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On the Minkowski-Hlawka theorem

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MATHEMATICS

ON THE MINKOWSKI-HLAWKA THEOREM

BY

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Let $R_n (n \geqq 2)$ be the real Euclidean space of points $x=(x_1, \ldots, x_n)$. Let O be the origin. The theorem of MINKOWSKI–HLAWKA asserts that, if K is a bounded star domain in R_n, of volume V and symmetric about O, there exists a lattice, which is admissible for K and whose determinant does not exceed $V/2\zeta(n)$, where $\zeta(n)=1+2^{-n}+3^{-n}+\ldots$. Various proofs of this theorem have been given [1, 2, 3, 4, 5]. In all these proofs the lattice found may have a rather contorted form: if S is any sphere about O and the volume of K has a fixed value, then, for suitably chosen K, these proofs lead to a lattice which does not have a basis contained in S. It is also clear that the proofs of SIEGEL [6] and WEIL [7] do not give us any information concerning the form of the lattices which have the required properties.

By applying the BRUNN–MINKOWSKI theorem, MAHLER [8] and DAVENPORT–ROGERS [9] obtained improvements of the theorem of MINKOWSKI–HLAWKA in the case of convex bodies. The results of these authors are as follows. For $n \geqq 2$, let c_n denote the lower bound of V/Δ for all centrally symmetric convex bodies in R_n, V being the volume and Δ denoting the critical determinant of K; it is the largest positive number such that, for all K, there exists a K-admissible lattice with determinant not exceeding V/c_n. By the theorem of MINKOWSKI–HLAWKA, $c_n \geqq 2\zeta(n)$. Now MAHLER found that

(1) $$c_2 \geqq 2\sqrt{3}, \quad c_n > 2\zeta(n)+1/6 \text{ for all } n,$$

whereas DAVENPORT–ROGERS proved that

$$\liminf_{n\to\infty} c_n \geqq c,$$

where $c=4.921\ldots$ is the solution of

(2) $$c \log c = 2(c-1) \qquad (c>1).$$

In this note I shall prove two theorems. Firstly, I shall show

Theorem 1. *Let c be defined by* (2). *Then $c_n > c$ for $n \geqq 5$.*

Next, I shall show that, for an arbitrary convex body K, by using the method of proof employed by MAHLER and DAVENPORT–ROGERS, one can find a lattice which has the properties discussed and, in addition, has a basis contained in some relatively small sphere about O. More precisely, I shall prove the following

Theorem 2. *Let the numbers* $\bar{c}_2, \bar{c}_3, \ldots$ *be given by*

$$(3) \qquad \bar{c}_2 = 3,\ \bar{c}_3 = 3.82,\ \bar{c}_4 = 4.41,\ \bar{c}_5 = 4.80,\ \bar{c}_6 = c \text{ for } n \geqq 6.$$

Let $\varkappa_n$ *be the volume of the n-dimensional unit sphere. Let* K *be a convex body in* R_n, *of volume* V *and symmetric about* O. *Then there exists a* K-*admissible lattice, whose determinant does not exceed* $V/\bar{c}_n$ *and which, after a suitable rotation about* O, *has a basis contained in the cube defined by*

$$(4) \qquad |x_i| < b\,(V/\varkappa_n)^{1/n} \quad (i = 1, \ldots, n),$$

where, for all n, *one may take* $b = 2.13$.

We note that, since $\varkappa_n = \pi^{n/2}/\Gamma\left(\frac{n+2}{2}\right)$, $\varkappa_n^{-1/n}$ is asymptotically equal to $\sqrt{n/(2\pi e)}$. Further, the cube defined by (4) is contained, also after an arbitrary rotation, in the sphere with centre at O and radius $\sqrt{n}\cdot b(V/\varkappa_n)^{1/n}$. So the assertion of theorem 2 can also be stated in the following somewhat weaker form: *there exists a* K-*admissible lattice, whose determinant does not exceed* $V/\bar{c}_n$ *and which has a basis contained in the sphere*

$$x_1^2 + \ldots + x_n^2 < (b_1 n V^{1/n})^2,$$

where b_1 *is some positive constant not depending on* K *and* n.

The proofs of the above theorems will be preceded by a number of lemmas. The first two of these lemmas are the main steps in the proof of Davenport–Rogers and are also fundamental for our purpose. For the proofs I refer to the paper mentioned above. Further, in particular, lemmas 4 and 5 deal with the volumes of the sections of a convex body or a star body by planes through O. Lemma 5 is perhaps of interest by itself.

The following notations will be used. For given K and real a, denote by $V(a)$ the $(n-1)$-dimensional volume of the section of K by $x_n = a$. For each $(n-1)$-dimensional hyperplane Π through O, let $v(\Pi)$ denote the $(n-1)$-dimensional volume of $K \cap \Pi$. Further, for each point x, write $|x| = (x_1^2 + \ldots + x_n^2)^{1/2}$. If $x \neq O$, then let Π_x denote the $(n-1)$-dimensional hyperplane, which passes through O and is orthogonal to the vector x. Finally, let ω_n denote the area of the unit sphere in R_n, so that

$$(5) \qquad \omega_n = n\varkappa_n \quad (n \geqq 2).$$

Lemma 1.[1]) *Let* $\mathscr{L}$ *be an* $(n-1)$-*dimensional lattice in the space*

[1]) Lemma 1 says a little more than lemma 2 in the paper of Davenport–Rogers, but follows immediately from the proof of that lemma. Lemma 2 follows from their lemma 3, if one puts $\beta = (1 + \delta/n)^{-n+1}$ and notes that

$$1 < \frac{n(1-\beta^{1/(n-1)})}{1-\beta^{n/(n-1)}} < n, \text{ if } 0 < \beta < 1.$$

$x_n = 0$, *with determinant* $d(\mathscr{L})$, *which has no point (except* O*) in* K. *Suppose that* $\alpha > 0$, *and*

(6) $$\sum_{t=1}^{\infty} V(\alpha t) < d(\mathscr{L}).$$

Then there exists a point g *of the form* $g = (g_1, \ldots, g_{n-1}, \alpha)$, *such that the lattice generated by* $\mathscr{L}$ *and* g *is admissible for* K.

Lemma 2. *Let* β *be a real number with* $0 < \beta < 1$ *and let* α *be defined by*

(7) $$\alpha = \frac{V}{2V(0)} \cdot \frac{n(1-\beta)^{1/(n-1)}}{1-\beta^{n/(n-1)}}.$$

Then

(8) $$\sum_{t=0}^{\infty} V(\alpha t) \leqq \beta V(0).$$

It is convenient to deduce from lemma 1 the following

Lemma 3. *The assertion of lemma* 1 *remains true, if* $\mathscr{L}$ *is allowed to have points on the boundary of* K *and if the condition* (6) *is replaced by*

(6') $$\sum_{t=1}^{\infty} V(\alpha t) \leqq d(\mathscr{L}).$$

Proof. Let $\mathscr{L}$ and α be such that (6') holds and that $\mathscr{L}$ has no point (except O) in the interior of K. For each positive integer r, let $\mathscr{L}_r$ be the lattice $(1+r^{-1})\mathscr{L}$. Then $\mathscr{L}_r$ has no point (except O) in K and (6) holds, with $\mathscr{L}_r$ instead of $\mathscr{L}$. So, by lemma 1, there exists a point $g^{(r)}$ in the plane $x_n = \alpha$, such that the lattice Λ_r generated by $\mathscr{L}_r$ and $g^{(r)}$ is admissible for K. Clearly there exists an increasing sequence of positive integers $r_1, r_2, \ldots$, such that Λ_{r_k}, for $k \to \infty$, converges to a lattice Λ generated by $\mathscr{L}$ and some point g in the plane $x_n = \alpha$. By a familiar argument [1]), Λ is admissible for K. This proves the lemma.

Lemma 4. *Let* K *be a convex body in* R_n, *of volume* $V = \varkappa_n$, *symmetric about* O. *Then there exists an* $(n-1)$*-dimensional hyperplane* Π, *which passes through* O, *such that*

(9) $$\tfrac{1}{2}\varkappa_n \leqq v(\Pi) \leqq \tfrac{1}{2}n\varkappa_n.$$

Proof. Let x' be a point on the boundary of K, for which $|x'|$ is maximal. Since $V = \varkappa_n$, K is not properly contained in the unit sphere, and so $|x'| \geqq 1$. Write $\Pi' = \Pi_{x'}$, and consider the cone C, with vertex at x' and with the intersection $K \cap \Pi'$ as a basis.

Since K is convex, C is contained in K. Hence, since K is also symmetric about O,

$$V = \varkappa_n \geqq 2V(C) = \frac{2}{n}|x'| \cdot v(\Pi') \geqq \frac{2}{n} v(\Pi'),$$

hence

$$v(\Pi') \leqq \tfrac{1}{2}n\varkappa_n.$$

[1]) See e.g. Mahler [10], proof of theorem 8.

Next, let x'' be a point on the boundary of K, such that $|x''|$ is minimal. Since the unit sphere is not properly contained in K, we now have $|x''| \leqq 1$. Write $\Pi'' = \Pi_{x''}$ and, for real a, denote by $W(a)$ the $(n-1)$-dimensional volume of the intersection of K and the plane which passes through ax'' and is parallel to Π''. In particular, $W(0) = v(\Pi'')$. In virtue of the BRUNN–MINKOWSKI theorem [1], the expression $\sqrt[n-1]{W(a)}$ is a concave function of a. Further, by the symmetry of K, this function is even. Hence $W(a) \leqq W(0)$, for all a, and so

$$V = \varkappa_n \leqq 2|x''| \cdot W(0) \leqq 2W(0) = 2v(\Pi'').$$

Hence

$$v(\Pi'') \geqq \tfrac{1}{2}\varkappa_n.$$

Since the point of intersection of the boundary of K and a straight line through O varies continuously, as the direction of this line varies, the volume $v(\Pi)$ varies continuously with Π. Then it follows from the above estimates for $v(\Pi')$ and $v(\Pi'')$ that Π may be chosen such that (9) holds.

When K is the n-dimensional unit sphere, then, for all Π, $v(\Pi)$ is equal to $\varkappa_{n-1}$. Here, since $\varkappa_n = \pi^{n/2}/\Gamma\left(\frac{n+2}{2}\right)$, the number $\varkappa_{n-1}$ is asymptotically equal to $\sqrt{n/(2\pi)} \cdot \varkappa_n$. One might conjecture that in lemma 4 the plane Π can always be chosen in such a way that $v(\Pi) = \varkappa_{n-1}$. But it seems difficult to decide whether this is true. In the following lemma I shall prove (for a much wider class of bodies) that a certain mean value of $v(\Pi)$ is at most equal to $\varkappa_{n-1}$. As a consequence, in lemma 4 the plane Π may be chosen such that instead of (9) we have

$$\tfrac{1}{2}\varkappa_n \leqq v(\Pi) \leqq \varkappa_{n-1}. \tag{9'}$$

Lemma 5. *Let K be a bounded star domain (not necessarily symmetric about O), of volume $\varkappa_n$. Let S_{n-1} be the sphere $x_1^2 + \ldots + x_n^2 = 1$. For $x \in S_{n-1}$, let $v(\Pi_x)$ be the $(n-1)$-dimensional volume of $K \cap \Pi_x$. Then we have*

$$\left[\frac{1}{\omega_n} \int_{S_{n-1}} \{v(\Pi_x)\}^{n/(n-1)}\, dx\right]^{(n-1)/n} \leqq \varkappa_{n-1}. \tag{10}$$

Proof. For $x \in S_{n-1}$, denote by $f(x)$ the uniquely determined positive number λ, for which λx belongs to the boundary of K. Clearly $f(x)$ is a positive, continuous function. Expressing the volume of K in terms of $f(x)$ we get

$$\varkappa_n = \frac{1}{n} \int_{S_{n-1}} \{f(x)\}^n\, dx.$$

For given $x \in S_{n-1}$, let us denote by $S_{n-2}(x)$ the set of points y with

$$y_1^2 + \ldots + y_n^2 = 1,\ y_1 x_1 + \ldots + y_n x_n = 0.$$

[1] See BONNESEN–FENCHEL [11], pp. 71 and 88.

Then for $v(\Pi_x)$ we have the expression

$$v(\Pi_x) = \frac{1}{n-1} \int_{S_{n-1}(x)} \{f(y)\}^{n-1} \, dy.$$

To the last integral we apply HÖLDER's inequality. This gives

$$v(\Pi_x) \leq \frac{1}{n-1} \Big[\int_{S_{n-2}(x)} \{f(y)\}^n \, dy \Big]^{(n-1)/n} \cdot \Big[\int_{S_{n-2}(x)} dy \Big]^{1/n}$$

$$= \frac{1}{n-1} \, \omega_{n-1}^{1/n} \Big[\int_{S_{n-2}(x)} \{f(y)\}^n \, dy \Big]^{(n-1)/n},$$

hence, on account of (5),

$$\{v(\Pi_x)\}^{n/(n-1)} \leq \frac{1}{n-1} \, \varkappa_{n-1}^{1/(n-1)} \int_{S_{n-2}(x)} \{f(y)\}^n \, dy.$$

The inequality (10) will follow if we can prove that

$$\int_{S_{n-1}} dx \int_{S_{n-2}(x)} \{f(y)\}^n \, dy = \omega_{n-1} \int_{S_{n-1}} \{f(y)\}^n \, dy; \tag{11}$$

for then we have

$$\int_{S_{n-1}} \{v(\Pi_x)\}^{n/(n-1)} \, dx \leq \frac{1}{n-1} \, \varkappa_{n-1}^{1/(n-1)} \, \omega_{n-1} \cdot n\varkappa_n = \varkappa_{n-1}^{n/(n-1)} \, \omega_n.$$

Let δ be a small positive number. We define a function $\phi(x, y)$ of two independent variables x, y, as follows:

$$\phi(x, y) = \begin{cases} \{f(y)\}^n & \text{if } |xy| \leq \delta \\ 0 & \text{if } |xy| > \delta \end{cases} \quad (x, y \in S_{n-1}),$$

where $xy = x_1 y_1 + \ldots + x_n y_n$. Since $f(y)$ is a continuous function of y, we certainly have

$$\int_{S_{n-1}} \int_{S_{n-1}} \phi(x, y) \, dx \, dy = \int_{S_{n-1}} \int_{S_{n-1}} \phi(x, y) \, dy \, dx.$$

We further have

$$\int_{S_{n-1}} \phi(x, y) \, dy \sim 2\delta \int_{S_{n-2}(x)} \{f(y)\}^n \, dy \quad \text{as } \delta \to 0,$$

$$\int_{S_{n-1}} \phi(x, y) \, dx = \{f(y)\}^n \int_{|xy| \leq \delta} dx \sim 2\delta \, \omega_{n-1} \{f(x)\}^n \quad \text{as } \delta \to 0.$$

From these relations (11) follows. This proves the lemma.

Lemma 6. *Let c be the solution of* (2). *Then, if $a > c$ and $n \geq 5$,*

$$\frac{2}{n} \frac{a^{n/(n-1)} - 1}{a^{1/(n-1)} - 1} > c. \tag{12}$$

Proof. Since the left-hand member of (12) can be written as a polynomial in $a^{1/(n-1)}$, with positive coefficients, it is an increasing function of a. Hence it is sufficient to prove that

$$\frac{2}{n} \frac{c^{n/(n-1)} - 1}{c^{1/(n-1)} - 1} \geq c \quad \text{for } n \geq 5. \tag{12'}$$

Put $y_n = c^{1/(n-1)} - 1$. Then $n = 1 + \log c / \log(y_n + 1)$ and so the inequality (12′) takes the form

$$(12'') \qquad \frac{2\log(y_n+1)}{\log c + \log(y_n+1)}\left(c + \frac{c-1}{y_n}\right) \geqq c \quad \text{for } n \geqq 5.$$

The quantity y_n is positive and tends to zero for $n \to \infty$. We now consider the function

$$f(y) = \frac{2\log(y+1)}{\log c + \log(y+1)}\left(c + \frac{c-1}{y}\right) \quad (y > 0).$$

Clearly, by (2),

$$\lim_{y \to +0} f(y) = \frac{2}{\log c}(c-1) = c.$$

We shall prove that $f(y)$ is a steadily increasing function of y for $0 < y < 0.5$.

Differentiating $f(y)$ and using (2) we get

$$f'(y) = -\frac{c-1}{y^2} \cdot \frac{2\log(y+1)}{\log c + \log(y+1)} + \left(c + \frac{c-1}{y}\right) \cdot \frac{2\log c}{(y+1)[\log c + \log(y+1)]^2}$$
$$= \frac{2(c-1)}{y^2(y+1)[\log c + \log(y+1)]^2}\,\psi(y),$$

with

$$\psi(y) = -(y+1)\log(y+1)\,[\log c + \log(y+1)] + 2y^2 + \left(2 - \frac{2}{c}\right)y.$$

We have $\psi(0) = 0$. Further,

$$\psi'(y) = -[\log^2(y+1) + (2 + \log c)\log(y+1) + \log c] + 4y + 2 - \frac{2}{c},$$
$$\psi''(y) = -\frac{1}{y+1}[2\log(y+1) + 2 + \log c] + 4 \quad (y \geqq 0).$$

Clearly $\psi'(0) = -\log c + 2 - \dfrac{2}{c} = 0$. Next,

$$2\log(y+1) + 2 + \log c < 0.4 + 2 + 1.6 = 4 \text{ for } 0 < y < 0.2,$$
$$2\log(y+1) + 2 + \log c < 1 + 2 + 1.6 < 4.8 \text{ for } 0.2 \leqq y < 0.5.$$

Hence $\psi''(y) > 0$, and so $\psi'(y) > 0$, for $0 < y < 0.5$. Then also $\psi(y) > 0$, and so $f'(y) > 0$ for $0 < y < 0.5$.

It follows that $f(y) > c$ for $0 < y < 0.5$. Since $0 < y_n < 0.5$ for $n \geqq 5$, this proves (12″) and so proves the lemma.

Lemma 7. *If a is a number with $c < a < 6$, and $n \geqq 6$, then*

$$(13) \qquad \frac{94}{100}\,a < \frac{2}{n}\,\frac{a^{n/(n-1)} - 1}{a^{1/(n-1)} - 1} < 6.$$

Proof. Put $n - 1 = m$, so that $m \geqq 5$. Write $a_m = \dfrac{2}{m+1}\,\dfrac{a^{(m+1)/m} - 1}{a^{1/m} - 1}$. Since a_m is a steadily increasing function of a, we have

$$a_m < \frac{2}{m+1}\,\frac{6^{(m+1)/m} - 1}{6^{1/m} - 1} = \frac{2}{m+1}\left(6 + \frac{5}{6^{1/m} - 1}\right)$$
$$< \frac{2}{m+1}\left(6 + \frac{5}{m^{-1}\log 6 + \frac{1}{2}m^{-2}\log^2 6}\right) = \frac{6}{m+1}\left(2 + \frac{10}{6\log 6} \cdot \frac{m}{1 + \frac{1}{2}m^{-1}\log 6}\right).$$

Hence $a_m < 6$, since

$$\frac{10}{6\log 6} = 0.9416\ldots < (1 - m^{-1})(1 + \tfrac{1}{2}m^{-1}\log 6) \text{ for } m \geqq 5.$$

Further, since $\frac{a-1}{a\log a}$ is steadily decreasing for $c<a$,

$$\frac{a_m}{a}=\frac{2}{m+1}\left(1+\frac{a-1}{a(a^{1/m}-1)}\right)>\frac{2}{m+1}\left(1+m\cdot\frac{a-1}{a\log a}\left(1-\frac{1}{2m}\log a\right)\right)$$

$$>\frac{2}{m+1}\left(1+\frac{5m}{6\log 6}-\frac{a-1}{2a}\right)>\frac{10}{6\log 6}>\frac{94}{100}.$$

This proves (13).

Proof of theorem 1. By (2), $c_2\geqq 2\sqrt{3}>1$. Suppose $n\geqq 3$, and that $c_{n-1}>1$. Let K be a convex body in R_n, of volume V, symmetric about O. Consider the section of K by $x_n=0$. It is an $(n-1)$-dimensional convex body, symmetric about O, of volume $V(0)$. By the definition of c_{n-1}, the critical determinant of this body is at most equal to $V(0)/c_{n-1}$, and so there exists an $(n-1)$-dimensional lattice $\mathscr{L}$ in the plane $x_n=0$, of determinant $d(\mathscr{L})=V(0)/c_{n-1}$, which has no point (except O) in the interior of K.

Put $\beta=c_{n-1}^{-1}$. Since we assumed that $c_{n-1}>1$, we have $0<\beta<1$. For this β let α be defined by (7). Then, from lemma 2,

$$\sum_{t=1}^{\infty} V(\alpha t)\leqq \beta V(0)=V(0)/c_{n-1}=d(\mathscr{L}).$$

In virtue of the lemmas 1 and 3 there now exists a point g of the form $g=(g_1,\ldots,g_{n-1},\alpha)$, such that the lattice Λ generated by $\mathscr{L}$ and g is admissible for K. It follows that the critical determinant of K, Δ say, is at most equal to $\alpha d(\mathscr{L})$. Hence

$$V/\Delta\geqq\frac{c_{n-1}V}{\alpha V(0)}=\frac{2c_{n-1}}{n}\frac{1-c_{n-1}^{-n/(n-1)}}{1-c_{n-1}^{-1/(n-1)}}=\frac{2}{n}\cdot\frac{c_{n-1}^{n/(n-1)}-1}{c_{n-1}^{1/(n-1)}-1}.$$

From the arbitrariness of K it then follows that

$$c_n\geqq\frac{2}{n}\frac{c_{n-1}^{n/(n-1)}-1}{c_{n-1}^{1/(n-1)}-1}. \tag{14}$$

From (14) it follows that $c_n>1$, since we assumed that $c_{n-1}>1$. Then it follows, by induction on n, that (14) holds for all $n\geqq 3$. Using the relation $c_2\geqq 2\sqrt{3}$ and applying (14) with $n=3,4,5$ we find that $c_5>c=4.921\ldots$ Hence, by (14) and lemma 6, we have $c_n>c$ for $n\geqq 5$. This proves the theorem.

Proof of theorem 2. We define **numbers** $d_2, d_3, \ldots$ as follows:

$$d_2=3,\ d_n=\frac{2}{n}\frac{d_{n-1}^{n/(n-1)}-1}{d_{n-1}^{1/(n-1)}-1}\ \text{for}\ n\geqq 3. \tag{15}$$

Clearly $d_n>1$ for all n. By induction on n, we shall prove the following assertion:

Let b be any number $\geqq\frac{200}{94}$. Then there exists a lattice Λ with the following properties:

1. *the lattice* Λ *is admissible for* K
2. *the determinant of* Λ *is equal to* V/d_n
3. Λ *has a basis contained in the cube*

$$W: \ |x_i| < b(V/\varkappa_n)^{1/n} \quad (i = 1, 2, \ldots, n)$$

4. *for each point* x *of the space there exists a point* $y \in \Lambda$, *such that* $x - y \in W$.

In proving this assertion it is no loss of generality to suppose that $V = \varkappa_n$.

First consider the case $n = 2$ and suppose that $V = \varkappa_2 = \pi$. Then there exists a point x' on the boundary of K at distance 1 from O. It is no loss of generality to suppose that x' is the point $(1, 0)$. Then the line $x_2 = \pi/3$ intersects K in a segment of length $\leqq 1$, since otherwise the volume of K would be greater than π. For a similar reason the line $x_2 = 2\pi/3$ does not intersect K. Consequently, there exists a K-admissible lattice Λ, generated by the point $(1, 0)$ and a point of the form $(a, \pi/3)$. Here we may take a such that $|a| \leqq \frac{1}{2}$. Hence Λ has a basis contained in the square

$$|x_i| \leqq \pi/3 < b \, (i = 1, 2).$$

Next, for each point x of the plane there exist integers u and v, such that $x - u \cdot (a, \pi/3) - v \cdot (1, 0)$ is contained in the square $|y_i| < b \, (i = 1, 2)$. Finally, Λ has determinant $d(\Lambda) = \pi/3$. This proves the assertion in the case $n = 2$.

Now let $n \geqq 3$ and suppose that the assertion is true, with n replaced by $n - 1$. Let K be a centrally symmetric convex in n dimensions, of volume $\varkappa_n$. As we already remarked above (see the relation (9′)), it follows from the lemmas 4 and 5 that there exists an $(n-1)$-dimensional plane Π through O, such that the $(n-1)$-dimensional volume of $K \cap \Pi$ is comprised between $\frac{1}{2}\varkappa_n$ and $\varkappa_{n-1}$. It is no loss of generality to suppose that Π is the plane $x_n = 0$, so that

$$\tag{16} \tfrac{1}{2}\varkappa_n \leqq V(0) \leqq \varkappa_{n-1}.$$

Hence $V(0)/\varkappa_{n-1} \leqq 1$. So, according to the induction hypothesis, there exists an $(n-1)$-dimensional lattice $\mathscr{L}$ in the plane $x_n = 0$ with the following properties:

1) $\mathscr{L}$ has no point (except O) in the interior of K
2) $\mathscr{L}$ has determinant $d(\mathscr{L}) = V(0)/d_{n-1}$
3) $\mathscr{L}$ has a basis contained in the cube

$$|x_i| < b (i = 1, 2, \ldots, n-1), \; x_n = 0$$

4) for each point x in the space $x_n = 0$ there exists a point $y \in \mathscr{L}$, such that $|x_i - y_i| < b$ for $i = 1, 2, \ldots, n-1$.

We now apply the lemmas 2 and 3. Take $\beta = 1/d_{n-1}$. Then $0 < \beta < 1$. With this value of β, let α be defined by (6). Then, by 2) and lemma 2,

$$\sum_{t=1}^{\infty} V(\alpha t) \leqq d(\mathscr{L}).$$

Hence, in virtue of 1) and lemma 3, there exists a point g of the form $g=(g_1, \ldots, g_{n-1}, \alpha)$, such that the lattice Λ generated by $\mathscr{L}$ and g is admissible for K. We shall prove that this lattice possesses also the properties 2, 3, 4.

Using 2) and the definitions of α and d_n we find that the determinant of Λ is given by

$$d(\Lambda)=\alpha d(\mathscr{L})=\frac{V}{2d_{n-1}}\,\frac{n(1-d_{n-1}^{-1/(n-1)})}{1-d_{n-1}^{-n/(n-1)}}=\frac{V}{2}\,\frac{n(d_{n-1}^{1/(n-1)}-1)}{d_{n-1}^{n/(n-1)}-1}=V/d_n.$$

Next, in virtue of 4), there exists a point $y\in\mathscr{L}$, such that $|g_i-y_i|<b$ for $i=1, 2, \ldots, n-1$. Put $g-y=x$. Then Λ is generated by $\mathscr{L}$ and x. Further $|x_i|<b$ for $i=1, 2, \ldots, n-1$ and $x_n=\alpha$. We prove that $\alpha<b$. We have $\alpha=\frac{V}{V(0)}\frac{d_{n-1}}{d_n}$. On account of (16), $\frac{V}{V(0)}\leqq 2$. From (15) one easily finds that $d_2<d_3<d_4<d_5<d_6<6$ and $d_6>c$. Then, by lemma 6, $d_n>c$ for $n>6$. Hence, by lemma 7, $\frac{94}{100}d_{n-1}<d_n<6$ for $n\geqq 7$. Hence $d_{n-1}/d_n<\frac{100}{94}$ for $n=3, 4, \ldots$. This shows that $\alpha<b$. It follows that Λ has a basis contained in the cube $|x_i|<b$ $(i=1, 2, \ldots, n)$. Finally, let x be an arbitrary point. There exists an integer u_n, such that $|x_n-u_n\alpha|<b$. In virtue of 4), there further exists a point $y\in\mathscr{L}$, such that $|x_i-u_ng_i-y_i|<b$ for $i=1, 2, \ldots, n-1$. Hence the point $x-u_ng-y$ is contained in the cube $|x_i|<b$ $(i=1, 2, \ldots, n)$.

This proves that Λ possesses the properties 1, 2, 3, 4. By induction on n, the truth of the assertion follows. Calculation gives $d_3>3.82$, $d_4>4.41$, $d_5>4.80$, whereas $d_n>c$ for $n\geqq 6$. From this the theorem follows.

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