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AMSTERDAM

ZW 1952-26r

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Reprinted from
Proceedings of the KNAW, Series A, 55(1952)
Indagationes Mathematicae, 14(1952), p 24-27

1952

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Reprinted from Proceedings, Series A, **55**, No. 1 and Indag. Math., **14**, No. 1, 1952

MATHEMATICS

A PROPERTY OF POSITIVE MATRICES

BY

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(Communicated by Prof. J. F. KOKSMA at the meeting of December 22, 1951)

In this paper we establish an elementary property of matrices which appeared of importance for practical applications. First we give a definition, which is convenient for our purpose.

A symmetric matrix (c_{jk}) of order n the elements of which are complex numbers ($c_{jk} = a_{jk} + ib_{jk}$; a_{jk} and b_{jk} real) is called positive if

$$\sum_{j,k=1}^{n} a_{jk} x_j x_k \geqq 0$$

for all real points $(x_1, \ldots, x_n)$.

For positive matrices we prove the following Theorem. If a positive matrix $C = (c_{jk})$ has determinant zero, a set of real numbers $\lambda_1, \ldots, \lambda_n$ not all zero, exists such that

$$\sum_{k=1}^{n} c_{jk} \lambda_k = 0 \quad \text{for } j = 1, \ldots, n. \tag{1}$$

We give several proofs of the theorem all using the following well known lemma.

Let T and C be square matrices of order n; let the elements of the non-singular matrix T be real. Put $T'CT = D = (d_{jk})$. If n real numbers $\lambda_1, \ldots, \lambda_n$, not all zero exist which satisfy the relations (1), then also n real numbers $\mu_1, \ldots, \mu_n$, not all zero, exist which satisfy the relations

$$\sum_{k=1}^{n} d_{jk} \mu_k = 0 \quad \text{for } j = 1, \ldots, n,$$

and conversely.

First proof (W. PEREMANS)

We first remark that a positive matrix (a_{jk}), all elements of which are real, has the folllowing properties:

$$a_{jj} \geqq 0 \text{ for } j = 1, \ldots, n \tag{2}$$

$$\text{if } a_{kk} = 0 \text{ for some } k, \text{ then } a_{kj} = a_{jk} = 0 \text{ for } j = 1, \ldots, n. \tag{3}$$

Since det $c_{jk} = 0$, there exist n complex numbers $\mu_1, \ldots, \mu_n$, not all zero, satisfying

$$\sum_{k=1}^{n} c_{jk} \mu_k = 0 \quad \text{for } j = 1, \ldots, n. \tag{4}$$

Suppose that no $\varrho \neq 0$ exists such that all numbers $\varrho\mu_j$ $(j = 1, \ldots, n)$ are real. Without loss of generality we may assume that $\mu_1 \neq 0$ and μ_2/μ_1 is not real. Put

$$\mu_j = \mu_j' + i\mu_j'' \quad (\mu_j', \mu_j'' \text{ real}; \quad j = 1, \ldots, n).$$

Then $\mu_1' \mu_2'' - \mu_2' \mu_1'' \neq 0$. The matrix

$$T = (t_{jk}) = \begin{bmatrix} \mu_1' & \mu_1'' & 0 & \ldots & 0 \\ \mu_2' & \mu_2'' & 0 & \ldots & 0 \\ \cdot & \cdot & 1 & \ldots & 0 \\ \cdot & \cdot & & \cdot & \\ \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & & \\ \mu_n' & \mu_n'' & 0 & \ldots & 1 \end{bmatrix}$$

is non singular and has real elements. Put $D = T'CT$. Then

$$(5) \qquad d_{j1} = \sum_{r,s=1}^{n} t_{rj}\, c_{rs}\, \mu_s'; \quad d_{j2} = \sum_{r,s=1}^{n} t_{rs}\, c_{rs}\, \mu_s'',$$

so

$$(6) \qquad d_{j1} + i d_{j2} = \sum_{r,s=1}^{n} t_{rj}\, c_{rs}\, \mu_s = 0 \qquad (j = 1, \ldots, n)$$

in virtue of (4).

If in (6) we take $j = 1, 2$ respectively, we get

$$d_{11} + i d_{12} = 0, \quad d_{21} + i d_{22} = 0,$$

hence from $d_{12} = d_{21}$ it follows $d_{11} = - d_{22}$.

From the above remark we first deduce $\operatorname{Re} d_{11} \geqq 0$ $\operatorname{Re} d_{22} \geqq 0$, hence $\operatorname{Re} d_{11} = \operatorname{Re} d_{22} = 0$, and further get

$$\operatorname{Re} d_{j1} = \operatorname{Re} d_{j2} = 0 \qquad (j = 1, \ldots, n).$$

Again using (6) we deduce

$$\operatorname{Im} d_{j1} = \operatorname{Im} d_{j2} = 0 \qquad (j = 1, \ldots, n),$$

hence

$$(7) \qquad d_{j1} = d_{j2} = 0 \qquad (j = 1, \ldots, n).$$

Put $(\nu_1, \ldots, \nu_n) = (1, 0, \ldots, 0)$. Then from (5) and (7) it follows

$$\sum_{k=1}^{n} d_{jk}\, \nu_k = 0 \qquad (j = 1, \ldots, n)$$

and from our lemma we see that n real numbers $\lambda_1, \ldots, \lambda_n$, not all zero, exist, which satisfy (1).

Second proof (H. J. A. Duparc)

Let $c_{jk} = a_{jk} + i\, b_{jk}$ $(j,\ k = 1, \ldots, n)$, $A = (a_{jk})$. Let q be the rank of A, so that $0 \leqq q \leqq n$. There exists a non-singular matrix T with real elements such that $T'AT$ is a diagonal matrix. Hence, in view of our lemma, it suffices to prove the theorem in the case that $A = (a_{jk})$ is a diagonal matrix, satisfying

$$\begin{aligned} a_{jj} &> 0 \qquad \text{for } j = 1, \dots, q. \\ a_{jj} &= 0 \qquad \text{for } j = q+1, \dots, n. \\ a_{jk} &= 0 \qquad \text{for } j \neq k. \end{aligned}$$

Introducing the numbers μ_j, μ_j', μ_j'' as in the first proof, it follows from (4):

$$\sum_{k=1}^{n} (\mu_k' + i\mu_k'') (a_{jk} + i\, b_{jk}) = 0 \qquad (j = 1, \dots, n).$$

Hence

$$\sum_{k=1}^{n} (\mu_k' a_{jk} - \mu_k'' b_{jk}) = \sum_{k=1}^{n} (\mu_k' b_{jk} + \mu_k'' a_{jk}) = 0 \qquad (j = 1, \dots, n)$$

so

$$\mu_j' a_{jj} = \sum_{k=1}^{n} \mu_k'' b_{jk}; \quad \mu_j'' a_{jj} = -\sum_{k=1}^{n} \mu_k' b_{jk} \qquad (j = 1, \dots, n). \tag{8}$$

From (8) and the symmetry of the matrix (b_{jk}) it follows

$$\sum_{j=1}^{n} \mu_j'^2 a_{jj} = \sum_{j,k=1}^{n} \mu_j' \mu_k'' b_{jk} = \sum_{j,k=1}^{n} \mu_k' \mu_j'' b_{jk} = -\sum_{j=1}^{n} \mu_j''^2 a_{jj},$$

hence

$$\sum_{j=1}^{n} (\mu_j'^2 + \mu_j''^2)\, a_{jj} = 0. \tag{9}$$

From $a_{jj} > 0$ for $j = 1, \dots, q$, we get for $j = 1, \dots, q$ the relations $\mu_j' = \mu_j'' = 0$, so $\mu_j = 0$.

Since not all n numbers μ_k $(k = 1, \dots, n)$ are equal to zero we infer $q < n$.

Consider the two sets of n numbers

$$\mu_1', \dots, \mu_n'; \; \mu_1'', \dots, \mu_n''. \tag{10}$$

These sets satisfy in virtue of (8) and (9) the relations

$$\sum_{k=1}^{n} \mu_k' b_{jk} = 0; \quad \sum_{k=1}^{n} \mu_k'' b_{jk} = 0 \qquad (j = 1, \dots, n);$$

since obviously we also have

$$\sum_{k=1}^{n} \mu_k' a_{jk} = \sum_{k=1}^{n} \mu_k'' a_{jk} = 0 \qquad (j = 1, \dots, n),$$

we get

$$\sum_{k=1}^{n} \mu_k' c_{jk} = 0; \quad \sum_{k=1}^{n} \mu_k'' c_{jk} = 0 \qquad (j = 1, \dots, n).$$

Finally we remark that at least one of the two sets (10) contains a number $\neq 0$, because not all $\mu_1, \ldots, \mu_n$ are equal to zero. This proves the theorem.

A *third proof* (C. G. LEKKERKERKER) may be sketched briefly. It consists in reducing by a simultaneous transformation both matrices (a_{jk}) and (b_{jk}) into a form as simple as possible. Let q be the rank of A; let A_1, B_1, B_2 be the matrices respectively consisting of the first q rows and columns of A, the first q rows and columns of B, the last n—q rows and columns of B. We can arrive successively at the following three stages:

1. A is a diagonal matrix, A_1 a unit matrix.

2. moreover B_1 is a diagonal matrix; let at this stage p be the rank of B_2.

3. in addition $b_{jk} = 0$ for the pairs of indices (j, k), which do not belong to one of the following two sets:

$$(1, 1), \ldots\ldots, (q + p, q + p),$$
$$(1, q + p + 1), (q + p + 1, 1), \ldots\ldots, (n - q - p, n), (n, n - q - p).$$

On account of det $(a_{jk} + i\, b_{jk}) = 0$ at least one pair (j, k) occurs in the second set (hence $q < n$), so that $b_{jk} = 0$. Then it is obvious that there exist n numbers $\lambda_1, \ldots, \lambda_n$ with the required properties.

N.B. The theorem was posed to the Mathematical Centre at Amsterdam by the Research Laboratory of N. V. Philips' Gloeilampenfabrieken, Eindhoven, Netherlands, and was conjectured by Prof. B. D. H. TELLEGEN in connection with his investigations on electric networks.

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