follows.

Let $\sigma: X \to X$ be an involution (i.e. $\sigma \circ \sigma =$ identity map); for $a \in X$, let $Ca = a \cup \sigma a$ (where $\sigma a = \{\sigma x : x \in a\}$). Then C is a quantifier in A.

It is immediate that $C\phi = \phi$ and that $a \subset Ca$, for all $a \in A$. We now show that $C(a \cap Cb) = Ca \cap Cb$, for arbitrary $a, b \in A$.

$$C(a \cap Cb) = (a \cap (b \cup \sigma b)) \cup \sigma(a \cap (b \cup \sigma b)) =$$

$$= ((a \cap b) \cup (a \cap \sigma b)) \cup (\sigma a \cap (b \cup \sigma b)) = \qquad \text{(as } \sigma \text{ is 1-1)}$$

$$= (a \cap b) \cup (a \cap \sigma b) \cup (\sigma a \cap b) \cup (\sigma a \cap \sigma b) =$$

$$= ((a \cup \sigma a) \cap b) \cup ((a \cup \sigma a) \cap \sigma b) =$$

$$= (a \cup \sigma a) \cap (b \cup \sigma b) = Ca \cap Cb.$$

This construction is applied in

<u>Example 4.</u> Let $X = \{\alpha, \beta, \gamma, \delta\}$; define $\sigma_1, \sigma_2 : X \to X$ as follows:

$$\sigma_1\alpha = \alpha, \quad \sigma_1\beta = \gamma, \quad \sigma_1\gamma = \beta, \quad \sigma_1\delta = \delta;$$

$$\sigma_2\alpha = \gamma, \quad \sigma_2\gamma = \alpha, \quad \sigma_2\beta = \delta, \quad \sigma_2\delta = \beta.$$

Let A be the boolean algebra of all subsets of X; if $a \in A$, let $C_i a = a \cup \sigma_i a$ (i=1,2).

As A is finite, Q(A) is finite, hence $C_1 \uparrow C_2$ exists. However, $C_1C_2 \neq C_2C_1$, for

$$C_1C_2\{\alpha\} = C_1\{\alpha, \gamma\} = \{\alpha, \beta, \gamma\};$$
$$C_2C_1\{\alpha\} = C_2\{\alpha\} = \{\alpha, \gamma\}.$$

(One verifies at once that $C_1 \uparrow C_2$ is the unit quantifier: $(C_1 \uparrow C_2)a = X$, for every $a \neq \emptyset$.)

Topologically the situation is very simple; the discrete space X "is" the Stone-space T(A) of A. The decomposition corresponding to C_1 is given by

$$X = \{\alpha\} \cup \{\beta, \gamma\} \cup \{\delta\};$$

the decomposition introduced by C_2 is given by

$$X = \{\alpha, \gamma\} \cup \{\beta, \delta\}.$$

There is only one decomposition of which both are refinements, namely the decomposition having X as its only component.

References

[1] N. BOURBAKI, Topologie Générale Ch.1, Third printing; A.S.I. 1142. Paris, 1961.

[2] Ch. DAVIS, Modal operators, equivalence relations and projective algebras. Amer.J.Math. 76 (1954) 217-249.

[3] P.R. HALMOS, Algebraic Logic I. Monadic Boolean algebras. Comp.Math. 12 (1955) 217-249.

[4] H. HERMES, Einführung in die Verbandstheorie. Berlin-Göttingen-Heidelberg, 1955.

[5] O. VARSAVSKY, Quantifiers and equivalence relations. Revista Matemática, Cuyana 2 (1956) 29-51.

[6] G.T. WHYBURN, Continuous decompositions. Amer.J.Math.71 (1949) 218 - 226.

[7] G.T. WHYBURN, Open and closed mappings. Duke Math.J. 17 (1950) 69-74.

A fairly complete list of the literature concerning quantifiers can be found in

[8] P.C. BAAYEN, Cylinderalgebra's. Report ZW 1962-004 of the Mathematical Centre, Amsterdam.