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AFDELING ZUIVERE WISKUNDE ZW 28/75 FEBRUARI

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CRITICAL POINTS OF THE DEGENERATE OPERATOR IN ELLIPTIC SINGULAR PERTURBATION PROBLEM

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Printed at the Mathematical Centre, 49, 2e Boerhaavestraat, Amsterdam.

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AMS(MOS) subject classification scheme (1970): 35B25, 35J25.

Critical points of the degenerate operator in elliptic singular perturbation problems

by

P.P.N. de Groen

ABSTRACT

In this report we deal with the asymptotic behaviour for $\varepsilon \downarrow 0$ of the solution of the elliptic boundary value problem

$$\varepsilon L_2\Phi + L_1\Phi = h \text{ in } G,\ \Phi \text{ prescribed at } \partial G,$$

on a bounded domain $G \subset \mathbb{R}^2$. When $\varepsilon \downarrow 0$, the uniformly elliptic operator $\varepsilon L_2 + L_1$ degenerates to the first order operator L_1 which has critical points in the interior of G, i.e. points at which the coefficients of the first derivatives vanish. We construct a formal first order approximation for the simple types of critical points of L_1 and we prove the validity under some restriction on the range of the zero-th order part of L_1. In a number of cases we get internal layers of nonuniformity (which extend to the boundary in the saddle-point case) near the critical points; this depends on the position of the characteristics of L_1 and their direction. At special points outside the range, in which we could prove validity, we observe "resonance", a sudden displacement of boundary layers; these points are connected with the spectrum of $\varepsilon L_2 + L_1$.

TABLE OF CONTENTS

1. INTRODUCTION.

In a closed and bounded domain $G \subset \mathbb{R}^2$ with a piecewise smooth boundary we consider the linear elliptic boundary value problem

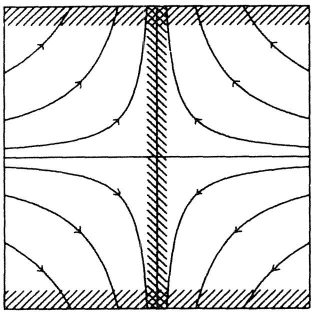
$$(1) \qquad \varepsilon L_2\Phi + L_1\Phi = h, \quad \Phi \text{ prescribed at } \partial G,$$
$$L_1 := p(x,y)\frac{\partial}{\partial x} + q(x,y)\frac{\partial}{\partial y} - \mu,$$

which depends on a small positive parameter ε and a real parameter μ. L_2 is a uniformly elliptic 2nd order partial differential operator: the quadratic form associated with its principal part is positive everywhere in G. L_1 is a first order partial differential operator which is allowed to have critical points in the interior of G, i.e. points at which the operator degenerates to zero-th order. Our aim is to study the asymptotic behaviour for $\varepsilon \downarrow 0$ of the solution Φ of (1). We will restrict ourselves to problems in which existence and unicity of the solution is ensured by a maximum principle (cf. [11]).

Most work on the singular perturbation problem (1) is done for an operator L_1 without singularities; e.g. ECKHAUS & DE JAGER [4] prove that then Φ is approximated by the solution of the reduced equation (set $\varepsilon = 0$ in (1)) in the major part of the domain and that a simple boundary layer of width $O(\varepsilon)$ is located along a part of the boundary. The characteristics of L_1 and the direction in which they are traversed play a decisive role in the location of the boundary layer. These characteristics are the curves $\{(x(s,t), y(s,t)) \mid t \text{ constant}\}$, directed in the sense of increasing s, where the transformation $(s,t) \mapsto (x,y)$ locally is a smooth diffeomorphism of the plane which transforms L_1 into $-\frac{\partial}{\partial s} + g(s,t)$ while the quadratic form associated with L_2 remains positive. At a point at which a characteristic *enters* the domain the approximation takes the given boundary value and its evolution inwards along the characteristic is governed by the reduced equation $L_1 u = h$. At the point at which the characteristic *leaves*, the value thus obtained has to be matched to the prescribed boundary value by a boundary layer term. The validity of such an approximation can be proved if neighbourhoods of characteristics which are tangent to the boundary or have more than two points in common are excluded. When a part of the boundary *coincides* with a characteristic, a parabolic boundary layer of width $O(\sqrt{\varepsilon})$ is formed along it, cf. [4] §4.2. GRASMAN [5] constructed an approximation which is valid also in a neighbourhood of a point at which a characteristic is tangent but does not enter. When a characteristic *enters* at a point where it is tangent to the boundary, we can construct an approximation by regularization as was proved in [6].

Singularities of L_1 make the problem much more complicated as was already pointed out by DE JAGER [9] and by the author [7]. We may expect that these singularities produce internal nonuniformities ("free boundary layers") since at such points the operator L_1 degenerates to zero-th order, though it will appear that this is not always the case. Here also the position of layers of nonuniformity depends of course on the direction in which the characteristics of L_1 are traversed. The approximation again takes the boundary value at a point at which a characteristic of L_1 enters the domain and its evolution along the characteristics is governed by the reduced equation. At parts of the boundary where characteristics leave the domain an (ordinary) boundary layer occurs. If characteristics end in a singularity an internal nonuniformity is formed in a neighbourhood of that point. Characteristics also can start at a singularity, in which case we do not know a priori at what value the solution of the reduced equation has to start. Furthermore, characteristics having a large distance at entrance may run very close to each other eventually; since the boundary values at entrance can differ considerably, we have to expect in this case too internal nonuniformities in which these differences are matched.

We will construct here (formal) first order approximations to the solution of (1), when G contains a single nondegenerate singularity of L_1, cf. §2.a; the validity of the approximation will be proved by the maximum principle, when μ is larger than some bound, determined by the spectrum of $\varepsilon L_2 + L_1$. At special values of μ below this bound the formal approximations exhibit the phenomenon of "resonance", as it was called among others in [14] and [15] for the analogous type of singular perturbation problems in ordinary differential equations. In order to exclude additional difficulties caused by the form of the boundary, we always assume that the characteris-



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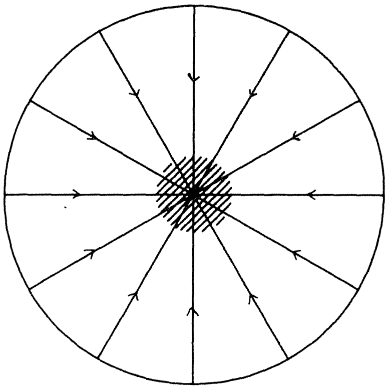

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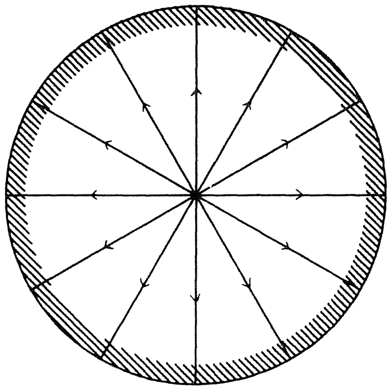

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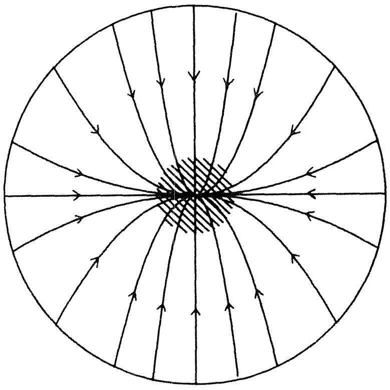

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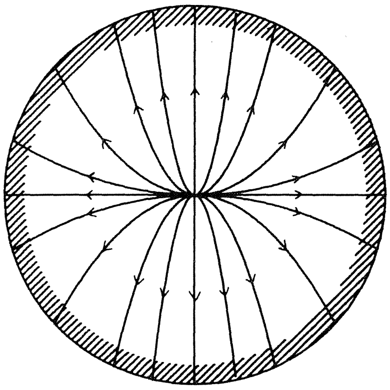

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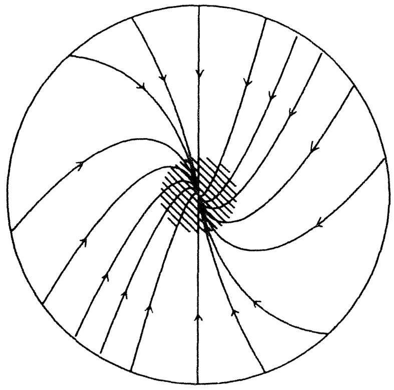

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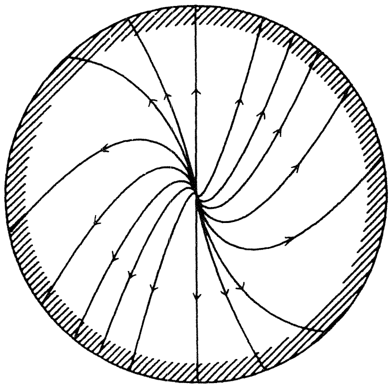

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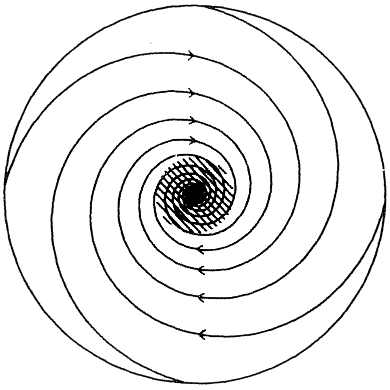

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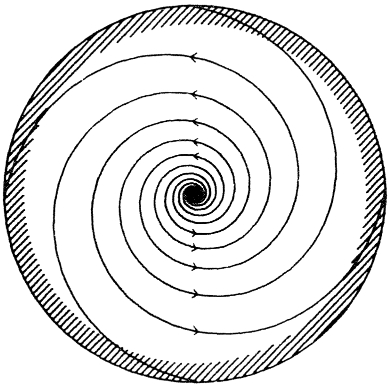

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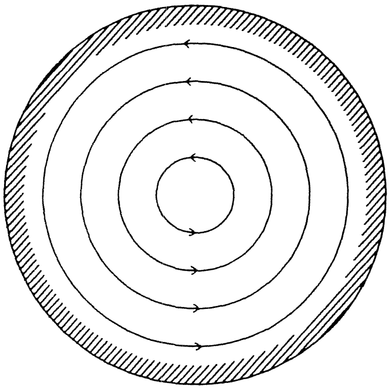

Figure 1.
Types of critical points; the curves denote characteristics with their direction and the shading \\\\ the position of the free and //// of the ordinary boundary layers; the numbers refer to §2.a.

tics of L_1 are nowhere tangent to the boundary (except in §7.c). After some preliminaries on types of singularities of L_1 and on the maximum principle in §2, we treat the case L_1 has a saddle-point in §§3-4, a node in §§5-6 and a vortex in §7. In §8 we deal with cases in which G contains several singularities of L_1 and with the phenomenon of "resonance" and in the appendix §9 we derive a number of inequalities on special functions we need. A qualitative summary of the results can be read from the figures (page 3, fig. 1) in which the domain G and some characteristics are drawn. The shadings /// and \\\ indicate the positions of the ordinary and the free boundary layers respectively. The numbers and the + and - sign refer to the different types of L_1 as displayed in §2.a.

2. PRELIMINARIES.

a. The *singularities* of the first order partial differential operator $L_1 := p\frac{\partial}{\partial x} + q\frac{\partial}{\partial y} + r$ are connected with the singular points of the system of ordinary differential equations

$$(1) \qquad \frac{dx}{ds} = p(x,y), \quad \frac{dy}{ds} = q(x,y);$$

its integral curves are the characteristics of L_1. Let the origin be a non-degenerate singular point of (1), i.e. $p(0,0) = q(0,0) = 0$ while the determinant of the Jacobian matrix J of (1) does not vanish at the origin. The type of the singularity is determined by $J(0,0)$, for which we have the following standard forms (cf. [8]):

1. saddle-point: $J(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}$ with $\lambda < 0$,
2. nodes: a. $J(0,0) = \pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$,

   b. $J(0,0) = \pm\begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}$ with $0 < \lambda < 1$,

   c. $J(0,0) = \pm\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$,
3. vortices: a. $J(0,0) = \pm\begin{pmatrix} 1 & \lambda \\ -\lambda & 1 \end{pmatrix}$ with $\lambda \neq 0$,

   b. $J(0,0) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

STERNBERG proved in [12] that (1) can be transformed into a linear equation by a smooth coordinate transformation, if the eigenvalues of $J(0,0)$ are in the same open complex half plane which does not contain the origin and if one of the eigenvalues is not an integral ($\neq 1$) multiple of the other. Hence we may assume without loss of generality that J is constant in the cases 2a, 2c, and 3a and in case 2b if $\lambda \notin \mathbb{N}\backslash\{1\}$. It is noted, that in general (1) cannot be linearized by a sufficiently smooth transformation, if the singularity is of saddlepoint-type or is a vortex of type 3b.

b. *The maximum principle* will be the tool, by which we will prove the validity of the constructed approximations. Let L be a 2nd order uniformly elliptic operator on a bounded domain G, whose zero-th order part is nonpositive, and let the quadratic form associated with it be positive. The maximum principle may be formulated as follows: "If a twice continuously differentiable function Φ attains a positive maximum in an interior point P of G, then $L\Phi(P) \leq 0$." (cf. [11] ch. 2, th.6). From this one easily derives

LEMMA 2.1. *If Φ and Ψ are twice continuously differentiable and if $|\Phi| < \Psi$ at the boundary of G and $|L\Phi| < -L\Psi$ in the interior of G, then also $|\Phi| < \Psi$ in the interior of G.*

REMARK. Ψ is called a barrier function for Φ.

PROOF. If $\Phi - \Psi$ attains a positive maximum in the interior of G, then $L(\Phi-\Psi) \leq 0$ according to the maximum principle. This is in contradiction to the assumption $L\Phi > L\Psi$ and so $\Phi - \Psi$ does not attain a positive maximum in the interior of G. Since $\Phi - \Psi$ is negative at the boundary, we have $\Phi < \Psi$ everywhere in G. In the same way we prove $-\Phi < \Psi$. □

When the zero-th order part of L is positive somewhere, then we can prove the following generalization (cf. [11] ch. 2, th.10): "If there exists a positive function W such that $LW \leq 0$ in all of G and if $L\Phi \geq 0$, then Φ/W does not have a positive maximum in the interior of G." This results in the generalization:

LEMMA 2.2. *If L is an elliptic second order operator whose associated quadratic form is positive in the domain G and if there exists a positive func-*

tion W *such that* $LW \le 0$ *on all of* G*, then every* C^2 *function* Φ *which satisfies* $|L\Phi| \le -LW$ *in the interior of* G *and* $|\Phi| \le W$ *at the boundary satisfies also* $|\Phi| \le W$ *everywhere in* G.

PROOF. The functions $U^{\pm} := -W \pm \Phi$ satisfy $LU^{\pm} \ge 0$. According to the maximum principle $U^{\pm}/W$ does not attain a positive maximum in the interior of G. Since they are nonpositive at the boundary by definition, they are nonpositive everywhere and the theorem follows. □

c. *notation*. The *entier* of a real number α, denoted by $[\alpha]$ is the largest integer smaller than or equal to α.

Let $f(x,\varepsilon)$ and $g(\varepsilon)$ be real functions with $x \in \mathbb{R}^n$ and $\varepsilon \in \mathbb{R}$, then we define

$$f(x,\varepsilon) = \mathcal{O}(g(\varepsilon)) \qquad \text{for } \varepsilon \downarrow \alpha$$

if there exists constants $M > 0$ and $\varepsilon_0 > \alpha$ such that $|f(x,\varepsilon)| \le M|g(\varepsilon)|$ for all $\varepsilon \in (\alpha,\varepsilon_0]$; this equation holds *uniformly* for $x \in G \subset \mathbb{R}^n$ if M and ε_0 can be chosen independent of x if $x \in G$. We define also the small o,

$$f(x,\varepsilon) = o(g(\varepsilon)) \qquad \text{for } \varepsilon \downarrow \alpha \text{ by} \qquad \lim_{\varepsilon\downarrow\alpha} f(x,\varepsilon)/g(\varepsilon) = 0.$$

A real function f on a domain $G \subset \mathbb{R}^n$ is called (of class) C^k for $k=0,1,2$, etc., if its k-th derivatives are continuous in G. It is called *uniformly Hölder continuous* with exponent $\alpha \in (0,1)$ or C^{α} if

$$|f(x) - f(y)| = \mathcal{O}(\|x-y\|^{\alpha}) \qquad \text{for } y \to x \text{ uniformly for } x \in G.$$

It is called C^{α} with $\alpha \ge 0$ if its $[\alpha]$-th derivatives are $C^{\alpha-[\alpha]}$.

The symbol $(\alpha)_n$ is defined for complex α and nonnegative integer n by

$$(\alpha)_n := \alpha(\alpha+1) \ldots (\alpha+n-1) = \Gamma(\alpha+n)/\Gamma(\alpha).$$

Frequently we will use in estimates the function R_a, defined by

$$(2) \qquad R_a(t) := 1 \text{ if } |t| \le 1 \quad \text{and} \quad R_a(t) := |t|^a \text{ if } |t| > 1.$$

The *boundary of a domain* G is denoted by ∂G.

We will denote consistently parts of the approximations to be constructed by the following letters (perhaps with indices, tildes etc.):

u for parts of the outer expansion

v for parts of the free boundary layer terms

w for parts of the ordinary boundary layer terms.

3. AN EXAMPLE OF THE SADDLE-POINT CASE; PERTURBATION BY Δ.

We study the boundary value problem

$$L_\varepsilon \Phi = \varepsilon \Delta \Phi + x\frac{\partial \Phi}{\partial x} - y\frac{\partial \Phi}{\partial y} = h(x,y) \tag{1a}$$

on the square

$$G := \{(x,y) \in \mathbb{R}^2 \mid |x| \le 1, |y| \le 1\}$$

with boundary conditions

$$\Phi(x,\pm 1) := f_\pm(x) \quad \text{and} \quad \Phi(\pm 1,y) := g_+(\pm y) \pm g_-(\pm y). \tag{1b}$$

The boundary functions are continuous at the cornerpoints, i.e.

$$f_\pm(1) = g_+(\pm 1) + g_-(\pm 1) \quad \text{and} \quad f_\pm(-1) = g_+(\mp 1) - g_-(\mp 1). \tag{1c}$$

The degree of smoothness of $f_\pm$, $g_\pm$ and h will be determined later on; ε is a small positive parameter.

Our aim is to construct an asymptotic approximation for $\varepsilon \downarrow 0$ to the solution Φ of (1a-c) which is uniform with respect to $(x,y) \in G$. This construction is performed in a number of steps. First bounds for the solution will be given and information will be drawn from it on the location of the boundary layers. Then the outer expansion and the boundary layer terms will be calculated in the homogeneous case (i.e. $h \equiv 0$) and correctness of the constructed approximation will be proved. Finally the inhomogeneous problem is reduced to the homogeneous one.

a. *Bounds for the solution* can be derived with aid of lemma 2.1. As a barrier function we use the solution ψ of (cf. §9.a)

$$\psi'' + \xi\psi' = -1, \qquad \psi(\pm 1/\sqrt{\varepsilon}) = 0;$$

it is bounded by $1 - \log \varepsilon$ and has the asymptotic representation

$$\psi(x/\sqrt{\varepsilon}) = -\log |x| + \mathcal{O}(\varepsilon/x^2) \qquad \text{for } \frac{x^2}{\varepsilon} \to \infty$$

as is proved in (9.5). If K_1 is the maximum of $|f_\pm|$ and $|g_+ \pm g_-|$ and K_2 is the maximum of $|h|$, then we conclude from

$$|\Phi(x,y)| \le K_1 + K_2\psi(x/\sqrt{\varepsilon}) \qquad \text{for all } (x,y) \in \partial G$$

and

$$|L_\varepsilon\Phi(x,y)| = |h(x,y)| \ge -K_2 = L_\varepsilon(K_1+K_2\psi(x/\sqrt{\varepsilon}))$$

with aid of lemma 2.1, that $\Phi(x,y)$ is bounded uniformly on G by $K_1 + K_2\psi(x/\sqrt{\varepsilon})$.

A better bound can be obtained for the function

$$\phi(x,y) := \Phi(x,y) - g_+(xy) - xg_-(xy)$$

It is zero at $|x| = 1$ and because of the continuity conditions (1c) there exists a constant $K_3 \ge K_2$ such that

$$|\phi(x,y)| \le K_3\psi(x/\sqrt{\varepsilon}) \qquad \text{at } \partial G$$

and such that

$$|L_\varepsilon\phi(x,y)| = |h(x,y) - L_\varepsilon(g_+(xy) + xg_-(xy))| \ge -K_3 = L_\varepsilon K\ \psi(x/\sqrt{\varepsilon}).$$

Hence by lemma 2.1

(2) $\quad |\Phi(x,y) - g_+(xy) - xg_-(xy)| \leq K_3\psi(x/\sqrt{\varepsilon})$ everywhere in G.

Since the C^∞-function ψ satisfies $\psi(\pm 1/\sqrt{\varepsilon}) = 0$ we may conclude, that the derivatives of Φ at $x = \pm 1$ are bounded independent of ε and that no boundary layer is to be expected there (at least in a first order approximation).

b. The *outer expansion* u is, assuming $h \equiv 0$, a solution of the equation of the first order

$$x \frac{\partial u}{\partial x} - y \frac{\partial u}{\partial y} = 0.$$

The characteristics are the family of hyperbolae

$$xy = \text{constant}$$

(oriented with the direction of increasing $|y|$) and u is constant along each of them. Since we did not expect a boundary layer at $x = \pm 1$, u will satisfy the boundary conditions there, i.e.

(3a) $\quad u_0(x,y) = g_+(xy) + \frac{x}{|x|} g_-(xy).$

This expression in general does not satisfy the boundary conditions at $y =$ $= \pm 1$ and is not continuous at $x = 0$, which is the dividing line of the two families of characteristics originating at $x = \pm 1$. Hence (ordinary) boundary layers have to be located at $y = \pm 1$ and an internal region of nonuniformity or "free boundary layer" at $x = 0$. Though (3a) will appear to be the correct expression for the lowest order term of the outer expansion (i.e. the approximation outside the boundary layers), it does not have an expedient form since it is not continuous at $x = 0$. Later on we need this term to be of class C^3, hence we subtract a part that is not C^3 and try to find a free boundary layer term which approximates this part outside the boundary layer. We define

$$g_-^*(x) := g_-(x) - g_-(0) - xg_-'(0) - \tfrac{1}{2}x^2 g_-''(0) - \tfrac{1}{6}x^3 g_-'''(0)$$

and take

(3b) $$u_1(x,y) := g_+(xy) + \frac{x}{|x|}\, g_-^*(xy)$$

which is of class C^3 everywhere on G.

c. *The free boundary layer term* v is calculated in the local coordinates (ξ,y) with $x = \xi\sqrt{\varepsilon}$. In these coordinates L_ε takes the form

$$L_\varepsilon = \frac{\partial^2}{\partial\xi^2} + \xi\frac{\partial}{\partial\xi} - y\frac{\partial}{\partial y} + \varepsilon\frac{\partial^2}{\partial y^2};$$

v is, still assuming $h \equiv 0$, the solution of the lowest order part

(4a) $$\left(\frac{\partial^2}{\partial\xi^2} + \xi\frac{\partial}{\partial\xi} - y\frac{\partial}{\partial y}\right) v = 0,$$

with asymptotic behaviour

(4b) $$v(x/\sqrt{\varepsilon},y) = \frac{x}{|x|}\{g_-(0) + xyg_-'(0) + \tfrac{1}{2}x^2y^2g_-''(0) + \tfrac{1}{6}x^3y^3g_-'''(0)\} + \mathcal{O}(1) \quad (\varepsilon\downarrow 0).$$

By the substitution

(5a) $$v(\xi,y) = v_0(\xi) + yv_1(\xi) + y^2v_2(\xi) + y^3v_3(\xi)$$

the equation (4a) together with the condition (4b) separates into

(5b) $$M_kv_k := v_k'' + \xi v_k' - kv_k = 0, \quad k=0,1,2,3,$$

with the asymptotic condition

(5c) $$v_k(x/\sqrt{\varepsilon}) \sim \frac{x}{|x|}\frac{x^k}{k!}\, g_-^{(k)}(0).$$

The solutions, given in §9.b, are

$$v_k(\xi) = \frac{1}{k!} \varepsilon^{\frac{1}{2}k} g_-^{(k)}(0)\ F_k^s(\xi)$$

and we find

$$\text{(5d)} \qquad v(\xi,y) = \sum_{k=0}^{3} \frac{1}{k!} g_-^{(k)}(0)\ \varepsilon^{\frac{1}{2}k} y^k F_k^s(\xi),$$

where s denotes the minus-sign if k is even and the plus-sign if k is odd.

d. In *the ordinary boundary layers* $u_1 + v$ is matched to the boundary conditions at the upper and lower boundaries. Since the construction in the lower boundary layer is exactly the same as in the upper one, it will be assumed that with a term w_i in the upper boundary layer also its counterpart $\hat{w}_i$ in the lower boundary layer is known. In the upper boundary layer we take the local coordinates (x,η) in order to match u_1 to the boundary condition f_+ and the local coordinates (ξ,η) to match v to zero, where $x = \xi\sqrt{\varepsilon}$ and $y = 1-\varepsilon\eta$.

In the coordinates (x,η) L_ε takes the form

$$L_\varepsilon = \frac{1}{\varepsilon}\left(\frac{\partial^2}{\partial\eta^2} + \frac{\partial}{\partial\eta}\right) + x\frac{\partial}{\partial x} - \eta\frac{\partial}{\partial\eta} + \varepsilon\frac{\partial^2}{\partial x^2}$$

and we have to solve $w_1(x,\eta) + \varepsilon w_2(x,\eta) + \ldots.$ from the equations

$$\left(\frac{\partial^2}{\partial\eta^2} + \frac{\partial}{\partial\eta}\right) w_1 = 0$$

with

$$w_1(x,0) = f_+(x) - u_0(x,1) \quad \text{and} \quad \lim_{\eta\to\infty} w_1(x,\eta) = 0$$

and

$$\left(\frac{\partial^2}{\partial\eta^2} + \frac{\partial}{\partial\eta}\right) w_2 = -\left(x\frac{\partial}{\partial x} - \eta\frac{\partial}{\partial\eta}\right) w_1$$

with

$$w_1(x,0) = 0 \quad \text{and} \quad \lim_{\eta\to\infty} w_2(x,\eta) = 0.$$

This results in

(6a) $$w_1(x,\eta) = \exp(-\eta)\ \{f_+(x) - u_1(x,1)\}$$

and

(6b) $$w_2(x,\eta) = -\eta\exp(-\eta)\ \{\tfrac{1}{2}\eta(u_1(x,1) - f_+(x)) + x\frac{\partial u_1(x,1)}{\partial x} + - xf_+'(x) + u_1(x,1) - f_+'(x)\}.$$

In the coordinates (ξ,η) the operator L_ε takes the form

$$L_\varepsilon = \frac{1}{\varepsilon}\left(\frac{\partial^2}{\partial\eta^2} + \frac{\partial}{\partial\eta}\right) + \frac{\partial^2}{\partial\xi^2} + \xi\frac{\partial}{\partial\xi} - \eta\frac{\partial}{\partial\eta}$$

and we now have to solve $z_1(\xi,\eta) + \varepsilon z_2(\xi,\eta)$ from the equations

$$\left(\frac{\partial^2}{\partial\eta^2} + \frac{\partial}{\partial\eta}\right) z_1 = 0$$

with

$$z_1(\xi,0) = -v(\xi,1) \quad \text{and} \quad \lim_{\eta\to\infty} z_1(\xi,\eta) = 0$$

and

$$\left(\frac{\partial^2}{\partial\eta^2} + \frac{\partial}{\partial\eta}\right) z_2 = -\left(\frac{\partial^2}{\partial\xi^2} + \xi\frac{\partial}{\partial\xi} - \eta\frac{\partial}{\partial\eta}\right) z_1,$$

with

$$z_2(\xi,0) = 0 \quad \text{and} \quad \lim_{\eta\to\infty} z_2(\xi,\eta) = 0.$$

This results in

(7a) $$z_1(\xi,\eta) = -v(\xi,1)\ \exp(-\eta)$$

and

(7b) $$z_2(\xi,\eta) = -\eta\exp(-\eta)\ \{v(\xi,1) + \xi\frac{\partial v(\xi,1)}{\partial\xi} + \frac{\partial^2 v(\xi,1)}{\partial\xi^2} + \tfrac{1}{2}\eta v(\xi,1)\} =$$

$$= -\eta\exp(-\eta)\ \{v(\xi,1) + v_1(\xi) + 2v_2(\xi) + 3v_3(\xi) + \tfrac{1}{2}\eta v(\xi,1)\}.$$

Hence the ordinary boundary layer solution is

(8) $$w(x,\eta) = w_1(x,\eta) + \varepsilon w_2(x,\eta) + z_1(x/\sqrt{\varepsilon},\eta) + z_2(x/\sqrt{\varepsilon},\eta).$$

e. *A proof of the validity*, still assuming $h \equiv 0$, of the first order approximation A_0, consisting of the functions constructed in c-d-e,

$$A_0(x,y,\varepsilon) := u_1(x,y) + v(x/\sqrt{\varepsilon},y) + w(x,\frac{1-y}{\varepsilon}) + \hat{w}(x,\frac{1+y}{\varepsilon}),$$

will now be given with aid of the maximum principle.
In the interior of G we have the following estimates:

(9a) $$L_\varepsilon u_1 = \varepsilon\Delta u_1 = \mathcal{O}(\varepsilon) \quad \text{uniformly in } G;$$

(9b) $$L_\varepsilon v = \varepsilon\frac{\partial^2}{\partial y^2} v = \mathcal{O}(\varepsilon) \quad \text{uniformly in } G,$$

since by (9.18-19a) the functions v_k are $\mathcal{O}(1)$ uniformly in G;

(9c) $$L_\varepsilon w = \varepsilon\frac{\partial^2}{\partial x^2}(w_1 + \varepsilon w_2) + \varepsilon(x\frac{\partial}{\partial x} - \eta\frac{\partial}{\partial\eta})w_2 + \varepsilon(\varepsilon\frac{\partial^2}{\partial x^2} + x\frac{\partial}{\partial x} - \eta\frac{\partial}{\partial\eta})\ z_2 =$$

$$= \mathcal{O}(\varepsilon) \quad \text{uniformly in } G,$$

as easily follows from the definition of w_1, w_2 and z_2 and from (9.18-20).
At the upper boundary ($y=1$) $\hat{w}$ is exponentially small and

$$u_1(x,1) + v(x/\sqrt{\varepsilon},1) + w(x,0) = f_+(x),$$

as follows from the construction of these functions, hence

$$A_0(x,1,\varepsilon) = f_+(x) + \mathcal{O}(\varepsilon).$$

At the right-hand boundary (x=1) we have with aid of (9.16)

$$u_1(1,y) + v(1/\sqrt{\varepsilon},y) = g_+(y) + g_-(y) + \mathcal{O}(\varepsilon)$$

and from (1c) we see that in particular

$$f_+(1) - u_1(1,1) - v(1/\sqrt{\varepsilon}) = f_+(1) - g_+(y) - g_-(y) + \mathcal{O}(\varepsilon) = \mathcal{O}(\varepsilon),$$

such that

$$w_1(1,(1-y)/\varepsilon) = \mathcal{O}(\varepsilon)$$

and analogously

$$\hat{w}_1(1,(1+y)/\varepsilon) = \mathcal{O}(\varepsilon).$$

At the other parts of the boundary we have analogous estimates; hence at the boundary of G we have the uniform estimate

$$\text{(10)} \qquad A_0(x,y,\varepsilon) = \Phi(x,y) + \mathcal{O}(\varepsilon) \quad \text{for } (x,y) \in \partial G.$$

With aid of lemma 2.1 and the barrier function

$$K_\varepsilon(1 + \psi(x/\sqrt{\varepsilon}))$$

in which the constant K is related to the constants in the order terms in (9a-c) and (10) we derive:

THEOREM 3.1. *Let* Φ *be the solution of the boundary value problem* (1) *with* $h \equiv 0$ *and let the boundary value functions* $f_\pm$ *and* $g_\pm$ *be of class* C^3, *then* Φ *is approximated by* A_0 *uniformly in* G,

$$\Phi(x,y) - A_0(x,y,\varepsilon) = \mathcal{O}(\varepsilon\psi(x/\sqrt{\varepsilon})) = \begin{cases} \mathcal{O}(\varepsilon) & \textit{for } x \geq \delta > 0, \\ \mathcal{O}(\varepsilon \log \varepsilon) & \textit{for all } (x,y) \in G. \end{cases}$$

f. When *the inhomogeneous term* $h(x,y)$ in (1a) is not zero, we construct a function B_0, bounded by $K_2(1 + \psi(x/\sqrt{\varepsilon}))$, where K_2 equals the maximum of $|h|$, such that

$$L_\varepsilon B_0 = h + \mathcal{O}(\varepsilon \log \varepsilon), \quad \text{uniformly on } G.$$

This reduces the problem to the case dealt with before.

The reduced equation $L_1 u = h$ has solutions of the form

(11a) $\int^x h(t,\frac{xy}{t})\,\frac{dt}{t}$ and (11b) $-\int^y h(\frac{xy}{t},t)\,\frac{dt}{t}$,

the first expression being in general discontinuous at the line $x = 0$ and the latter at $y = 0$. To the same purpose as in § 3.b where a part of the boundary condition was subtracted in order to get a sufficiently smooth outer expansion, h is split into different parts. Define the difference expressions

(12a) $(\mathcal{D}_x h)(x,y) := (h(x,y)-h(0,y))/x$ and $(\mathcal{D}_y h)(x,y) := (h(x,y)-h(x,0))/y.$

When h is C^2 the operators $\mathcal{D}_x$ and $\mathcal{D}_y$ commute; $\mathcal{D}_x^2 h$, $\mathcal{D}_y^2 h$ and $\mathcal{D}_x\mathcal{D}_y h$ are continuous and equal the respective second derivatives at (0,0) and $x^2(\mathcal{D}_x^2 h)(x,y)$ and $y^2(\mathcal{D}_y^2 h)(x,y)$ are C^1 but $xy(\mathcal{D}_x\mathcal{D}_y h)(x,y)$ is still C^2. Furthermore we have

(12b) $h(x,y) = h(0,y) + x(\mathcal{D}_x h)(x,y).$

Continuation of this rule, when h is C^8, results in the splitting

(12c) $h(x,y) = \sum_{j=0}^{3} x^j y^j (xh_j^+(x) + yh_j^-(x) + h_j^0) + x^4 y^4 H_4(x,y)$

in which the functions $h_j^\pm$, H_j and h_j^0 are defined by

$$\begin{aligned} h_j^+(x) &:= (\mathcal{D}_x^{j+1}\mathcal{D}_y^j h)(x,0), \\ h_j^-(y) &:= (\mathcal{D}_y^{j+1}\mathcal{D}_x^j h)(0,y), \\ H_j(x,y) &:= (\mathcal{D}_x^j\mathcal{D}_y^j h)(x,y), \\ h_j^0 &:= H_j(0,0). \end{aligned}$$

Clearly all parts of (12c) are of class C^5 at least.

We substitute each part of expression (12c) in (11a) or (11b) and try to find a lower bound of integration such that the integral is of class C^3. Such a lower bound does not exist for $h_j x^j y^j$, j=0,1,2 or 3 and for these parts we have to take in account the behaviour of their responses in the free boundary layer. For the remaining parts we find, if h is C^8

$$(13) \qquad u_2(x,y) := x^4y^4 \int_{x/|x|}^{x} H_4(t,xyt^{-1})t^{-1}dt + \\ + \sum_{k=0}^{3} (xy)^k \left\{\int_0^x h_k^+(t)dt - \int_0^y h_k^-(t)dt\right\}.$$

This expression is of class C^3 everywhere in G, hence it satisfies

$$(14) \qquad (\varepsilon\Delta + L_1)u_2 = h(x,y) - \sum_{k=0}^{3} h_k^0 x^k y^k + O(\varepsilon).$$

Outside the line x = 0 the function u_2 is much smoother; since $x^4y^4H_4$ is C^5, u_2 is C^5 too for $x \neq 0$, so $u_2(\pm1,y)$, which have to be subtracted from the boundary values given at $x = \pm1$, satisfy the smoothness condition on the boundary values of theorem 3.1.

g. *The part* $\tilde{v}$ *of the free boundary layer term* due to the inhomogeneous term h is a solution of the equation

$$(15a) \qquad \left(\frac{\partial^2}{\partial\xi^2} + \xi\frac{\partial}{\partial\xi} - y\frac{\partial}{\partial y}\right)\tilde{v} = \sum_{k=0}^{3} h_k^0 x^k y^k \qquad (x=\xi\sqrt{\varepsilon}).$$

Again we take

$$\tilde{v}(\xi,y) = \sum_{k=0}^{3} y^k \tilde{v}_k(\xi);$$

this results in the equations, cf. (5b),

$$(15b) \qquad M_k\tilde{v}_k = h_k^0\, \varepsilon^{\frac{1}{2}k}\xi^k.$$

From the equations, (with integer p, $0 \le p \le k$)

$$(16a)\qquad M_k(\xi^p\psi) = -\xi^p + (p-k)\xi^p\psi + p(p-1)\xi^{p-2}\psi + 2p\xi^{p-1}\psi',$$

$$(16b)\qquad M_k(\xi^p\psi') = -(k+p+1)\xi^p\psi' + p(p-1)\xi^{p-2}\psi' - 2p\xi^{p-1},$$

$$(16c)\qquad M_k(\xi^p) = (p-k)\xi^p + p(p-1)\xi^{p-2}$$

it can be inferred, that an exact solution of (15) consists of a suitable linear combination of $\xi^k\psi$, $\xi^{k-1}\psi'$, ξ^{k-2}, $\xi^{k-2}\psi$, ... ψ and ψ'. Hence

$$(16d)\qquad \begin{aligned}\tilde{v}_0 &= -h_0^0\psi\\ \tilde{v}_1 &= -h_1^0\sqrt{\varepsilon}(\xi\psi+\psi')\\ \tilde{v}_2 &= -h_2^0\varepsilon\xi^2\psi+....\\ \tilde{v}_3 &= -h_3^0\varepsilon\sqrt{\varepsilon}\xi^3\psi+....;\end{aligned}$$

since the lower order terms in $\tilde{v}_2$ and $\tilde{v}_3$ are of order $\mathcal{O}(\varepsilon)$ uniformly on G (e.g. $\varepsilon^{\frac{1}{2}k}\xi^{k-1}\psi' = \mathcal{O}(\varepsilon)$ for $k \geq 2$ as follows from (9.6)) and since we have the estimate (from (16), (9.6, 7 and 9))

$$M_k\varepsilon^{\frac{1}{2}k}\xi^k\psi(\xi) = \mathcal{O}(\varepsilon) \qquad \text{uniformly on } G \text{ for } k \geq 2,$$

we may skip the lower order terms in $\tilde{v}_2$ and $\tilde{v}_3$. Clearly we have for $\tilde{v}$, constructed thus:

$$(17)\qquad L_\varepsilon\tilde{v} = \sum_{k=0}^{3} y^k M_k\tilde{v}_k + \varepsilon\frac{\partial^2\tilde{v}}{\partial y^2} = \sum_{k=0}^{3} h_k^0\, x^k y^k + \mathcal{O}(\varepsilon),$$

uniformly on G; besides we have at the left- and right-hand boundaries

$$(18)\qquad \tilde{v}(1/\sqrt{\varepsilon},y) = \mathcal{O}(\varepsilon),$$

since $\psi(1/\sqrt{\varepsilon}) = 0$ and $\psi'(1/\sqrt{\varepsilon}) = \mathcal{O}(\sqrt{\varepsilon})$, cf. (9.4).
The function $B(x,y,\varepsilon)$ is now defined by

$$B(x,y,\varepsilon) := u_2(x,y) + \tilde{v}(x/\sqrt{\varepsilon},y).$$

h. *The result* can be stated as follows. If $\tilde{A}(x,y,\varepsilon)$ is the approximation of theorem 3.1 to the boundary value problem

$$L_\varepsilon \Phi = 0$$

$$\Phi(x,\pm 1) = f_\pm(x) - u_2(x,\pm 1) - \tilde{v}(x/\sqrt{\varepsilon},\pm 1),$$

$$\Phi(\pm 1,y) = g_+(\pm y) \pm g_-(\pm y) - u_2(\pm 1,y),$$

then we may conclude from (14), (17) and (18) with aid of the barrier function $\varepsilon K(\psi + 1)$ and lemma 2.1,

THEOREM 3.2. *The solution* Φ *of the boundary value problem* (1,a-b-c) *has the uniform approximation for* $\varepsilon \downarrow 0$

$$\Phi(x,y) = \tilde{A}(x,y,\varepsilon) + B(x,y,\varepsilon) + \begin{cases} O(\varepsilon) & \text{if } |x| \geq \delta > 0, \\ O(\varepsilon \log \varepsilon), & \end{cases}$$

if $f_\pm$ *and* $g_\pm$ *are* C^3 *and* h *is* C^8.

i. *Remarks*.

1. Since ψ is of order unity outside a fixed neighbourhood of $x = 0$, the approximations of Φ in both theorems are of order $O(\varepsilon)$ outside a fixed neighbourhood of the line $x = 0$. Moreover it is clear from the construction that the approximation of Φ in this region can be written in the form

$$\Phi(x,y) = g_+(xy) + \frac{x}{|x|} g_-(xy) + \int_{(x/|x|)}^{x} h(t,\frac{xy}{t}) \frac{dt}{t} +$$

$$+ \exp(\frac{1-y}{\varepsilon}) \left\{ f_+(x) - g_+(x) - \frac{x}{|x|} g_-(x) - \int_{(x/|x|)}^{x} h(t,\frac{x}{t}) \frac{dt}{t} \right\} +$$

$$+ \exp(\frac{1+y}{\varepsilon}) \left\{ f_-(x) - g_+(-x) - \frac{x}{|x|} g_-(-x) - \int_{(x/|x|)}^{x} h(t,-\frac{x}{t}) \frac{dt}{t} \right\} +$$

$$+ O(\frac{\varepsilon}{x^2}), \quad \varepsilon \downarrow 0,$$

This agrees with the result obtained in [4] theorem VII.

2. Higher order approximations can be obtained for this special problem by iteration, if the parameters of the problem are sufficiently many times differentiable. The process however becomes prohibitively laborious: for a second order approximation the expansion of h in (12) has to be pursued up to $\mathcal{O}(x^{12}y^{12})$.

3. If $h_0^0 \neq 0$, the solution Φ tends to infinity at the line $x = 0$, when ε tends to zero, due to the singularity of L_1.

4. THE SADDLE-POINT CASE; PERTURBATION BY AN ELLIPTIC OPERATOR.

In this section we generalize the results of §3 to the Dirichlet problem for the operator $L_\varepsilon := \varepsilon L_2 + L_1$ on a bounded domain G, where L_2 is uniformly elliptic and L_1 has a single singularity of saddle-point type. We did succeed in determining the free boundary layer terms in the first approximation only if we assume that L_1 takes the form

(1a) $$L_1 := x\frac{\partial}{\partial x} - \lambda y\frac{\partial}{\partial y} - \mu \quad \text{with} \quad \lambda < 0 \quad \text{and} \quad \mu > -1,\ \mu \text{ constant.}$$

i.e. the coefficients of $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ are *linear* in x and y; we have to assume also that the coefficient $a(x,y)$ in L_2,

(1b) $$L_2 := a\frac{\partial^2}{\partial x^2} + 2b\frac{\partial^2}{\partial x\partial y} + c\frac{\partial^2}{\partial y^2} + \text{lower order terms}, \quad a > 0 \text{ and } b^2 < ac,$$

satisfies $a(0,y)$ is constant; by a rescaling of ε we may take $a(0,y) = 1$. If these assumptions about the coefficients are not made the separation of variables in the equation of the free boundary layer, cf. (3.4a - 5), is not possible and we are not able to calculate the free boundary layer terms explicitly. Positivity and ellipticity of L_2 are ensured by $a > 0$ and $b^2 < ac$. Since the lower order parts of L_2 do not add any new difficulty we will skip them in the sequel. We have to take $\mu > -1$; for smaller values of μ we cannot prove the validity of the constructed approximation by the maximum principle, since barrier functions do not exist when $\mu < 1$.

a. *The problem* now is to find an asymptotic approximation to the solution Φ of the Dirichlet-problem

$$(2a) \qquad L_\varepsilon \Phi = (\varepsilon L_2 + x\frac{\partial}{\partial x} - \lambda y\frac{\partial}{\partial y} - \mu)\Phi = h$$

on the unit-square

$$G = \{(x,y) \in \mathbb{R}^2 \mid |x| \le 1, \ |y| \le 1\}$$

with boundary conditions

$$(2b) \qquad \Phi(x,\pm 1) := f_\pm(x) \quad \text{and} \quad \Phi(\pm 1,y) := g_+(y) \pm g_-(y),$$

which are continuous at the cornerpoints (cf. 3.1c); furthermore $f_\pm$, $g_\pm$ and h are sufficiently smooth. For convenience the splitting of the boundary conditions at x = ±1 into a symmetric and an antisymmetric part is chosen somewhat differently from (3.1b).

As in §3 a free boundary layer has to be located along the line x = 0 and ordinary boundary layers along the lines y = ±1, the construction is not essentially different but the amount of calculations is increased.

REMARK. The construction can be generalized easily for a domain G^*, whose boundary can be parametrized by four curves, (α_+,t), (α_-,t), (t,β_+) and (t,β_-), in which $\alpha_\pm$ and $\beta_\pm$ are sufficiently smooth functions, satisfying

(3a) $\qquad \alpha_+(t) > 0 > \alpha_-(t) \quad$ and $\quad \beta_+(t) > 0 > \beta_-(t)$

(3b) $\qquad \pm\lambda t\alpha'_\pm(t) > \mp\alpha_\pm(t) \quad$ and $\quad \pm t\beta'_\pm(t) > \mp\beta(t)$ (non-tangency condition)

(3c) $\qquad$ in the j-th quadrant there is one point (x_j,y_j) at which two curves meet.

In order to make formulae not more complicated than necessary, we will indicate in some remarks how this generalization works. By the conditions (3) the tangent to ∂G jumps at the corner points (x_j,y_j). If the boundary is smooth and hence is tangent somewhere to a characteristic, we have to add the local analysis of GRASMAN [5] in a neighbourhood of that point.

b. *The outer expansion* is a solution of the equation

$$L_1 u = h, \quad u(\pm 1,y) = g_+(y) \pm g_-(y),$$

whose characteristics are the curves $|x|^\lambda y$ = constant. It is solved for $x \neq 0$ by

$$(4) \qquad u_0(x,y) = |x|^\mu g_+(|x|^\lambda y) + x|x|^{\mu-1} g_-(|x|^\lambda y) + \\ + \int_{(x/|x|)}^{x} h(t,y(\tfrac{x}{t})^\lambda)(\tfrac{x}{t})^\mu \frac{dt}{t}.$$

If $\mu > 3$ this is of class C^3 at $x = 0$ and no free boundary layer terms appear in a first order approximation; if $\mu \leq 3$, a suitable decomposition of $g_\pm$ and h has to be made. Define $\mathcal{D}_x$ and $\mathcal{D}_y$ as in (3.12a). Let I be the set of pairs of integers

$$I := \{(k,l) \mid l = 0,1,2 \text{ or } 3,\ k \in \mathbb{N} \cup \{0\},\ k = l\lambda+\mu \text{ and } k < \lceil 4-\mu \rceil\}$$

and define the constants h_l^0 by

$$h_l^0 := (\mathcal{D}_x^k \mathcal{D}_y^l h)(0,0) \quad \text{if } (k,l) \in I \quad \text{and} \quad h_l^0 := 0 \quad \text{otherwise}.$$

Defining furthermore, if h is sufficiently smooth, i.e. $h \in C^{m^*+n^*}$,

$$\begin{aligned}
h_k^+(x) &:= (\mathcal{D}_y^k \mathcal{D}_x^{m_k} h)(x,0) && \text{with} \quad m_k := \lceil k\lambda+\mu \rceil + 1, \\
h_k^-(y) &:= (\mathcal{D}_x^k \mathcal{D}_y^{n_k} h)(0,y) && \text{with} \quad n_k := \max\{0, \lceil (k-\mu)/\lambda \rceil + 1\}, \\
h^*(x,y) &:= x^{m^*} y^{n^*} (\mathcal{D}_x^{m^*} \mathcal{D}_y^{n^*} h)(x,y) && \text{with} \quad m^* := \max\{0, \lceil 4-\mu \rceil\} \text{ and} \\
& && \phantom{\text{with}} \quad n^* := n_{m^*},
\end{aligned}$$

we get the splitting

$$(5)\qquad h(x,y) = \sum_{k=0}^{n^*-1} x^{m_k} y^k h_k^+(x) + \sum_{k=0}^{m^*-1} x^k y^{n_k} h_k^-(y) + \sum_{(k,1)\in I} h_1^0 x^k y^1 + h^*(x,y).$$

When h is C^p then h^* is $C^{p-\max\{n^*,m^*,1\}+1}$ and the other terms of (5) are at least as smooth. We now define the part u_2 of the outer expansion by

$$(6)\qquad u_2(x,y) := \sum_{k=0}^{n^*-1} x^{m_k} y^k \int_0^1 t^{m_k-k\lambda-\mu-1} h_k^-(xt)\,dt +$$

$$- \sum_{k=0}^{m^*-1} x^k y^{n_k} \int_0^1 t^{n_k-(k-\mu)/\lambda-1} h_k^-(yt)\,dt +$$

$$+ \int_{(x/|x|)}^{x} h^*(t, yx^\lambda t^{-\lambda}) x^\mu t^{-\mu-1}\,dt.$$

If $h \in C^3 \cap C^p$ with $p \geq m^*+n^*$ it is easily seen that u_2 is C^3, furthermore u_2 is a solution of $L_1 u_2 = h - \Sigma_{(k,1)\in I} h_1^0 x^k y^1$, hence it satisifes

$$(7)\qquad L_\varepsilon u_2 = h - \sum_{(k,1)\in I} h_1^0 x^k y^1 + O(\varepsilon) \quad \text{uniformly in } G.$$

Outside the line x = 0 the function u_2 is as smooth as h^* is. Since u_2 needs not be zero at the boundary of G we have to subtract its value at ∂G from the given boundary value; in view of expansion (9a) $u_2(\pm 1,y)$ is required to be of class $C^{n^*} \cap C^3$, so h has to be of class C^p with $p := \max\{3, n^*+2, 2n^*-1, m^*+2, m^*+n^*\}$. When I is not empty, the approximation $\tilde{v}$ of a solution $L_\varepsilon V = \Sigma_{(k,1)\in I} h_1^0 x^k y^1$ might be nonzero at the boundary too. This approximation is a part of the free boundary layer expansion to be constructed in the next subsection and it has the asymptotic behaviour, cf. (15),

$$(8)\qquad \tilde{v}(x/\sqrt{\varepsilon}, y) = \gamma(x,y) + O(\varepsilon/x^2) \qquad (\varepsilon x^{-2} \to 0).$$

Define $G_\pm$ by

$$G_\pm(y) := g_\pm(y) + \tfrac{1}{2}\{u_2(1,y) + \gamma(1,y) \pm u_2(-1,y) \pm \gamma(-1,y)\},$$

it then remains to find a splitting of the solution of

$$L_1 u = 0, \qquad u(\pm 1,y) = G_+(y) \pm G_-(y)$$

into a part which is C^3 and a part which is the asymptotic form of a part of the free boundary layer expansion. So we expand $G_\pm$,

(9a) $$G_\pm(y) = \sum_{j=0}^{n^*-1} g_j^\pm y^j + G_\pm^*(y), \qquad g_j^\pm := G_\pm^{(j)}(0)/j!,$$

and $G_\pm^*$ is $\mathcal{O}(y^{n^*})$ for $y \to 0$. Finally we define as in (3.3b)

(9b) $$u_1(x,y) := |x|^\mu G_+^*(|x|^\lambda y) + x|x|^{\mu-1} G_-^*(|x|^\lambda y).$$

which is C^3 if $g_\pm$ are $C^3 \cap C^{[(3-\mu)/\lambda]+1}$; hence u_1 satisfies

(9c) $$L_\varepsilon u_1 = \mathcal{O}(\varepsilon) \quad \text{and} \quad u_1(\pm 1,y) = G_+^*(y) \pm G_-^*(y).$$

REMARK. In case of domain G^* we change the lower bound $\frac{x}{|x|}$ $(= \pm 1)$ in the integrals by $\alpha_\pm(y)$ and we define $\sigma_\pm$ as the inverse functions of $y \mapsto y|\alpha_\pm(y)|^\lambda$. If $p_\pm(y)$ are the boundary values given at the part of the boundary parametrized by $(\alpha_\pm(y),y)$, then

$$u(x,y) := (x/\alpha_\pm(\tau))^\mu p_\pm(\tau) \text{ with } \tau(x,y) := \begin{cases} \sigma_+(x^\lambda y) & \text{if } x > 0 \\ \sigma_-(|x|^\lambda y) & \text{if } x < 0 \end{cases}$$

solves $L_1 u = 0$ together with the boundary conditions at $(\alpha_\pm(y),y)$. The functions $G_\pm$ now are defined by

$$G_+(y) \pm G_-(y) := \{(x/\alpha_\pm(\tau))^\mu (p_\pm(\tau) + (u_2+\gamma)(\alpha_\pm(\tau),\tau))\}\big|_{x = \pm 1}.$$

c. *In the free boundary layer* we take the local coordinates (ξ,y) with $\xi = x/\sqrt{\varepsilon}$. The lowest order part P_0 of L_ε in these coordinates is

$$P_0 := \frac{\partial^2}{\partial \xi^2} + \xi\frac{\partial}{\partial \xi} - \lambda y \frac{\partial}{\partial y} - \mu.$$

We now have to solve

(11a) $\qquad P_0 v = 0,$

where $v(\xi,y)$ is determined by its asymptotic behaviour for $\xi \to \infty$,

(11b) $$v(x/\sqrt{\varepsilon},y) = |x|^{\mu} \sum_{k=0}^{n^*-1} (g_k^+ + \frac{x}{|x|} g_k^-)|x|^{\lambda k} y^k + \mathcal{O}(\varepsilon/x^2).$$

Because of the assumption $a(0,y) \equiv 1$ on the coefficient of $\partial^2/\partial x^2$ in L_2 the operator P_0 admits separation of variables as in (3.5ab) and (3.15ab). With aid of (9.10) and (9.16) we find as in (3.5d)

(11c) $$v(\xi,y) = \sum_{k=0}^{n^*-1} \varepsilon^{\frac{1}{2}k\lambda+\frac{1}{2}\mu} y^k \{g_k^+ F_{k\lambda+\mu}^+(\xi) + g_k^- F_{k\lambda+\mu}^-(\xi)\}.$$

Since $a(x,y)-1 = \mathcal{O}(x) = \mathcal{O}(\xi\sqrt{\varepsilon})$, we see with aid of the estimates (9.19) and (9.20) on $F_\alpha^\pm$ and their derivatives:

(12) $$(L_\varepsilon - P)v = \{(a-1)\frac{\partial^2}{\partial\xi^2} + 2\sqrt{\varepsilon}b\frac{\partial^2}{\partial\xi\partial y} + \varepsilon c\frac{\partial^2}{\partial y^2}\}v =$$

$$= \mathcal{O}(\varepsilon^{\frac{1}{2}+\frac{1}{2}\mu}\xi R_{\mu-2}(\xi) + \varepsilon^{\frac{1}{2}+\frac{1}{2}\mu}R_{\mu-1}(\xi) + \varepsilon^{1+\frac{1}{2}\mu}R_\mu(\xi)) =$$

$$= \begin{cases} \mathcal{O}(\varepsilon) & \text{if} \quad \mu \geq 1 \\ \mathcal{O}(\varepsilon^{\frac{1}{2}+\frac{1}{2}\mu}R_{\mu-1}(\xi)) & \text{if} \quad -1 < \mu < 1. \end{cases}$$

When I is not empty we have to consider also the equation

(13a) $$P_0\tilde{v} = \sum_{(k,l)\in I} h_l^0 x^k y^l, \qquad (x = \xi\sqrt{\varepsilon}),$$

with asymptotic behaviour (8b), $\tilde{v}(x/\sqrt{\varepsilon},y) = \gamma(x,y) + \mathcal{O}(\varepsilon x^{-2})$. To $\tilde{v}$ we can add any solution of the homogeneous equation (11a), but this changes $G_\pm$ and hence also condition (11b) in such a way that the sum $v + \tilde{v}$ is not affected. Since however all terms of the asymptotic expansion are estimated separately we add the condition that outside the free boundary layer $\tilde{v}$ is of the same order of magnitude as $v + \tilde{v}$ is, i.e. that $\gamma(x,y)$ is of order unity.

We get a particular solution of (13a) inserting $\tilde{v}(\xi,y) = \Sigma_{(k,l)\in I} y^l \tilde{v}_k(\xi)$. The equation then separates as in (3.15b) in

$$M_k\tilde{v}_1(\xi) = h_1^0 \varepsilon^{\frac{1}{2}k}\xi^k \qquad \text{with } (1,k) \in I.$$

The solutions are given in (3.16d),

$$\text{(13b)} \qquad \tilde{v}_k(\xi) = \varepsilon^{-\frac{1}{2}k}\xi^k\psi(\xi) \quad \text{if } k \neq 1 \text{ and } \quad v_1(\xi) = \varepsilon^{-\frac{1}{2}}(\xi\psi(\xi) + \psi'(\xi)).$$

Since $L_\varepsilon - P_0$ still contains derivatives with respect to ξ we cannot obtain as good a remainder of $(L_\varepsilon - P_0)\tilde{v}$ as in (3.17). In fact we find from (3.16a-c) and (9.6-8) the estimates

$$\text{(14)} \qquad (L_\varepsilon - P_0)y^1\tilde{v}_k(x/\sqrt{\varepsilon}) = \begin{cases} O(\sqrt{\varepsilon}R_{-1}(x/\sqrt{\varepsilon})) & \text{if } k = 0, \\ O(\varepsilon\psi(x/\sqrt{\varepsilon})) & \text{if } k = 1, \\ O(\varepsilon) & \text{if } k = 2,3. \end{cases}$$

From (9.4-5) we derive

$$\text{(15)} \qquad \tilde{v}(x/\sqrt{\varepsilon},y) = \sum_{(k,1)\in I} h_1^0 x^k y^1 \log x + O(\varepsilon/x^2), \quad \text{hence}$$

$$\gamma(x,y) = \sum_{(k,1)\in I} h_1^0 x^k y^1 \log x.$$

It has to be remarked that v and $\tilde{v}$ are C^∞-functions in both variables since they are finite sums of products of nonnegative integral powers of y and ξ with ψ and with confluent hypergeometric functions.

d. *In ordinary boundary layers*, located along the upper and lower boundaries, we have to match $u_1 + u_2 + v + \tilde{v}$ to the boundary condition $f_\pm$; for this we take the local coordinates (x,η) with $\eta = (1-y)/\varepsilon$ in the upper and $\eta = (y+1)/\varepsilon$ in the lower boundary layer. We assume again that every term w, constructed in the upper boundary layer has a counterpart $\hat{w}$ in the lower one, constructed in the same way.

In the upper boundary layer we have to solve (approximately)

$$\text{(16a)} \qquad L_\varepsilon w(x,\eta) = 0$$

with boundary conditions

(16b) $w(x,0) = \tilde{f}_+(x,x/\sqrt{\varepsilon})$ and $\lim_{\eta\to\infty} w(x,\eta) = 0.$

where $\tilde{f}_+$ is defined by

(16c) $\tilde{f}_+(x,\xi) := f_+(x) - u_1(x,1) - u_2(x,1) - v(\xi,1) - \tilde{v}(\xi,1).$

Since a C^1-function $Q(x,y)$ of two parameters satisfies

$$(\frac{\partial}{\partial x} + \frac{\partial}{\partial y})Q\Big|_{x=y} = \frac{d}{dx} Q(x,x),$$

we can deal with the parameters x and $\xi = x/\sqrt{\varepsilon}$, on which the boundary condition for $w(x,0)$ depends, as if they were independent variables, when we replace $\frac{\partial}{\partial x}$ in the expression for L_ε by $\frac{\partial}{\partial x} + \frac{1}{\sqrt{\varepsilon}}\frac{\partial}{\partial \xi}$. If we also substitute η in the coefficients of L_ε and expand these coefficients into powers of ε, we get

(17) $$L_\varepsilon = \frac{1}{\varepsilon} N_0 + \frac{1}{\sqrt{\varepsilon}} N_1 + N_2 + \sqrt{\varepsilon}N_3 + \varepsilon N_4,$$

in which

$$N_0 := \tilde{c}_0 \frac{\partial^2}{\partial \eta^2} - \lambda\frac{\partial}{\partial \eta},$$

$$N_1 := (2b_0 \frac{\partial^2}{\partial \xi \partial \eta} - x\frac{\partial}{\partial \xi}),$$

$$N_2 := \eta\tilde{c}_1 \frac{\partial^2}{\partial \eta^2} + \tilde{a}_0 \frac{\partial^2}{\partial \xi^2} + 2\tilde{b}_0 \frac{\partial^2}{\partial x \partial \eta} + x\frac{\partial}{\partial x} + \lambda\eta\frac{\partial}{\partial \eta} - \mu,$$

$$N_3 := 2\eta\tilde{b}_1 \frac{\partial^2}{\partial \xi \partial \eta},$$

$$N_4 := \eta^2\tilde{c}_2 \frac{\partial^2}{\partial \eta^2} + \eta\tilde{a}_1 \frac{\partial^2}{\partial \xi^2} + \tilde{a}\frac{\partial^2}{\partial x^2} + 2\eta\sqrt{\varepsilon}\tilde{b}_2 \frac{\partial^2}{\partial \xi \partial \eta} + 2\eta(\tilde{b}_1+\eta\tilde{b}_2)\frac{\partial^2}{\partial x \partial \eta};$$

the coefficients and their expansions are defined by

$$\tilde{c}(x,\varepsilon\eta) := c(x,1-\varepsilon\eta)-2b(x,1-\varepsilon\eta)+a(x,1-\varepsilon\eta) = \tilde{c}_0(x)+\varepsilon\eta\tilde{c}_1(x)+\varepsilon^2\eta^2\tilde{c}_2(x,\varepsilon\eta),$$

$$\tilde{b}(x,\varepsilon\eta) := 2a(x,1-\varepsilon\eta)-2b(x,1-\varepsilon\eta) = \tilde{b}_0(x)+\varepsilon\eta\tilde{b}_1(x)+\varepsilon^2\eta^2\tilde{b}_2(x,\varepsilon\eta),$$

$$\tilde{a}(x,\varepsilon\eta) := a(x,1-\varepsilon\eta) = \tilde{a}_0(x)+\varepsilon\eta\tilde{a}_1(x,\varepsilon\eta).$$

A necessary condition for this expansion is that the coefficients of L_2 are C^2.

Now we set $w(x,\eta) = \Sigma_{k=0}^{3} \varepsilon^{\frac{1}{2}k} w_k(x,\xi,\eta)$ (with $\xi = x/\sqrt{\varepsilon}$) and solve successively for k=0,1,2 and 3, treating formally the parameters ξ and x as independent ones,

(18a) $$N_0 w_k = -\sum_{j=0}^{k-1} N_{k-j} w_j$$

with boundary conditions at $\eta = 0$

$$w_0(x,\xi,0) = \tilde{f}_+(x,\xi) \quad \text{and} \quad w_k(x,\xi,0) = 0 \quad \text{for } k=1,2,3$$

and for large η (i.e. outside the boundary layer)

$$\lim_{\eta\to\infty} w(x,\xi,\eta) = 0 \quad \text{for } k=0,1,2,3.$$

By this we get the remainder

(18b) $$L_\varepsilon \sum_{k=0}^{3} \varepsilon^{\frac{1}{2}k} w_k = \sum_{l=0}^{3} \varepsilon^{1+\frac{1}{2}l} \sum_{k=l+1}^{4} N_k w_{l-k+4}.$$

From (18a) we find

(18c) $$w_0(x,\xi,\eta) = \tilde{f}_+(x,\xi)\,\exp\{-\lambda\eta/\tilde{c}_0(x)\}$$

and the calculation of the other w_k, which all contain the same exponential factor as w_0 does, is straightforward, cf. [4] §3.4. A necessary condition in order that the iteration (18a) makes sense is that the functions w_k are sufficiently smooth. The coefficients of N_k for k=0,1,2,3 and $\tilde{f}$ are C^∞ with respect to ξ and η and so do the solutions w_k of (18a). When the coefficients of L_2 are C^3, the coefficients of N_0 and N_1 are C^3 with respect to x and those of N_2 and N_3 are C^2; furthermore $\tilde{f}$ and its ξ-derivatives are C^3 with respect to x. Since N_2 contains only a first x-derivative and N_0, N_1 and N_3 do not contain any x-derivative, it follows from (18a) that w_0, w_1 and their ξ- and η-derivatives are C^3 in x and w_2 and w_3 are C^2, so (18b) is continuous in all its variables. It satisfies the estimate

$$\text{(19a)} \qquad L_\varepsilon \sum_{k=0}^{3} \varepsilon^{\frac{1}{2}k} w_k = \begin{cases} O(\varepsilon) + O(\varepsilon\psi(x/\sqrt{\varepsilon})) & \text{when } \mu = 0, \\ O(\varepsilon) + O(\varepsilon^{1+\frac{1}{2}\mu} R_\mu(x/\sqrt{\varepsilon})) & \text{otherwise.} \end{cases}$$

In order to prove this formula, we first estimate the part $N_4 w_0$ of it. From (16c), (12c), (13b), (9.9) and (9.18) we find:

$$\tilde{f}(x,\xi) = \begin{cases} O(1) + O(\psi(x/\sqrt{\varepsilon})), & \text{when } \mu = 0 \quad \text{for } \varepsilon \downarrow 0, \\ O(1) + O(\varepsilon^{\frac{1}{2}\mu} R_\mu(x/\sqrt{\varepsilon})), & \text{otherwise} \quad \text{for } \varepsilon \downarrow 0. \end{cases}$$

By (9.19-20) and (9.8) we find that all ξ-derivatives of $\tilde{f}$ and its first, second and third x-derivatives are of the same order at worst. All derivatives of the exponential factor of w_0 (cf. 18c), that occur in $N_4 w_0$, are bounded by a constant uniformly in $(x,\eta) \in [-1,1] \times [0,\infty)$. Although the coefficients of N_4 are of order η^2, the exponential factor in w_0 ensures existence of a bound for $N_4 w_0$, which is uniform in $\eta \in [0,\infty)$. So $N_4 w_0$ is of the same order as $\tilde{f}$ is. In exactly the same way we show that all other terms of (19) are of this order too; hence (19a) is proved.

Finally we show that this local first order approximation w in the upper boundary layer is small at the other parts of the boundary. From (16c) and the continuity of the boundary conditions, cf. (2b), at the cornerpoints we find that $\tilde{f}_+(\pm 1, \pm 1/\sqrt{\varepsilon}) = f_+(\pm 1) - g_+(1) \mp g_-(1) + O(\varepsilon) = O(\varepsilon)$, hence from (18c) we see that $w_0(\pm 1, \pm 1/\sqrt{\varepsilon}, (1-y)/\varepsilon) = O(\varepsilon)$. By equation (18a) with $k = 1$ we see that w_1 contains the factor $\partial\tilde{f}_+/\partial\xi$ linearly and that the remaining part is uniformly bounded for all ε. By (9.8) and (9.19-20) we see that each term of $v + \tilde{v}$, which is itself of order $O(\varepsilon^{\frac{1}{2}\delta}\xi^\delta)$ for $|\xi| \geq 1$ and for some δ, has a ξ-derivative of order $O(\varepsilon^{\frac{1}{2}\delta}\xi^{\delta-1})$ for $|\xi| \geq 1$ and hence of order $O(\sqrt{\varepsilon})$ at $\xi = \pm 1/\sqrt{\varepsilon}$. So $\partial f_+/\partial\xi$ $(= \partial v/\partial\xi + \partial\tilde{v}/\partial\xi)$ is of order $O(\sqrt{\varepsilon})$ at $\xi = \pm 1/\sqrt{\varepsilon}$ and hence $w_1(\pm 1, \pm 1/\sqrt{\varepsilon}, (1-y)/\varepsilon)$ too. Since furthermore w_2 and w_3

are bounded at $x = \pm 1$ independent of ε and y, we have derived:

(19b) $$w(\pm 1,(1-y)/\varepsilon) = \mathcal{O}(\varepsilon) \quad \text{uniformly in } y, \text{ for } \varepsilon \downarrow 0.$$

From the exponential factor in w we see furthermore that w is of an order smaller than every positive power of ε at the lower boundary,

(19c) $$w(x,2/\varepsilon) = o(\varepsilon^n) \quad \text{for every } n \in \mathbb{N}.$$

<u>REMARK</u>. In case of domain G^* we take the boundary layer coordinates (x,η) with $\eta = (\beta_+(x)-y)/\varepsilon$ in the upper and $\eta = (\beta_-(x)+y)/\varepsilon$ in the lower boundary layer.

e. *The validity* of the formal approximation Ψ, which is the sum of the outer expansion and the free and ordinary boundary layer terms constructed in the subsection b-c-d,

(20) $$\Psi(x,y) := u_1(x,y) + u_2(x,y) + v(x/\sqrt{\varepsilon},y) + \tilde{v}(x/\sqrt{\varepsilon},y) + \\ + w(x,(1-y)/\varepsilon) + \hat{w}(x,(1+y)/\varepsilon),$$

will be proved by the maximum principle (cf. section 2.b). This approximation satisfies, cf. (7), (9c), (12), (14) and (19a),

$$L_\varepsilon\Psi = h + \mathcal{O}(\varepsilon) + \mathcal{O}(\varepsilon^{\frac{1}{2}+\frac{1}{2}\mu}R_{\mu-1}(x/\sqrt{\varepsilon})) + \\ \left(+ \mathcal{O}(\varepsilon\psi(x/\sqrt{\varepsilon})) \text{ if } \lambda + \mu \text{ is integral}\right), \qquad (\varepsilon\downarrow 0),$$

uniformly with respect to $(x,y) \in G$; at the boundary of G it satisfies

$$\Psi(x,\pm 1) = f_\pm(x) + o(\varepsilon^n) \quad \text{for every } n \in \mathbb{N}, \text{ cf. (18a) and (19c)},$$
$$\Psi(\pm 1,y) = g_+(y) \pm g_-(y) + \mathcal{O}(\varepsilon), \quad \text{cf. (9c), (11b) and (19b)}.$$

When Φ is the solution of the boundary value problem (2), these formulae result in the uniform estimates, valid for $\varepsilon \downarrow 0$, (21a) $\quad L_\varepsilon(\Phi-\Psi) = \mathcal{O}(\varepsilon) + \mathcal{O}(\varepsilon^{\frac{1}{2}+\frac{1}{2}\mu}R_{\mu-1}(x/\sqrt{\varepsilon})) + \Big(+\ \mathcal{O}(\varepsilon\psi(x/\sqrt{\varepsilon}))$ if $\lambda+\mu$ is integral$\Big)$

in the interior of G and

(21b) $\quad (\Phi - \Psi)\big|_{\partial G} = \mathcal{O}(\varepsilon), \qquad (\varepsilon\downarrow 0),$

at the boundary. With aid of lemma 2.1 or 2.2 and a suitable barrier function we derive from this an estimate on the difference of the solution Φ and the constructed approximation Ψ. When $\mu \geq 1$ we apply lemma 2.1 using $C_1\varepsilon$ as barrier function, if $\lambda + \mu$ is nonintegral, and $C_1\varepsilon + C_2\varepsilon\psi(x/\sqrt{\varepsilon})$, if $\lambda + \mu$ is integral; in here C_1 and C_2 are suitably chosen constants. When $\mu < 1$ the terms $\mathcal{O}(\varepsilon\psi)$, if present, and $\mathcal{O}(\varepsilon)$ in (21a) are contained in the third one. By (9.19-20) we deduce

(22) $\quad L_\varepsilon F_\nu^+(x/\sqrt{\varepsilon}) = (\nu-\mu)F_\nu^+(x/\sqrt{\varepsilon}) = \mathcal{O}(\sqrt{\varepsilon}R_{\nu-1}(x/\sqrt{\varepsilon})) \quad$ if $|x| \leq 1$,

which by (9.18a) is negative if $-1 < \nu < \mu$. Hence, when $0 < \mu < 1$ we can apply lemma 2.1 with a constant times $\varepsilon^{\frac{1}{2}+\frac{1}{2}\mu}F_{\mu-1}^+(x/\sqrt{\varepsilon})$ as barrier function; from (9.18a) it follows that this function indeed majorizes the $\mathcal{O}$-term of (21a). When $-1 < \mu \leq 0$ we apply lemma 2.2 with barrier function $C\varepsilon^{\frac{1}{2}+\frac{1}{2}\mu}F_\nu^+(x/\sqrt{\varepsilon})$ in which $-1 < \nu < \mu$; formula (22) gives the additional negativity condition, which has to be satisfied. By formula (9.18a) this barrier function majorizes the $\mathcal{O}$-term of (21a) for every ν with $-1 < \nu < \mu \leq 0$. We now can state the theorem we have proved:

THEOREM 4.1.

If G is the unit square $(-1,1) \times (-1,1)$,

L_2 *is a uniformly elliptic 2^{nd} order operator on G with C^3 coefficients,*

$L_2 := a\partial^2/\partial x^2 + 2b\partial^2/\partial x\partial y + c\partial^2/\partial y^2$ + *lower order terms, with* $a(0,y) \equiv 1$,

Φ *is the solution of the boundary value problem on G:*

$(\varepsilon L_2 + x\partial/\partial x - \lambda y\partial/\partial y - \mu)\Phi = h,\ \lambda > 0,\ \varepsilon > 0,\ \mu > -1,$

with boundary conditions

$\Phi(x,\pm 1) = f_\pm(x),\quad \Phi(\pm 1,y) = g_+(y) \pm g_-(y),$ Φ *continuous at* $(\pm 1,\pm 1)$,

where $f_\pm \in C^3$, $g_\pm \in C^3 \cap C^{[(3-\mu)/\lambda]+1}$ *and* $h \in C^p$ *with*

$p := \max\{3,[([4-\mu]-\mu)/\lambda]+2,2[([4-\mu]-\mu)/\lambda]+1,[6-\mu],[5-\mu]+\lceil([4-\mu]-\mu)/\lambda\rceil\},$

Ψ is the formal approximation of Φ, defined in (20),
then Ψ is an asymptotic approximation of Φ for $\varepsilon \downarrow 0$ which is uniformly valid in G of order

(23a) $\Phi - \Psi = \mathcal{O}(\varepsilon)$ *if* $\mu \geq 1$ *and* $\lambda + \mu$ *nonintegral*

(23b) $\Phi - \Psi = \mathcal{O}(\varepsilon\psi(x/\sqrt{\varepsilon})) = \begin{cases} \mathcal{O}(\varepsilon), |x| \geq x_0 > 0, \\ \mathcal{O}(\varepsilon \log \varepsilon), \textit{otherwise}, \end{cases}$ *if* $\mu \geq 1$ *and* $\lambda + \mu$ *integral*

(23c) $\Phi - \Psi = \mathcal{O}(\varepsilon^{\frac{1}{2}+\frac{1}{2}\mu}R_{\mu-1}(x/\sqrt{\varepsilon})) = \begin{cases} \mathcal{O}(\varepsilon), |x| \geq x_0 > 0, \\ \mathcal{O}(\varepsilon^{\frac{1}{2}+\frac{1}{2}\mu}), \textit{otherwise}, \end{cases}$ *if* $0 < \mu < 1$,

(23d) $\Phi - \Psi = \mathcal{O}(\varepsilon^{\frac{1}{2}+\frac{1}{2}\mu}R_{\nu}(x/\sqrt{\varepsilon})) = \begin{cases} \mathcal{O}(\varepsilon^{\frac{1}{2}(1+\mu-\nu)}), |x| \geq x_0 > 0, \\ \mathcal{O}(\varepsilon^{\frac{1}{2}+\frac{1}{2}\mu}), \textit{otherwise}, \end{cases}$ *if* $-1 < \nu < \mu \leq 0$.

In here R_ν satisfies $R_\nu(t) = \begin{cases} 1 & \text{if } |t| \leq 1 \\ t^\nu & \text{if } |t| \geq 1 \end{cases}$ as defined in (2.2). In the estimate (23d) we can still choose the parameter ν freely within the constraint $-1 < \nu < \mu$; the smaller we choose ν, the better the order-estimate becomes.

The size of the remainder in (23) increases as μ decreases; the remainder in the free boundary layer becomes of order unity when μ decreases to -1 and no estimate is obtained when $\mu = -1$. This indicates that the region $\mu \leq -1$ is beyond reach of the method of proving the validity used here, in which the absolute error is estimated. Estimates of the relative error however might be expected to have a larger μ-range, for Ψ is of order $\mathcal{O}(\varepsilon^{\frac{1}{2}\mu}R_\mu(x/\sqrt{\varepsilon}))$ and hence by (23) the relative error is $\mathcal{O}(\sqrt{\varepsilon})$ independent of μ inside the free boundary layer and $\mathcal{O}(\varepsilon^{\frac{1}{2}(1+\mu-\nu)})$ outside for every $\nu \in (-1,\mu) \cap [\mu-1,\mu)$. This conjecture however is false; in the next subsection we will show that the boundary value problem (2) has (nonzero) eigenfunctions in the range $\mu \leq -1$ and that hence barrier functions satisfying the conditions of lemma 2.2 do not exist in this case.

f. *The spectrum* σ_ε of the boundary value problem (2) is the set of complex numbers μ, for which the homogeneous boundary value problem (2), i.e. $f_\pm = g_\pm = h = 0$, has a nontrivial solution. (Since the inverse operator is compact for every $\varepsilon > 0$, every point of the spectrum is an eigenvalue.) We will study here the spectrum of the operator T_ε, defined on the space B of C^2-functions on G which are zero at ∂G,

$$(24) \qquad T_\varepsilon u := \varepsilon\Delta u + x\frac{\partial u}{\partial x} - y\frac{\partial y}{\partial y} \quad \text{with } u \in B.$$

The eigenfunction equation $T_\varepsilon u = \mu u$, $u|_{\partial G} = 0$ is reduced by $u(x,y,\varepsilon) = X(x,\varepsilon)Y(y,\varepsilon)\exp(y^2/2\varepsilon)$ to

$$S_\varepsilon X := \varepsilon X'' + xX' = \mu_1 X, \quad X(\pm 1,\varepsilon) = 0,$$

$$S_\varepsilon Y := \varepsilon Y'' + yY' = \mu_2 Y, \quad Y(\pm 1,\varepsilon) = 0,$$

with $\mu = \mu_1 + \mu_2 + 1$. S_ε is a selfadjoint operator on the space of functions f on $(-1,1)$, which have an absolutely continuous first derivative, which satisfy $f(-1) = f(1) = 0$ and have the norm $\|f\|^2 = \int_{-1}^{1} |f(t)|^2\exp(t^2/2\varepsilon)dt$. So its spectrum $\sigma(S_\varepsilon)$ is real.

Since S_ε is invariant under the operation $x \mapsto -x$, its eigenfunctions are symmetric or antisymmetric; the eigenfunctions are solutions of

$$(25) \qquad \varepsilon X'' + xX' - \lambda X = 0,$$

hence X equals ${}_1F_1(-\tfrac{1}{2}\lambda;\tfrac{1}{2};-x^2/2\varepsilon)$ or $x\,{}_1F_1(\tfrac{1}{2}-\tfrac{1}{2}\lambda;3/2;-x^2/2\varepsilon)$. The first solution has no zero's in the range $0 < x < \infty$, when $\lambda \geq -1$, and k zero's, when $-1-2k \leq \lambda < 1-2k$; when λ tends to $-1-2k$ from below, the largest zero is sent to infinity. Hence the equation ${}_1F_1(-\tfrac{1}{2}\lambda;\tfrac{1}{2};-1/2\varepsilon) = 0$ has exactly one solution, $\lambda_{2k-1}(\varepsilon)$, in the λ-interval $-1-2k \leq \lambda < 1-2k$ and $\lim_{\varepsilon\downarrow 0} \lambda_{2k-1}(\varepsilon) = -2k-1$. In the same way the equation ${}_1F_1(\tfrac{1}{2}-\tfrac{1}{2}\lambda;3/2;-1/2\varepsilon) = 0$ has exactly one solution $\lambda_{2k}(\varepsilon)$ in the interval $-2-2k \leq \lambda < -2k$, which satisfies $\lim_{\varepsilon\downarrow 0} \lambda_{2k}(\varepsilon) = -2k$. Hence $\sigma(S_\varepsilon) = \{\lambda_k(\varepsilon) \mid k \in \mathbb{N}\}$ and $\lim_{\varepsilon\downarrow 0} \lambda_k(\varepsilon) = -k$.

The statement on the zero's of the solutions of (25), cf. 9.10, are proved exploiting the relations (9.16-17) and the fact that $F_\lambda^+(t)$ and $F_\lambda^-(t)$

are solutions of the 2^{nd} order ordinary differential equation (9.10), which is analytic in t and λ; hence $F_\lambda^\pm(t)$ are meromorphic functions of λ and t by definition (9.15) with poles for $\lambda = -1,-3$ etc. and $\lambda = -2,-4$ etc. respectively. Furthermore their zero's are simple and interlace. Since F_λ^+ and its derivative do not have a common zero, we can solve $F_\lambda^+(t) = 0$ resulting in an at most denumerable set of zero's $t_k(\lambda)$, which are themselves meromorphic in λ, which do not cross each other and do not have finite accumulation-points (for fixed λ). When λ is restricted to a compact subset A of an interval $(-1-2k,1-2k)$, $k \in \mathbb{N}$, F_λ^+ satisfies uniformly $F_\lambda^+(t) = |t|^\lambda(1+\mathcal{O}(t^{-2}))$, so the union of the ranges of all $t_k(\lambda)$ with $\lambda \in A$ is compact. Hence F_λ^+ has only a finite number of zero's on A and all these zero's are bounded analytic functions on some neighbourhood of A. Since they cannot vanish or coalesce, the number of zero's of F_λ^+ is constant for $1-2k > \lambda > -1-2k$, $k \in \mathbb{N}$ or $\lambda > -1$. A zero can disappear at infinity only, when λ tends to $1-2k$ for some $k \in \mathbb{N}$. For the same reason the number of zero's of F_λ^- is constant, when $\lambda > -2$ or $-2k > \lambda > -2k-2$ $(k\in\mathbb{N})$.

From (9.12) we see that F_λ^+ does not have zero's, when $-1 < \lambda < 0$, and that F_λ^- has a zero at $t = 0$ only for $0 > \lambda > -2$. Since F_λ^- has at least one extremum in $(0,\infty)$ for $0 > \lambda > -2$, by (17) F_λ^+ has at least one zero in $(0,\infty)$ for $-1 > \lambda > -3$. When $-1 > \lambda > -2$, F_λ^- has no zero's in $(0,\infty)$, hence F_λ^+ cannot have more than one, for the zero's of F_λ^+ and F_λ^- interlace. So F_λ^+ has exactly one zero in $(0,\infty)$ in the larger λ-range $-1 > \lambda > -3$ also. Now F_λ^+ has at least one extremum in $(0,\infty)$ for $-1 > \lambda > -3$ and we can continue the story ad infinitum proving the assertion on the number of zero's. When $\lambda = 1-2k$ $(k\in\mathbb{N})$ $\exp(-t^2/2)\,{}_1F_1(-\frac{1}{2}\lambda,\frac{1}{2},-t^2/2)$ equals - apart from a constant factor - the (2k-2)-th Hermite-polynomial which has k - 1 positive zero's, so the largest positive zero of ${}_1F_1(-\frac{1}{2}\lambda,\frac{1}{2},-\frac{1}{2}t^2)$ with $-2k-1 < \lambda < 1-2k$ is sent to infinity when λ tends to $1 - 2k$ from below, while the other converge to the zero's of the Hermite-polynomial and remain bounded.

The spectrum of T_ε is the set $\{\mu(\varepsilon) \mid \mu(\varepsilon) = \lambda_k(\varepsilon) + \lambda_m(\varepsilon) + 1;\ k,m \in \mathbb{N}\}$. Its largest element tends to -1 (from below), when $\varepsilon \downarrow 0$. To this largest eigenvalue, $2\lambda_1(\varepsilon) + 1$, belongs the eigenfunction

$$E_1(x,y,\varepsilon) := \exp(y^2/2\varepsilon)\,{}_1F_1(-\tfrac{1}{2}\lambda_1(\varepsilon);\tfrac{1}{2};-x^2/2\varepsilon)\,{}_1F_1(-\tfrac{1}{2}\lambda_1(\varepsilon);\tfrac{1}{2};-y^2/2\varepsilon),$$

which is nonnegative on G; hence

$$(T_\varepsilon-\mu)E_1 = (2\lambda_1(\varepsilon) +1 - \mu)E_1,$$

which is nonnegative on G if $\mu < -1$ and ε sufficiently small. When a barrier function W would exist, which satisfies $W > 0$ and $(T_\varepsilon-\mu)W \geq 0$ on all of G for all sufficiently small $\varepsilon > 0$ and $\mu < -1$, this contradicts the generalized maximum principle, cf. §2.b, for by this principle E_1 cannot have a positive maximum.

In this section we have constructed an approximation to the inverse operator $(T_\varepsilon-\lambda)^{-1}$ for $\lambda > -1$. Since this inverse operator is meromorphic in λ and has poles for $\lambda \in \sigma(T_\varepsilon)$, while the constructed approximation does not, this approximation cannot be valid near these poles. It remains an open problem whether we can indicate subspaces of the range of T_ε in which the construction remains valid. By comparison theorems we can prove that the largest element of the spectrum of $\varepsilon L_2 + x\frac{\partial}{\partial x} - \lambda y\frac{\partial}{\partial y}$ tends to -1 also, when $\varepsilon \downarrow 0$, cf. [16].

g. *In the general situation*, where L_1 has such a saddlepoint-singularity at the origin that it cannot be reduced to the form of (1a), we can do much less than before. Namely we cannot calculate explicitly the approximation in the free boundary layer.

What we can prove is that in a part of the domain, which has a nonzero distance to the free boundary layer independent of ε, the outer expansion together with the matching terms in the ordinary boundary layer approximates the solution to the same order as Φ approximates Ψ in (23) outside the free boundary layer. In order to prove this we first estimate the solution on the whole domain by some suitable barrier function (as used before). Then we locate an artificial boundary roughly half-way between the chosen subdomain and the free boundary layer, which is C^3 and C^3-connected to ∂G. At this artificial boundary we pose some arbitrary C^3 boundary condition which is C^3-connected to the condition at ∂G and which is bounded by the estimates on the solution. Finally we calculate the outer expansion and the ordinary boundary layer terms as is done here and e.g. in [4] and prove validity by the barrier function used before. The influence of the part of the boundary layer term, that originates at the artificial boundary, is exponentially small in the originally chosen subdomain, independent of the boundary condition chosen at that part of the boundary; hence it can be skipped in the approximation of the solution without change in order of the remainder.

About the behaviour in the free boundary layer we can say little. When L_1 has the form $L_1 := xp(x,y)\frac{\partial}{\partial x} - \lambda yq(x,y)\frac{\partial}{\partial y} - \mu$ with $\lambda > 0$, $\mu > 0$ and $p(0,0) = q(0,0) = 1$, we *conjecture* that the solution of $L_1 u = h$, $u(\pm 1,y) = g_{\pm}(y)$ is of order $O(x^{\mu})$ for $x \to 0$ and of Hölder-class $C^{\mu-\delta}$ (with respect to x) for every $\delta > 0$. Then we can regularize (cf. [6]) u, i.e. convoluate it with some suitable ε-dependent C^{∞}-function which approximates the Dirac-δ-measure when $\varepsilon \downarrow 0$. The resulting function, together with its matching terms to the given boundary condition in the ordinary boundary layers, approximates both the solution of the boundary value problem as well as the outer expansion plus its matching terms in the ordinary boundary layers. The order of approximation is $O(\varepsilon^{\frac{1}{2}\mu-\frac{1}{2}\delta})$. Assuming the truth of the conjecture mentioned above, we can prove this by the same method as is used in [4].

5. CHARACTERISTICS OF STABLE NODAL TYPE

In this section we study the Dirichlet-problem for the operator $L_\varepsilon := \varepsilon L_2 + L_1$ on a bounded domain G where L_2 is a uniformly elliptic 2^{nd} order operator of positive type. L_1 is a first order operator with one singularity in the interior of G which is a non-degenerate node and we may take for it (cf. §2.a)

$$L_1 := x\frac{\partial}{\partial x} + \lambda y\frac{\partial}{\partial y} - \mu \quad \text{or} \quad L_1^* := x\frac{\partial}{\partial x} + (x+y)\frac{\partial}{\partial y} - \mu$$

with $\lambda > 0$ and $\mu > 0$; also we may take $\lambda \le 1$. The boundary of G is smooth and nowhere tangent to the characteristics of L_1. We start with the simplest problem exhibiting these features and then try to generalize the results obtained.

In this section we will use both rectangular and polar coordinates (x,y) and (r,ϕ) to describe the points of the domain. They are related by $(x,y) = (r\cos\phi, r\sin\phi)$ and we will change from one set to the other without further notice.

a. Let G be the unit disk in $\mathbb{R}^2$, $L_2 := \Delta$ and $L_1 := x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$. We try to approximate for $\varepsilon \downarrow 0$ the solution Φ of

$$(1) \qquad L_\varepsilon \Phi := \varepsilon L_2 \Phi + L_1 \Phi = 0, \quad \Phi(\cos\phi, \sin\phi) = f(\phi)$$

where f is of class C^2 in ϕ.

The characteristics of L_1 are straight lines pointing to the origin, along which the solution u of the reduced equation $L_1 u = 0$ is constant. We see that u cannot satisfy the boundary conditions and also be continuous at the origin. It appears that we have to take $u(r\cos\phi, r\sin\phi) = f(\phi)$ and that a region of nonuniformity arises around the origin. In fact we will prove:

$$(2) \qquad \Phi(r\cos\phi, r\sin\phi) = f(\phi) + O(\varepsilon r^{-2}).$$

With the barrier function $K\varepsilon r^{-2}$ we estimate $\Phi - f$ on the annulus $2\sqrt{\varepsilon} \le r \le 1$. From the maximum principle it follows that Φ is bounded by $\max|f(\phi)|$; furthermore we have

$$L_\varepsilon r^{-2} = -2r^{-2}(1-2\varepsilon r^{-2}) \le -r^{-2} \quad \text{for } r > 2\sqrt{\varepsilon}.$$

Hence we can choose K such that

$$|\Phi_\varepsilon - f| \le K\varepsilon r^{-2} \quad \text{for } r = 1 \text{ and } r = 2\sqrt{\varepsilon}$$

and such that

$$L_\varepsilon K\varepsilon r^{-2} < L_\varepsilon(\Phi_\varepsilon - f) = \varepsilon r^{-2} f'' < -L_\varepsilon K\varepsilon r^{-2},$$

from which (2) follows using the maximum principle again.

So we may conclude that there is no loss of boundary conditions at all;

due to the focussing effect of L_1 the value at the boundary is propagated nearly unmodified to the origin along the rays of the circle and it is there that all difficulties crop up. If we try to find a solution near the origin by usual boundary layer techniques and we take the "natural" coordinate stretching $\rho = r/\sqrt{\varepsilon}$ (the only significant one in the sense of ECKHAUS [3]), we find that this stretching does not produce an equation of simpler type nor simplified boundary conditions.

By separation of variables in polarcoordinates we can find the exact solution expressed as the infinite sum S of confluent hypergeometric functions, cf. §9.d and exponentials

$$(3) \qquad S_0(f;\rho,\phi) := \sum_{k=-\infty}^{\infty} a_k f_k \rho^{|k|} e^{ik\phi} {}_1F_1(\tfrac{1}{2}|k|;|k|+1;-\tfrac{1}{2}\rho^2)$$

where $\rho := r/\sqrt{\varepsilon}$, f_k are the fourier coefficients of f,

$$f_k := \frac{1}{2\pi}\int_0^{2\pi} f(\theta)e^{-ik\theta}d\theta$$

and a_k are normalizing factors

$$a_k^{-1} := \varepsilon^{-\frac{1}{2}|k|} {}_1F_1(\tfrac{1}{2}|k|;|k|+1;-1/2\varepsilon).$$

Clearly S_0 converges absolutely and uniformly in each disc. The central region of nonuniformity has a very complex structure, for each term of the Fourier-expansion the behaviour in the "free boundary layer" is different.

REMARK. If G is a more general domain with smooth (C^2) boundary which is nowhere tangent to the characteristics and $\Phi(r\cos\phi, r\sin\phi) = f(\phi)$ at the boundary, then it is easily seen that (2) remains valid and that we have

$$(4) \qquad \Phi(r\cos\phi, r\sin\phi) = S_0(f;r/\sqrt{\varepsilon},\phi) + \mathcal{O}(\varepsilon)$$

uniformly on G.

b. The most general problem in which the region of non-uniformity can be treated similar to case (a) occurs when we have $\lambda = 1$, and hence

$L_1 := r\frac{\partial}{\partial r} - \mu$ $(\mu > 0)$. By a linear transformation of the x-y-plane we can make the principal part of L_2 equal to Δ at the origin. For G we take the unit disc again, a generalization analogous to the remark in (a) being obvious. So we have the boundary value problem

$$(5) \qquad (\varepsilon L_2 + r\frac{\partial}{\partial r} - \mu)\Phi = h, \quad \Phi(\cos\phi, \sin\phi) = f(\phi).$$

First we conclude from the maximum principle, using a constant as barrier function, (when $\mu = 0$ we have to take $\chi(r/\sqrt{\varepsilon})$ as defined in (9.22)), that Φ is uniformly bounded by a constant (by $K_1 + K_2\chi(r/\sqrt{\varepsilon})$). As in the saddle-point case we now try to find a particular solution of the equation $L_1 u =$ $= h$, hence

$$u = r^{\mu}\int^{r} h(\rho\cos\phi, \rho\sin\phi)\,\rho^{-1-\mu}d\rho$$

and bounds of integration are to be found such that it is C^2. Therefore we expand h, which is of class C^4,

$$h(x,y) = \sum_{0\le k+l\le 2} h_{kl}x^k y^l + h^*(x,y), \qquad h_{kl} = \frac{1}{k!l!}\frac{\partial^{k+l}h}{\partial x^k\partial y^l}(0,0),$$

such that the remainder h^* is of order $O(r^2)$ for $r \downarrow 0$ and we continue h^* in the plane such that it is zero outside a fixed neighbourhood of G. In order to define a particular solution of $L_1 u = h$ which is C^2 in all of G we have to distinguish between different values of μ.

If $\mu > 2$ then

$$(6a) \qquad u_2(x,y) := \sum_{0\le k+l\le 2}\frac{h_{kl}x^k y^l}{k+l-\mu} - \int_1^{\infty} h^*(xt,yt)t^{-1-\mu}dt$$

and if $\mu \le 2$

$$(6b) \qquad u_2(x,y) := \sum_{\substack{0\le k+l\le 2\\ k+l\ne\mu}}\frac{h_{kl}x^k y^l}{k+l-\mu} + \int_0^{1} h^*(xt,yt)t^{-1-\mu}dt.$$

In this way u_2 becomes a C^2-function. When μ equals one of the integers 0,1 or 2, (6b) is a solution of

(6c) $$L_1 u_2 = h - \sum_{k+l=\mu} h_{kl} x^k y^l.$$

For the remainder of h we cannot find a particular solution (of the reduced equation) which is C^2 (it contains always a factor $r^\mu \log r$), so we have to go into the central region of nonuniformity for it. We take the local coordinates (ξ,η) with $\xi = x/\sqrt{\varepsilon}$, $\eta = y/\sqrt{\varepsilon}$ and define ρ by $\rho^2 = \xi^2 + \eta^2$ (hence $\rho = r/\sqrt{\varepsilon}$). We now expand L_ε formally into powers of ε:

$$L_\varepsilon := \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} + \xi \frac{\partial}{\partial \xi} + \eta \frac{\partial}{\partial \eta} - \mu + \sqrt{\varepsilon}(\ldots.);$$

in the remainder the coefficients of the first derivatives are uniformly bounded by C and the coefficients of the second derivatives by $C\rho$, where C is some positive constant, independent of ξ,η and ε. By definition of (9.22) we have

$$\left(\frac{1}{\rho}\frac{\partial}{\partial \rho}\rho\frac{\partial}{\partial \rho} + \rho\frac{\partial}{\partial \rho}\right)\chi(\rho) = -1,$$

hence by (9.24-26) the function v, defined by

(8) $$v(x,y) := -\sum_{k+l=\mu} h_{kl} x^k y^l \chi(r/\sqrt{\varepsilon})$$

satisfies

(9) $$L_\varepsilon v = \sum_{k+l=} h_{kl} x^k y^l + \begin{cases} O(\varepsilon^{\frac{1}{2}} R_{-1}(r/\sqrt{\varepsilon})) & \text{if } \mu = 0 \text{ or } \mu = 1 \\ O(\varepsilon \chi(r/\sqrt{\varepsilon})) & \text{if } \mu = 2 \end{cases}$$

and it is zero at ∂G.

It now remains to construct an approximation to the problem

$$L_\varepsilon \tilde{\Phi} = 0,\ \tilde{\Phi}(\cos\phi, \sin\phi) = \tilde{f}(\phi) := f(\phi) - u_2(\cos\phi, \sin\phi).$$

The reduced equation

$$L_1 u = 0,\ u(\cos\phi, \sin\phi) = \tilde{f}(\phi)$$

is solved by

$$(10)\qquad u(r\cos\phi,\ r\sin\phi) = r^{\mu}\tilde{f}(\phi).$$

If $\mu > 2$ this is of class C^2, hence $L_\varepsilon(u_0+u_1) = h + \mathcal{O}(\varepsilon)$. Since $u_0 + u_1$ equals $f(\phi)$ at the boundary we conclude using lemma 2.1 and ε times a constant as barrier

THEOREM 5.1.
If $\mu > 2$, f *is of class* C^2 *and* h *of class* C^4, *then the solution* Φ *of* (5) *has the uniform asymptotic approximation*

$$(11)\qquad \Phi = u_1 + u_2 + \mathcal{O}(\varepsilon) \qquad (\varepsilon \downarrow 0).$$

It is remarkable that this first approximation for Φ does not show singular layers at all, though (2) has the appearance of a singular perturbation problem. The reason is that the factor r^{μ} in (10) smoothes out the angular dependence of the solution of the reduced equation on the boundary data. This effect is not restricted to the case $\mu > 2$; we have

THEOREM 5.2.
If $0 < \mu \le 2$, $f \in C^2$ *and* $h \in C^4$, *then the solution* Φ *of* (5) *has the asymptotic approximation*

$$(12)\qquad \Phi(x,y) = u_1(x,y) + u_2(x,y) + \left(\tfrac{1}{2}\sum_{k+l=\mu} h_{kl}x^k y^l \log(x^2+y^2) \text{ if } \mu = 1 \text{ or } 2\right)$$

$$+\begin{cases} \mathcal{O}(\varepsilon^{\frac{1}{2}\mu}) & \textit{if}\ \ \mu > 0,\ \mu \neq 1,2 \\ \mathcal{O}(\varepsilon^{\frac{1}{2}\mu}\log\varepsilon) & \textit{if}\ \ \mu = 1 \text{ or } 2 \end{cases} \ \textit{uniformly in } G$$

$$\quad \mathcal{O}(\varepsilon) \qquad \textit{if}\ \ r \ge \delta > 0\ \ \textit{and}\ \ \mu > 0.$$

In the proof of the uniform estimate we regularize the approximation and prove with aid of lemma 2.1 and $C\varepsilon^{\frac{1}{2}\mu}$ as barrier function for some constant C, that this regularization is an approximation of Φ, cf. §4.9 and [6]. The estimate on the annulus $0 < \delta \le r \le 1$ is proved in the same way as formula (2).

c. For an analysis of the structure of the nonuniformity in the center we again take the local coordinate $\rho = r/\sqrt{\varepsilon}$; the reduced equation in these coordinates is

$$\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \rho\frac{\partial}{\partial\rho} - \mu\right)v = 0.$$

The asymptotic behaviour of its solution v has to be

$$v(\rho,\phi) = \varepsilon^{\frac{1}{2}\mu}\rho^{\mu}\tilde{f}(\phi)(1 + o(1)) \qquad (\rho \to \infty,\ \varepsilon \downarrow 0).$$

Hence by separation of variables we find (cf. §9.d)

$$v(\rho,\phi) = S_\mu(\tilde{f};\rho,\phi) := \sum_{k=-\infty}^{\infty} a_k\tilde{f}_k\rho^{|k|}e^{ik\phi}{}_1F_1(\tfrac{1}{2}|k|-\tfrac{1}{2}\mu;|k|+1;-\tfrac{1}{2}\rho^2). \tag{13}$$

where

$$\tilde{f}_k := \frac{1}{2\pi}\int_0^{2\pi}\tilde{f}(\phi)e^{-ik\phi}d\phi \quad\text{and}\quad a_k^{-1} := \varepsilon^{-\frac{1}{2}|k|}{}_1F_1(\tfrac{1}{2}|k|-\tfrac{1}{2}\mu;|k|+1;-1/2\varepsilon).$$

In order to prove that $S_\mu + u_2$ approximates the solution Φ of (2) with aid of the maximum principle, as usual, it remains to find a bound for $\varepsilon(L_2-\Delta)S_\mu(\tilde{f};r/\sqrt{\varepsilon},\phi)$ or equivalently for $\varepsilon\mathcal{D}S_\mu$ where $\mathcal{D}$ represents the operators $\frac{\partial}{\partial r}$, $\frac{1}{r}\frac{\partial}{\partial\phi}$, $r\frac{\partial^2}{\partial r^2}$, $\frac{\partial^2}{\partial r\partial\phi}$ and $\frac{1}{r}\frac{\partial^2}{\partial\phi^2}$. From (9.34) we have:

$$\varepsilon r\frac{\partial^2}{\partial r^2}S_\mu(f;r/\sqrt{\varepsilon},\phi) \le \frac{4}{3}\frac{\sqrt{\varepsilon}R_{\mu-1}(r/\sqrt{\varepsilon})}{R_\mu(1/\sqrt{\varepsilon})}\sum_{k=-\infty}^{\infty}k^2|f_k| \tag{14}$$

$$\le \frac{4}{3}\varepsilon^{\frac{1}{2}+\frac{1}{2}\mu}R_{\mu-1}(r/\sqrt{\varepsilon})\cdot\max\{f''(\phi)\ |\ \phi\in[0,2\pi]\}.$$

It is easily seen that we have at worst the same estimate for the other operators too.

Now we can state the theorem:

THEOREM 5.3. *If G is the unit disc,*

L_2 is a uniformly elliptic operator on G with C^2 coefficients, which equals Δ at $(0,0)$,

Φ is the solution of the boundary value problem

$$(\varepsilon L_2 + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - \mu)\Phi = h, \qquad \Phi(\cos\phi, \sin\phi) = f(\phi),$$

where $h \in C^4$ and $f \in C^2$, and $\tilde{f}(\phi) := f(\phi) - u_2(\cos\phi, \sin\phi)$, then Φ has the asymptotic approximation

$$(15) \qquad \Phi(r\cos\phi, r\sin\phi) = u_2(r\cos\phi, r\sin\phi) + v(r\cos\phi, r\sin\phi) + S_\mu(\tilde{f}; r/\sqrt{\varepsilon}, \phi)$$

$$+ \begin{cases} O(\varepsilon) & \text{if } \mu > 1,\ \mu \neq 2, \\ O(\varepsilon\chi(r/\sqrt{\varepsilon})) & \text{if } \mu = 2, \\ O(\varepsilon^{\frac{1}{2}+\frac{1}{2}\mu} R_{\mu-1}(r/\sqrt{\varepsilon})) & \text{if } -1 < \mu < 1, \\ O(\varepsilon^{\frac{1}{2}} R_{-1}(r/\sqrt{\varepsilon})) & \text{if } \mu = 1, \\ O(\varepsilon^{\frac{1}{2}(1+\mu-\nu)}) \quad \text{for } r \geq \delta > 0 & \text{if } -2 < \nu < \mu \leq 1. \end{cases}$$

where R_μ as defined in (2.2) *and χ in* (9.22).

REMARK. For $-2 < \mu \leq -1$ no estimate is gotten of the absolute error in the region of nonuniformity, while the relative error is still of order $O(\sqrt{\varepsilon})$.

In order to prove this we use lemma 2.1 when $\mu \geq 0$ and lemma 2.2 otherwise. As barrier function we take $C_1\varepsilon$ if $\mu > 1$, $\mu \neq 2$ and $C_2\varepsilon + C_2\varepsilon\chi(r/\sqrt{\varepsilon})$ if $\mu = 2$, with suitably chosen constants C_1 and C_2; if $-2 < \mu < 1$ we use the barrier function $\varepsilon C_1 W_\nu$,

$$W_\nu(r,\varepsilon) := {}_1F_1(-\tfrac{1}{2}\nu; 1; -r^2/2\varepsilon)/{}_1F_1(-\tfrac{1}{2}\nu; 1; -1/2\varepsilon),$$

with $\nu = -1$ if $\mu = 1$, $\nu = \mu - 1$ if $-1 < \mu < 1$ and $\nu \in (-2,\mu)$ if $-2 < \mu \leq -1$. This function satisfies, cf. (9.14c) and (9.33-34):

$$(\varepsilon\Delta + r\frac{\partial}{\partial r} - \mu)W_\nu = (\nu-\mu)W_\nu < 0 \quad \text{if} \quad \nu < \mu$$

$$\varepsilon(L_2 - \Delta)W_\nu = O(\varepsilon^{\frac{1}{2}} W_\nu)$$

$$0 < C\varepsilon^{\frac{1}{2}\nu} R_\nu(r/\sqrt{\varepsilon}) \leq W_\nu(r/\sqrt{\varepsilon}) \leq \varepsilon^{\frac{1}{2}\nu} R_\nu(r/\sqrt{\varepsilon}) \quad \text{if} \quad -2 < \nu < 0,$$

where C is a positive constant, $C < 1$. The theorem then is a consequence of the estimates (6c), (9) and (14).

Here also the μ-range in which we can prove the validity of the approximation (15) is restricted to $\mu > -2$; -2 is the limit of the largest eigenvalue $\mu_1(\varepsilon)$ of

$$(16) \qquad (\varepsilon L_2 + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y})\Phi = \mu\Phi, \quad \Phi(\cos\phi, \sin\phi) = 0.$$

In order to calculate the spectrum of (16) we mention without proof that the regular solution $\rho^k {}_1F_1(\frac{1}{2}k-\frac{1}{2}\mu;k+1;-\rho^2/2)$ of equation (9.27) has l positive zero's ($l \in \mathbb{N}$) when $-k-2l-2 \le \mu < -k-2l$, the largest of which is sent to infinity if μ increases to $-k-2l$. So, there are functions $\mu_{kl}(\varepsilon)$ (with $\mu_{kl}(\varepsilon) \uparrow -k-2l$ if $\varepsilon \downarrow 0$) such that ${}_1F_1(\frac{1}{2}k-\frac{1}{2}\mu;k+1;-1/2\varepsilon) = 0$. Hence $\mu_{kl}(\varepsilon)$, $k \in \mathbb{N} \cup \{0\}$, $l \in \mathbb{N}$ is in the spectrum of (16) and its eigenspace is spanned by

$$e^{\pm ik\phi} r^k {}_1F_1(\tfrac{1}{2}k-\tfrac{1}{2}\mu_{kl}(\varepsilon);k+1;-r^2/2\varepsilon).$$

d. If the parameter λ in $L_1 = x\frac{\partial}{\partial x} + \lambda y\frac{\partial}{\partial y} - \mu$ is not equal to one (assume $0 < \lambda < 1$) we cannot give detailed information on the structure of the central nonuniformity. The best result we can attain now is the analogue of the theorems 5.1 and 5.2. So we have the problem on the unit circle

$$(17) \qquad L_\varepsilon\Phi = [\varepsilon L_2 + x\frac{\partial}{\partial x} + \lambda y\frac{\partial}{\partial y} - \mu]\Phi = h,$$
$$\Phi(\cos\phi, \sin\phi) = f(\phi), \qquad \text{(f is } 2\pi\text{-periodic)},$$

where $0 < \lambda < 1$ and $0 < \mu$.

The characteristics of L_1 are the curves $\alpha|x|^\lambda = \beta y$, ($\alpha,\beta \in \mathbb{R}$). We now assume that h is of class $C^{2+[\mu/\lambda]}$ and define for nonnegative integers k,l with $k+\lambda l \le \min(2,\mu)$ the constants

$$h_{k,l} := \frac{1}{k!l!}\frac{\partial^{k+l}h}{\partial x^k \partial y^l}(0,0).$$

The remainder h^* in the Taylor-expansion of h,

(18) $$h^*(x,y) := h(x,y) - \sum_{k+\lambda l\le\min\{2,\mu\}} h_{kl}x^k y^l \quad \text{if} \quad x^2+y^2\le 1,$$

is continued outside G by a C^2-function which is zero if $x^2+y^2\ge 2$. A particular solution of $L_1 u = h$ is

(19) $$u_2(x,y) := \sum_{\substack{k+\lambda l\ne\mu\\ k+\lambda l\le\min(\mu,2)}} h_{kl}\frac{x^k y^l}{k+\lambda l-\mu} + \sum_{k+\lambda l=\mu} \tfrac{1}{2}h_{kl}x^k y^l\log(x^2+|y|^{2/\lambda}) +$$
$$+\begin{cases} \int_0^1 h^*(xs,ys^\lambda)s^{-1-\mu}ds & \text{if} \quad \mu\le 2,\\ -\int_1^\infty h^*(xs,ys^\lambda)s^{-1-\mu}ds & \text{if} \quad \mu > 2.\end{cases}$$

By this definition we ensure that u_2 is C^2 everywhere in G, except when $0\le\mu\le 2$ and $k+\lambda l=\mu$ for some pair(s) of integers (k,l); in that case u_2 is of Hölderclass $C^{\mu-\delta}$ for every positive δ (uniformly in G) and C^2 in any closed part of G not containing the origin.

Since u_2 need not be zero at ∂G, we correct for its contribution and take

$$\tilde{f}(\phi) := f(\phi) - u_2(\cos\phi, \sin\phi).$$

Define σ_1 and σ_2 to be the inverse functions of $x \mapsto x(1-x^2)^{-1/2\lambda}$ for $x\in(-1,1)$ and of $y\mapsto y(1-y^2)^{-\frac{1}{2}\lambda}$ respectively and then define for $t>0$ and $|x|<2t^{1/\lambda}$

(20a) $$u_1(x,\pm t) := t^{\mu/\lambda}\{1-\sigma_1^2(xt^{-1/\lambda})\}^{-\mu/2\lambda}\tilde{f}(\operatorname{arccot}\frac{\sigma_1(xt^{-1/\lambda})}{\sqrt{\{1-\sigma_1^2(xt^{-1/\lambda})\}}} \mp \frac{\pi}{2}+\frac{\pi}{2})$$

and for $t>0$ and $|y|<2t^\lambda$

(20b) $$u_1(\pm t,y) := t^{\mu}\{1-\sigma_2^2(yt^{-\lambda})\}^{-\frac{1}{2}\mu}\tilde{f}(\arctan\frac{\sigma_2(yt^{-\lambda})}{\sqrt{\{1-\sigma_2^2(yt^{-\lambda})\}}} + \frac{\pi}{2}\mp\frac{\pi}{2}),$$

where for arctan and arccot we take the principal value, which have ranges $(-\frac{\pi}{2}, \frac{\pi}{2})$ and $(0,\pi)$ respectively. Both expressions for $u_1(x,y)$ satisfy $L_1u_1 = 0$, $u_1(\cos\phi, \sin\phi) = \tilde{f}(\phi)$, on their domains of definition and hence coincide on the intersection, for the solution is uniquely determined along a characteristic by the initial value at the boundary. Since u_2 is C^2 at ∂G, u_1 is also C^2 in any closed part of G not containing the origin if $f \in C^2$; in a neighbourhood of the origin u_1 is of Hölderclass C^{μ} if $0 < \mu \le 2$ and C^2 if $\mu > 2$ and u_1 is unbounded if $\mu < 0$ and of order $O(|x|^{\mu}+|y|^{\mu/\lambda})$. By regularization of $u_1 + u_2$ (if $\mu \le 2$) and with aid of lemma 2.1 with barrier-function a constant times ε, $\varepsilon^{\frac{1}{2}\mu-\delta}$ or $\varepsilon^{\frac{1}{2}\mu}$ we obtain:

THEOREM 5.4.
If Φ is the solution of the boundary value problem on the unit disc

$$(21) \qquad (\varepsilon L_2 + x\frac{\partial}{\partial x} + \lambda y\frac{\partial}{\partial y} - \mu)\Phi = h, \qquad \Phi(\cos\phi, \sin\phi) = f(\phi),$$

where $\mu > 0$ and $0 < \lambda \le 1$, L_2 as in theorem 5.3, $f \in C^2$ and $h \in C^{2+\lceil\mu/\lambda\rceil}$, then Φ is approximated uniformly by $u_1 + u_2$ when $\varepsilon \downarrow 0$ and

$$(22) \qquad \Phi(x,y) = u_1(x,y) + u_2(x,y) + \begin{cases} O(\varepsilon) & \text{if } \mu > 2, \\ O(\varepsilon^{\frac{1}{2}\mu-\delta}) & \text{if } k+\lambda l = \mu \le 2; k,l \in \mathbb{N} \cup \{0\}, \\ O(\varepsilon^{\frac{1}{2}\mu}) & \text{otherwise if } 0 < \mu < 2. \end{cases}$$

As we stated in formula (2) for the special case $\lambda = 1$ and $\mu = 0$, we can derive a better estimate on the remainder of (22) outside a neighbourhood of the origin; we will prove:

THEOREM 5.5.
With the same assumptions as theorem 5.4, except that $\mu > -1-\lambda$, we have the estimate

$$(23) \qquad \Phi(x,y) = u_1(x,y) + u_2(x,y) + O(\varepsilon) \quad \text{if } x^2 + y^2 \ge \delta > 0$$

which holds uniformly on every annulus $0 < \delta \le x^2 + y^2 \le 1$, which does not contain the origin.

PROOF. First we give a rough estimate of Φ on all of G using a constant C_1 as barrier function in lemma 2.1 if $\mu > 0$ and a constant C_2 times $V_\nu(x,y,\varepsilon)$ if $-1-\lambda < \mu \le 0$ in lemma 2.2. In here V_ν is defined by

$$V_\nu(x,y,\varepsilon) := \varepsilon^\nu {}_1F_1(-\tfrac{1}{2}\nu,\tfrac{1}{2},-x^2/2\varepsilon)\,{}_1F_1(-\tfrac{1}{2}\nu,\tfrac{1}{2},-\lambda y^2/2\varepsilon)$$

and by (9.15a), (9.18a) and (9.20) it satisfies, if $\nu > -1$,

$$(\varepsilon L_2 + x\frac{\partial}{\partial x} + \lambda y\frac{\partial}{\partial y} - \mu)V_\nu = (\mu - \nu - \lambda\nu)V_\nu + \mathcal{O}(\varepsilon^{\frac{1}{2}}V_\nu).$$

We conclude from this that Φ is uniformly bounded by C_1 when $\mu > 0$ and by $C_2\varepsilon^\nu$ when $-1 < \nu < \mu/(1+\lambda) \le 0$. When $\mu > 0$ we now prove the statement, using lemma 2.1 and the barrier function $C_1\varepsilon r^{-2}$ on the annular region $4\varepsilon \le x^2+y^2 \le 1$. When $\mu \le 0$ we prove by lemma 2.2, that the remainder is bounded by the barrier function $C_2\varepsilon x^{-2+\nu}$ in the sub-domain $x^2 > 12\varepsilon$ and by $C_2\varepsilon y^{\nu-2/\lambda}$ in the subdomain $y^2 > 12\varepsilon/\lambda$.

REMARKS.

1. Here also we have obtained the largest possible μ-range, for the largest eigenvalue of $\varepsilon L_2 + x\frac{\partial}{\partial x} + \lambda y\frac{\partial}{\partial y}$ on G tends to $-1 - \lambda$ when $\varepsilon \downarrow 0$. This can be proved by comparison of this eigenvalue to the largest eigenvalues of $\varepsilon\Delta + x\frac{\partial}{\partial x} + \lambda y\frac{\partial}{\partial y}$ on the squares $|x| \le 1$, $|y| \le 1$ and $|x| \le \frac{1}{2}\sqrt{2}$, $y \le \frac{1}{2}\sqrt{2}$.
2. For $L_1 := x\frac{\partial}{\partial x} + (x+y)\frac{\partial}{\partial y} - \mu$ we can prove the analogues of the theorems 5.4 and 5.5 if $\mu > -1$; we will not give the details in here.

6. AN UNSTABLE NODE

We now take for $L_1 := -x\frac{\partial}{\partial x} - \lambda y\frac{\partial}{\partial y} - \mu$, which has an unstable nodal singularity while L_2 remains uniformly elliptic of positive type. So we have the problem

$$(1) \qquad (\varepsilon L_2 + L_1)\Phi = h, \quad \Phi(\cos\phi, \sin\phi) = f(\phi),$$

where $L_1 = -x\frac{\partial}{\partial x} - \lambda y\frac{\partial}{\partial y} - \mu$ with $0 < \lambda \le 1$ and $\mu > 0$. The character of the solution is now totally different from the former case. It is even exponentially small outside a neighbourhood of the boundary if $h \equiv 0$, as is easily seen using lemma 2.1 and the barrier function W_ν,

(2) $$W_\nu(x,y) := \exp[\nu(x^2 + y^2 - 1)/\varepsilon],$$

for we have, if ν is a sufficiently small positive number,

$$L_\varepsilon W_\nu < 0 \quad \text{on all of the unit disc.}$$

So we may expect that the only nonuniformity in Φ is an ordinary boundary layer along the unit circle.

The response to the right-hand-side in the reduced equation is

(3) $$u_2(x,y) := -\int_0^1 h(tx,t^\lambda y)t^{\mu-1}dt, \quad \mu > 0,$$

which is easily seen to be of the same differentiability class as h is. This particular solution is not zero at the boundary, so we define

$$\tilde{f}(\phi) := f(\phi) - u_2(\cos\phi, \sin\phi);$$

it now remains to approximate the solution of

$$L_\varepsilon \Phi = 0, \quad \Phi|_{\partial G} = \tilde{f}.$$

By the above considerations already we know that its outer expansion is identically zero. In order to calculate the boundary layer terms we take new coordinates (t,ϕ) such that

$$x = e^{-t}\cos\phi \text{ and } y = e^{-\lambda t}\sin\phi,\ L_1 = +\frac{d}{dt} - \mu \text{ and}$$
$$L_2 = \alpha_1(t,\phi)\frac{\partial^2}{\partial t^2} + 2\alpha_2(t,\phi)\frac{\partial^2}{\partial t\partial\phi} + \alpha_3(t,\phi)\frac{\partial^2}{\partial\phi^2} + \text{lower order terms},$$

where L_2 is again elliptic with α_1 and α_3 positive and G is transformed in the halfplane $t > 0$ with periodic boundary condition. By substitution of the local coordinate $\tau := t/\varepsilon$ and formal expansion into powers of ε we find as lowest order part of $L_\varepsilon w = 0$ the ordinary differential equation

$$\alpha_1(0,\phi)\frac{d^2w}{d\tau^2} + \frac{dw}{d\tau} = 0,$$

with $w(0,\phi) = \tilde{f}(\phi)$ and $\lim_{\tau\to\infty} w(\tau,\phi) = 0$ and its solution is

$$w(\tau,\phi) = f(\phi)\exp(-\tau/\alpha_1(0,\phi)).$$

Let $z(t)$ be a C^∞-function which is one for $t < \frac{1}{2}$ and zero for $t > 1$, then we can prove in the same way as [4] theorem VII and [6] theorem I:

THEOREM 6.1.
The solution Φ *of the boundary value problem* (1) *has the uniform asymptotic approximation for* $\varepsilon \downarrow 0$

$$\Phi(x,y) = u_2(x,y) + z(t(x,y))w(t(x,y)/\varepsilon,\arg(x,y)) + O(\varepsilon)$$

if f,h *and the coefficients of* L_2 *are* C^3 *and if* $\mu > 0$.

REMARKS.
1. Higher order approximations can easily be made by iteration of the process if f and h are more smooth than mentioned in theorem 6.1, cf. [4].
2. When $L_1 = x\frac{\partial}{\partial x} + (x+y)\frac{\partial}{\partial y}$ we get a result that is completely analogous to theorem 6.1.
3. The largest eigenvalue of $\varepsilon L_2 + x\frac{\partial}{\partial x} + \lambda y\frac{\partial}{\partial y}$ tends to zero from below, when ε tends to zero from above, hence the μ-range of theorem 6.1 is the largest possible, cf. [16].

7. FOCAL POINTS

In this section we study the operator L_1 defined by

$$L_1 := (\nu x-\lambda y)\frac{\partial}{\partial x} + (\nu y+\lambda x)\frac{\partial}{\partial y} - \mu = \nu r\frac{\partial}{\partial r} + \lambda\frac{\partial}{\partial\phi} - \mu,$$

in which we can take without loss of generality $\nu = \pm 1$ and $\lambda > 0$, or $\nu = 0$ and $\lambda = 1$ (vortex), cf. §2.a. The characteristics of L_1 are spirals

$(r = e^t,\ \phi = \lambda t + \phi_0)$ inward or outward directed or circles. The cases $\nu = \pm 1$ are completely analogous to the stable and unstable nodes and in the case $\nu = 0$ we will restrict ourselves to a purely characteristic boundary. L_2 is still a uniformly elliptic operator of positive type. We have the boundary value problem

(1) $$(\varepsilon L_2 + L_1)\Phi = h \text{ in the unit disc, and } \Phi(\cos\phi, \sin\phi) = f(\phi) \text{ at its boundary, where } f \text{ is } 2\pi\text{-periodic.}$$

a. When $\nu = 1$, L_1 has a stable focus and the nonuniformity of the solution is located at the origin. The solution of the reduced equation is

$$u(r,\phi) = r^{\mu} f(\phi - \lambda \log r) - r^{\mu}\int_r^1 h(\zeta\cos(\phi-\lambda\log\zeta), \zeta\sin(\phi-\lambda\log\zeta))\zeta^{-1-\mu}d\zeta.$$

Completely analogous to theorem 5.4 we have

THEOREM 7.1. *The solution of the boundary value problem*

(2) $$(\varepsilon L_2 + (x-\lambda y)\frac{\partial}{\partial x} + (y+\lambda x)\frac{\partial}{\partial x} - \mu)\Phi = h, \quad \Phi(\cos\phi, \sin\phi) = f(\phi),$$

where $\lambda > 0$, $\mu > 0$, L_2 *is uniformly elliptic of positive type and* f,h *and the coefficients of* L_2 *are* C^2, *has the uniform asymptotic approximation (for* $\varepsilon \downarrow 0$*)*

$$\Phi(r\cos\phi, r\sin\phi) = u(r,\phi) + \begin{cases} O(\varepsilon) & \text{if } \mu \geq 2, \\ O(\varepsilon^{\frac{1}{2}\mu}) & \text{if } 0 < \mu < 2. \end{cases}$$

The analogue of theorem 5.5 is

THEOREM 7.2. *With the same conditions of theorem* 7.1, *except that* $\mu > -2$, *the solution* Φ *of* (2) *has the approximation (for* $\varepsilon \downarrow 0$*)*

(3) $$\Phi(r\cos\phi, r\sin\phi) = u(r,\phi) + O(\varepsilon r^{-2+\delta}), \quad r^2 \geq 8\varepsilon,$$

in which $\delta = 0$ *if* $\mu > 0$ *and* $-2 < \delta < \mu$ *if* $\mu \leq 0$.

PROOF. For $-2 < \mu \leq 0$ a rough estimate of Φ is derived from lemma 2.2 with aid of the barrier function

$$\varepsilon^{\frac{1}{2}\delta}\,{}_1F_1(-\tfrac{1}{2}\delta,1,-r^2/2\varepsilon), \qquad -2 < \delta < \mu \leq 0.$$

Hence Φ is of order $\mathcal{O}(\varepsilon^{\frac{1}{2}\delta})$ if $-2 < \delta < \mu \leq 0$. With aid of the barrier function $\varepsilon r^{-2+\delta}$ on the subdomain $8\varepsilon \leq r^2 \leq 1$ we then derive formula (3) in case $0 < \mu \leq 0$. When $\mu > 0$, we use εr^{-2} as barrier function on the same subdomain. □

<u>REMARK</u>. Here also the largest eigenvalue of $\varepsilon L_2 + (x-\lambda y)\frac{\partial}{\partial x} + (y-\lambda x)\frac{\partial}{\partial y}$ tends to -2 from below when $\varepsilon \downarrow 0$; when $L_2 = \Delta$ the largest eigenvalue and the eigenfunctions are equal to those of $\varepsilon\Delta + r\frac{\partial}{\partial r}$, cf. (5.16).

Analogous to equation (5.1) the boundary value problem

$$(\varepsilon\Delta + r\frac{\partial}{\partial r} + \lambda\frac{\partial}{\partial\phi} - \mu)\Phi = 0, \; \Phi(\cos\phi,\, \sin\phi) = f(\phi)$$

can be solved exactly in terms of an infinite sum of confluent hypergeometric functions multiplied by the Fourier-coefficients of f. As in §5.c this sum can be used to obtain a better approximation (near the origin) of the solution of (1) and to derive a theorem which is essentially the same as theorem 5.3.

b. When $\nu = -1$, L_1 has an unstable focus. By the barrier function W_δ, defined in (6.2) we can estimate the solution of $(\varepsilon L_2 + L_1)\Phi = 0$, when δ is chosen sufficiently small; we conclude from this estimate that only an ordinary boundary layer occurs.

The solution of the reduced equation is

$$(4) \quad u(r\cos\phi,\, r\sin\phi) = r^{-\mu}\int_0^r h(\zeta\cos(\phi+\lambda\log\zeta),\zeta\sin(\phi-\lambda\log\zeta)\zeta^{\mu-1}d\zeta,$$

which is as much times differentiable as h is. In order to calculate the boundary layer terms we take new coordinates (t,ϕ) such that

$$x = e^{-t}\cos(\lambda t+\phi), \quad y = e^{-t}(\sin\lambda t+\phi), \quad L_1 = \frac{d}{dt} - \mu \quad \text{and}$$

$$L_2 = \tilde{\alpha}_1(t,\phi)\frac{\partial^2}{\partial t^2} + 2\tilde{\alpha}_2(t,\phi)\frac{\partial^2}{\partial t\partial\phi} + \tilde{\alpha}_3(t,\phi)\frac{\partial^2}{\partial\phi^2} + \text{lower order terms},$$

where L_2 again is elliptic with α_1 and α_3 positive. G is transformed to the halfplane $t > 0$. Taking the local coordinate $\tau := \varepsilon t$ and expanding into powers of ε we find for the lowest order part of the equation $L_\varepsilon w = 0$:

$$\tilde{\alpha}_1(0,\phi)\frac{d^2w}{d\tau^2} + \frac{dw}{d\tau} - \mu w = 0$$

with boundary conditions

$$w(0,\phi) = f(\phi) \quad \text{and} \quad \lim_{\tau\to\infty} w(\tau,\phi) = 0.$$

Defining $\gamma(\phi) := \frac{1}{2} + \frac{1}{2}\sqrt{(1+4\mu\tilde{\alpha}_1(0,\phi))}$, we find the solution

$$w(\tau,\phi) = f(\phi)\exp(-\tau\beta(\phi)).$$

In the same way as theorem 6.1 we prove:

THEOREM 7.3. *The solution of the elliptic boundary value problem*

(5) $$(\varepsilon L_2 - (x+\lambda y)\frac{\partial}{\partial x} - (y-\lambda x)\frac{\partial}{\partial y} - \mu)\Phi = h, \quad \Phi(\cos\phi, \sin\phi) = f(\phi),$$

where f,h *and the coefficients of* L_2 *are* C^3, L_2 *is of positive type and* $\mu > 0$ *has the uniform asymptotic approximation (for* $\varepsilon \downarrow 0$*)*

(6) $$\Phi(x,y) = u(x,y) + z(t(x,y))w(t(x,y)/\varepsilon,\arg(x,y)) + O(\varepsilon),$$

where $\arg(x,y)$ *is the argument of* (x,y) *and* z *is a* C^∞*-function,* $z(t) \equiv 1$ *if* $t \leq \frac{1}{2}$ *and* $z(t) = 0$ *if* $t \geq 1$.

REMARK. The largest eigenvalue of $\varepsilon L_2 - (x+\lambda y)\frac{\partial}{\partial x} - (y-\lambda x)\frac{\partial}{\partial y}$ tends to zero, when $\varepsilon \downarrow 0$.

c. When $\nu = 0$, L_1 has a vortex point at the origin and the characteristics are neither converging to it nor diverging from it; furthermore the problem now has a so-called characteristic boundary. From the barrier function

$$\tilde{W}_\delta(x,y) = \exp\{\delta(x^2 + y^2 - 1)/\sqrt{\varepsilon}\}$$

which satisfies for sufficiently small positive δ in all of the unit disc G

$$L_\varepsilon \tilde{W}_\delta < 0 \quad \text{while } \tilde{W}_\delta > 0,$$

we see that the solution of $(\varepsilon L_2 + L_1)\Phi = 0$ with $\mu > 0$ is exponentially small in the interior of G and that a boundary layer is located along the boundary, whose width is at most of order $O(\sqrt{\varepsilon})$.

A particular solution of the reduced equation of (1), $L_1 u = h$, which is regular at the origin, is

$$u(x,y) = (1-\exp 2\pi\mu) \int_0^{2\pi} h(x \cos \psi + y \sin \psi, -x \sin \psi + y \cos \psi)e^{\mu\psi} d\psi.$$

Since $L_\varepsilon u = h + O(\varepsilon)$ uniformly in G it remains to approximate the solution of

$$L_\varepsilon \Phi = 0,\ \Phi(\cos \phi,\ \sin \phi) = \tilde{f}(\phi) := f(\phi) - u(\cos \phi,\ \sin \phi),$$

which is already known to be asymptotically equal to zero (for $\varepsilon \downarrow 0$) outside a thin layer along the boundary. Substituting polar coordinates (r,ϕ), stretching the r-variable near the boundary by $r = 1-\rho\sqrt{\varepsilon}$ and expanding L_ε, expressed in local coordinates, into powers of ε, we get

$$L_\varepsilon = M_0 + \sqrt{\varepsilon}M^* = \gamma(\phi)\frac{\partial^2}{\partial\rho^2} - \frac{\partial}{\partial\phi} - \mu + \sqrt{\varepsilon}M^*.$$

In M^* are collected all terms whose coefficients are of order $O(\sqrt{\varepsilon})$ at least; $\gamma(\phi)$ is defined by

$$\gamma(\phi) := a(\cos \phi,\ \sin \phi)\cos^2\phi + 2b(\cos \phi,\ \sin \phi)\cos \phi \sin \phi + \\ + c(\cos \phi,\ \sin \phi)\sin^2\phi;$$

due to the ellipticity of L_2, γ is strictly positive. So we have got in lowest order the parabolic boundary value problem

$$\mathcal{M}_0 v = 0, \quad v(0,\phi) = \tilde{f}(\phi), \quad \lim_{\rho\to\infty} v(\rho,\phi) = 0 \quad \text{and} \quad v(\rho,\phi) = v(\rho,\phi+2\pi).$$

By Fourier-sine transform (cf. [11]: XIII 5.32)

$$(\mathcal{S}v)(\xi,\phi) := w(\xi,\phi) = \int_0^\infty v(\rho,\phi)\sin \rho\xi \, d\rho,$$

which is self-inverse, i.e. $\mathcal{S}^2 v = \frac{1}{2}\pi v$, we get the equation

$$\frac{\partial w}{\partial \phi} + (\mu+\xi^2\gamma(\phi))w = \xi\gamma(\phi)\tilde{f}(\phi) \quad \text{with} \quad w(\xi,\phi+2\pi) = w(\xi,\phi),$$

which is solved by

$$w(\xi,\phi) = \xi(1-\exp\{-2\pi(\mu+p\xi^2)\})^{-1}\int_\phi^{\phi+2\pi}\tilde{f}(\tau)\gamma(\tau)\exp\{\mu\phi-\mu\tau-\xi^2\int_\phi^\tau \gamma(\sigma)d\sigma\}d\tau,$$

where $p := \frac{1}{2\pi}\int_0^{2\pi}\gamma(\phi)d\phi$. By two partial integrations we find that

$$\tilde{w}(\xi,\phi) := \xi\tilde{f}(\phi)\gamma(\phi)(\mu+\xi^2\gamma(\phi))^{-1} - w(\xi,\phi)$$

is such that $\tilde{w}$, $\frac{\partial\tilde{w}}{\partial\phi}$ and $\frac{\partial^2\tilde{w}}{\partial\phi^2}$ and all their ξ-derivatives are bounded by a constant times $\xi(\mu+\xi^2\gamma(\phi))^{-2}$. Hence $\tilde{v} := \frac{2}{\pi}\mathcal{S}\tilde{w}$ is C^2 at least and it decreases together with its first and second derivatives faster to zero than every negative power of ρ (for $\rho \to \infty$). From ([10] ch XI) at last we find

$$v(\rho,\phi) = \tilde{f}(\phi)\exp\{-\rho\sqrt{\mu/\gamma(\phi)}\} + \tilde{v}(\rho,\phi)$$

and from the considerations above it is clear that it satisfies

$$\mathcal{M}^* v = \mathcal{O}(1) \text{ uniformly in } \rho \text{ and } \phi.$$

By lemma 2.1, using a constant times $\sqrt{\varepsilon}$ as a barrier function, we find

THEOREM 7.4. *The solution* Φ *of the elliptic boundary value problem*

$$(\varepsilon L_2 + x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} - \mu)\Phi = h, \quad \Phi(\cos\phi, \sin\phi) = f(\phi),$$

where $\mu > 0$, L_2 *elliptic and of positive type and* f,h *and the coefficients of* L_2 *are of class* C^2, *has the uniform asymptotic approximation for* $\varepsilon \downarrow 0$

$$\Phi(r\cos\phi, r\sin\phi) = u(r\cos\phi, r\sin\phi) + v((1-r)/\sqrt{\varepsilon},\phi)z(r) + O(\sqrt{\varepsilon}),$$

in which z *is a* C^∞*-function,* $z(t) \equiv 0$ *if* $t \le \frac{1}{3}$ *and* $z(t) \equiv 1$ *if* $t \ge \frac{2}{3}$.

By inspection of the eigenfunctions of the operator Δ on the unit circle with Dirichlet-boundary-conditions, namely $J_k(r\nu_{kj})e^{ik\phi}$ with $k \in \mathbb{Z}$, J_k the Besselfunction of order k and ν_{kj} the j-th positive zero of $J_k(\nu)$, we see that these are eigenfunctions of $\varepsilon\Delta + x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$ $(= \varepsilon\Delta + \frac{\partial}{\partial\phi})$ too with eigenvalues $ik + \varepsilon\nu_{kj}^2$. Since the set of eigenfunctions of Δ is complete we did find all eigenvalues of $\varepsilon\Delta + \frac{\partial}{\partial\phi}$. In the limit $\varepsilon \downarrow 0$ the spectrum fills all of the halflines $\{\mu+ik \mid k\in Z, \mu\in\mathbb{R}, \mu\ge 0\}$ indicating that in theorem 7.4 we cannot go below $\mu = 0$.

8. CONCLUSIONS

The analysis of degenerations of second order elliptic boundary value problems to first order with critical points can be continued in two ways. We can deal with the more complex situation, where the domain contains several (simple and isolated) critical points, and we can try to enlarge the μ-range.

a. When the bounded domain $G \subset \mathbb{R}^2$ contains several simple and isolated critical points of $L_1 := p(x,y)\frac{\partial}{\partial x} + q(x,y)\frac{\partial}{\partial y} - \mu$ with $\mu > 0$, we first make a picture of the characteristics of L_1 and their direction. At a point P, which is on a characteristic that *enters* G somewhere at the boundary and that does not pass through a critical point before hitting P, the outer expansion approximates the solution Φ of

(1) $\qquad \varepsilon L_2\Phi + L_1\Phi = h, \qquad \Phi|_{\partial G}$ prescribed,

up to $\mathcal{O}(\varepsilon)$. In order to prove this we restrict ourselves to a subdomain $G^* \subset G$ which contains an open neighbourhood of the part of the characteristic through P between the point of entrance and P. Furthermore we assume that $\partial G \cap \partial G^*$ is connected, that all characteristics in G^* enter through $\partial G \cap \partial G^*$, leave through $\partial G^* \backslash \partial G$ and are nowhere tangent to ∂G^* and that $\partial G^* \cap \partial G$ and $\partial G^* \backslash \partial G$ are smooth arcs. By lemma 2.1 with a constant as barrierfunction we conclude that Φ is bounded at ∂G. Hence we can immediately apply [4] theorem VII to the restriction of (1) to G^*, proving the statement.

A more detailed description of the approximation can be given in some cases by combination of the results obtained before; in other cases additional difficulties come in, such that a detailed analysis of the internal nonuniformities remains as yet impossible. We will elucidate this by an example.

Let L_1 be the first order operator $L_1 := sx(x-1)\frac{\partial}{\partial x} + sy\frac{\partial}{\partial y} - 1$ with $s = \pm 1$ and let G be a domain containing (0,0) and (1,0) in its interior, such that ∂G is nowhere tangent to the characteristics of L_1. We consider the boundary value problem

$$(2) \qquad (\varepsilon\Delta + L_1)\Phi = h, \qquad \Phi \text{ prescribed at } \partial G,\ \varepsilon > 0.$$

L_1 has a saddle-point at (0,0) and a node at (1,0). This node is attracting if $s = +1$ and repulsive if $s = -1$. Now we divide G into two parts by a curve $(\alpha_1(y),y)$ with $0 < \alpha_1(y) < 1$, which is nowhere tangent to the characteristics of L_1 and such that the restriction of L_1 to the part G_1 of G left of this curve can be linearized (in the sense of §2.a). A second curve $(\alpha_2(y),y)$ with $0 < \alpha_2(y) < \alpha_1(y)$, which is nowhere tangent to the characteristics of L_1, divides G into two parts, the right-hand part of which is called G_2.

When $s = -1$, we can put an arbitrary boundary condition for Φ at $(\alpha_2(y),y)$ and apply theorem 6.1 to the restriction of (2) to G_2; this results in the approximation $\Phi(x,y) = A(x,y,\varepsilon) + \mathcal{O}(\varepsilon)$ in G_2. At the curve $(\alpha_1(y),y)$ we now put the boundary condition $\Phi(x,y) = A(x,y,\varepsilon)$ and apply theorem 4.1 to the restriction of (2) to G_1. In this way we find a uniform approximation to the solution Φ of (2) on all of G if $s = -1$.

When $s = +1$, the direction of the characteristics is inverted, hence we start with the restriction of (1) to G_1, putting an arbitrary boundary condition on Φ at $(\alpha_1(y),y)$. By theorem 3.1 we get a uniform approximation $B(x,y,\varepsilon)$ to Φ up to $\mathcal{O}(\varepsilon)$. At the curve $(\alpha_2(y),y)$ we now get the boundary condition $\Phi(x,y) = B(x,y,\varepsilon)$. To the restriction of (2) to G_2 however we cannot apply theorem 5.3 since the boundary condition near $(\alpha_2(0),0)$ possibly has a derivative of order $\mathcal{O}(\frac{1}{\varepsilon})$. A uniform approximation of order $\mathcal{O}(\varepsilon)$ can be obtained only in a subdomain of G, which does not contain a neighbourhood of the node.

As we showed in this example the construction of an approximation to the solution of the boundary value problem with several critical points is done in general by reduction of the problem to a number of problems on overlapping subdomains, which have to be solved in a definite order, prescribed by the direction of the characteristics.

b. When the parameter μ is below the bound, imposed by the spectrum, we do not know how to prove the validity of some formal process of construction of an approximation. The problems in here are quite analogous to those met in the papers [14] and [15] (among others) in which the analogous boundary value problems for singular ordinary differential equations with turning-point behaviour are analyzed.

In the stable nodal case for instance $S_\mu(f;r/\sqrt{\varepsilon},\phi)$, cf. (5.13), is the exact solution of

$$(3) \qquad (\varepsilon\Delta + r\frac{\partial}{\partial\phi} - \mu)\Phi = 0, \qquad \Phi(\cos\phi,\, \sin\phi) = f(\phi),$$

if μ is not in the spectrum of $\varepsilon\Delta + r\frac{\partial}{\partial\phi}$. When we expand this for $\mu > -3$ and μ fixed we get for sufficiently small ε

$$S_\mu(f;r/\sqrt{\varepsilon},\phi) = \begin{cases} r^\mu f(\phi)(1 + \mathcal{O}(\varepsilon r^{-2})) & \text{if } \mu > -3 \text{ and } \mu \neq -2, \\ r^{-2}(f(\phi)-f_0)(1 + \mathcal{O}(\varepsilon r^{-2})) + f_0 \exp((1-r^2)/2\varepsilon) & \\ & \text{if } \mu = -2, \end{cases}$$

where f_0 is the mean of f. The reason is that the asymptotic behaviour of ${}_1F_1(\alpha;\gamma;-|s|)$ for $s \to \infty$ is discontinuous in α and γ at $\alpha - \gamma = 0,1,2$, etc., cf. (9.13) and (9.29). In particular we find that the term with index zero in the sum S_{-2} (cf. 5.13) equals $f_0 \exp((1-r^2)/2\varepsilon)$, which is exponentially large. This discontinuity in the asymptotic behaviour of S_μ at $\mu = -2$ *has to occur* because of the fact that -2 is the limit of the largest eigenvalue $\mu_1(\varepsilon)$ of $\varepsilon\Delta + r\frac{\partial}{\partial r}$; $S_{\mu_1(\varepsilon)}$ does not exist for any $\varepsilon > 0$ if the mean value of f is not zero. The same phenomenon occurs at all limit points (for $\varepsilon \downarrow 0$) of the eigenvalues, cf. [16].

In the unstable nodal case we find the analogue. The solution of

$$(4) \qquad (\varepsilon\Delta - r\frac{\partial}{\partial r} - \mu)\Phi = 0, \quad \Phi(\cos\phi, \sin\phi) = f(\phi)$$

is $\Phi = \exp((r^2-1)/\varepsilon) S_{\mu+2}(f; r/\sqrt{\varepsilon}, \phi)$ and its asymptotic behaviour near $\mu = 0$ for sufficiently small ε is

$$\Phi(r\cos\phi, r\sin\phi) = \begin{cases} f(\phi)\exp((r^2-1)/2\varepsilon) + O(\varepsilon) & \text{if } \mu \neq 0, \\ f_0 + (f(\phi)-f_0)\exp((r^2-1)/2\varepsilon) + O(\varepsilon), & \text{if } \mu = 0. \end{cases}$$

In this case too this phenomenon - called "Resonance" by ACKERBERG & O'MALLEY [14] in the case of ordinary differential equations, cf. also [15] - occurs at every limit point of the spectrum ($\varepsilon \downarrow 0$). This type of discontinuity is exhibited by all formal approximations constructed here near the limit-points (for $\varepsilon \downarrow 0$) of the spectrum (where we could not prove validity of the formal approximation). From operator-theoretical point of view it is clear that the solution of (1.1) itself has to exhibit such discontinuity at a limit point (for $\varepsilon \downarrow 0$) of its spectrum. Let $\Phi(f;x,y;\mu,\varepsilon)$ be the solution of

$$(5) \qquad (\varepsilon L_2 + L_1 - \mu)\Phi = 0, \qquad \Phi_{|\partial G} = f$$

and let $\mu(\varepsilon)$ be an eigenvalue of $\varepsilon L_2 + L_1$ with $\lim_{\varepsilon\downarrow 0} \mu(\varepsilon) = \mu(0)$. At the spectral point $\mu(\varepsilon)$ the solution Φ of (5) does exist only when the class of boundary functions f is restricted (alternative of FREDHOLM), moreover it is not unique of it exists. When we approximate $\mu(0)$ via another path $\mu^*(\varepsilon)$

in the μ-ε plane, which does not cross an eigenvalue of $\varepsilon L_2 + L_1$ the solution $\Phi(f;x,y;\mu^*(\varepsilon),\varepsilon)$ does exist for all $f \in C^0(G)$ and is unique for all positive ε. In the limit for $\varepsilon \downarrow 0$ however it has to reflect in its behaviour also the nonexistence occuring on the neighbouring path $\mu(\varepsilon)$ when f is not in the restricted class.

9. APPENDIX

a. The asymptotic behaviour of ψ and its derivatives are deduced from the integral representation

$$(1) \qquad \psi(\xi) := \int_{\xi}^{1/\sqrt{\varepsilon}} \int_0^t \exp \tfrac{1}{2}(s^2-t^2)\,ds\,dt.$$

It is easily seen to be the solution of

$$(2) \qquad y'' + \xi y' = -1 \quad \& \quad y(\pm 1/\sqrt{\varepsilon}) = 0.$$

Since ψ is symmetric all formulae are stated for positive argument only. The derivative is

$$(3) \qquad \psi'(\varepsilon) = \int_0^{\xi} \exp \tfrac{1}{2}(s^2-\xi^2)\,ds$$

and for large argument this gives with aid of partial integration

$$\psi'(\xi) = e^{-\frac{1}{2}\xi^2} \int_1^{\xi} e^{\frac{1}{2}s^2} ds + O(e^{-\frac{1}{2}\xi^2})$$

$$= e^{-\frac{1}{2}\xi^2}\{[e^{\frac{1}{2}s^2}(\frac{1}{s} + \frac{1}{s^3})]_1^{\xi} + \int_1^{\xi} 3s^{-4} e^{\frac{1}{2}s^2} ds\} + O(e^{-\frac{1}{2}\xi^2})$$

$$= -\xi^{-1} - \xi^{-3} + O(\xi^{-4}) \qquad (\xi \to \infty),$$

hence

$$(4) \qquad \psi'(\xi) = -\frac{1}{\xi} + O(\xi^{-3}).$$

Repeated partial integration (cf. DE BRUIJN [1] ch.2 §5) results in the series

$$\psi'(\xi) \approx -\sum_{n=0}^{\infty} \frac{(2n)!}{2^n n!}\, \xi^{-2n-1}.$$

Integration of (4) gives

$$\psi(\xi) = -\log \xi + \text{constant} + \mathcal{O}(\xi^{-2}), \qquad (\xi \to \infty),$$

and, since $\psi(1/\sqrt{\varepsilon}) = 0$ we conclude

$$(5) \qquad \psi(x/\sqrt{\varepsilon}) = -\log |x| + \mathcal{O}(\frac{\varepsilon}{x^2}), \qquad (\varepsilon \downarrow 0).$$

Substitution $s = \xi - t$ in (3) gives

$$\psi'(\xi) = \int_0^{\xi} \exp(\tfrac{1}{2}t^2 - t\xi)dt.$$

Since $-t\xi + \frac{1}{2}t^2 \le -\frac{1}{2}t\xi$ for $0 < t < \xi$ we get

$$0 \le -\psi'(\xi) \le \int_0^{\xi} \exp(-\tfrac{1}{2}t\xi)dt = \frac{2}{\xi}(1-\exp(-\tfrac{1}{2}\xi^2)).$$

In the same way we get

$$|\psi''(\xi)| \le \exp(-\tfrac{1}{2}\xi^2) + 2\xi^{-2}(1-\exp(-\tfrac{1}{2}\xi^2)).$$

This results in the estimates

$$\left.\begin{array}{ll} (6) & |\psi'(\xi)| \le 2\xi R_{-2}(\xi) \\ (7) & |\psi''(\xi)| \le 2R_{-2}(\xi) \end{array}\right\} \text{ for all } \xi \in \mathbb{R}.$$

For higher order derivatives we obtain analogously constants K_n, such that

$$(8) \qquad |\psi^{(n)}(\xi)| \le K_n\, R_{-n}(\xi).$$

Integration of (6) gives

$$(9) \qquad |\psi(\xi)| \le 2\int_0^1 t dt + 2\int_1^{1/\sqrt{\varepsilon}} \frac{dt}{t} = 1 - \log \varepsilon;$$

in view of (5) this is only of value for small ξ.

b. The functions F_a^+ and F_a^- are defined as the symmetric and antisymmetric solutions of the equation

$$(10) \qquad y'' + ty' - ay = 0 \qquad (y' = \frac{dy}{dt})$$

with the asymptotic behaviour

$$y(t) \sim t^a \quad \text{for } t \to +\infty.$$

Substitution $s = -\frac{1}{2}t^2$ in (10) results in the confluent hypergeometric equation

$$(11) \qquad s\ddot{y} + (\tfrac{1}{2}-s)\dot{y} + \frac{a}{2}y = 0, \qquad (\dot{y} = \frac{dy}{ds}).$$

Independent solutions of it are ${}_1F_1(-\frac{1}{2}a;\frac{1}{2};s)$ and $\sqrt{s}\,{}_1F_1(-\frac{1}{2}a+\frac{1}{2},3/2,s)$. Here ${}_1F_1$ denotes the confluent hypergeometric or Kummer's function, cf. [10] ch. VI. For $\gamma > \alpha > 0$ it has the integral representation

$$(12) \qquad {}_1F_1(\alpha;\gamma;s) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} \int_0^1 (1-\sigma)^{\gamma-\alpha-1} \sigma^{\alpha-1} e^{s\sigma} d\sigma$$

and for large negative values of the argument it has the asymptotic behaviour

$$(13) \qquad {}_1F_1(\alpha;\gamma;-|s|) = \{\frac{\Gamma(\gamma)}{\Gamma(\gamma-\alpha)} |s|^{-\alpha} + e^{i\pi(\alpha-\gamma)} \frac{\Gamma(\gamma)}{\Gamma(\alpha)} e^{-|s|} |s|^{\alpha-\gamma}\}(1+\mathcal{O}(s^{-1}));$$

its second term is important only when $\gamma - \alpha = 0,-1,-2,$ etc.

From the integral representation (12) we easily derive some estimates on ${}_1F_1$. If $\alpha > 0$, $\gamma-\alpha \ge 1$ and $s \ge 0$, we have

$$0 < \int_0^1 (1-\sigma)^{\gamma-\alpha-1}\sigma^{\alpha-1}e^{-s\sigma}d\sigma \le \int_0^\infty \sigma^{\alpha-1}e^{-s\sigma}d\sigma = \Gamma(\alpha)s^{-\alpha}$$

and if $\alpha > 0$, $0 < \gamma-\alpha < 1$ and $s \ge 0$, then

$$0 < (\gamma-\alpha)\int_0^1 (1-\sigma)^{\gamma-\alpha-1}\sigma^{\alpha-1}e^{-s\sigma}d\sigma =$$

$$\gamma\int_0^1 (1-\sigma)^{\gamma-\alpha}\sigma^{\alpha-1}e^{-s\sigma}d\sigma - s\int_0^1 (1-\sigma)^{\gamma-\alpha}\sigma^{\alpha}e^{-s\sigma}d\sigma \le \gamma\Gamma(\alpha)s^{-\alpha},$$

hence we have for $\alpha > 0$, $\gamma-\alpha > 0$ and $s \ge 0$

(14a) $$0 \le {}_1F_1(\alpha,\gamma,-s) \le \Gamma(\gamma+1)s^{-\alpha}/\Gamma(\gamma-\alpha+1).$$

We can extend this inequality to the case $\alpha \le 0$, $\gamma-\alpha > 0$ and $\gamma \ne 0,-1,-2$, etc.; take $n := [1-\alpha]$, then we have for $s > 0$

$$|(d/ds)^n {}_1F_1(\alpha,\gamma,-s)| = |(\alpha)_n {}_1F_1(\alpha+n,\gamma+n,-s)/(\gamma)_n| \le$$
$$\le (\gamma+n)(\alpha)_n\Gamma(\gamma)s^{-\alpha-n}/\Gamma(\gamma-\alpha+1)$$

and integrating this n times, using ${}_1F_1(\alpha,\gamma,0) = 1$, we find a constant C, depending on γ and α, such that

(14b) $$|{}_1F_1(\alpha,\gamma,-s)| \le C(1+s^{-\alpha}) \quad \text{for } s \ge 0,\ \alpha \le 0,\ \gamma-\alpha > 0 \text{ and } \gamma \ne 0,-1,-2, \text{ etc.}$$

If $0 < \gamma-\alpha \le 2$ and $\alpha > 0$ we have for $s \ge 0$

$$\int_0^1 (1-\sigma)^{\gamma-\alpha-1}\sigma^{\alpha-1}e^{-s\sigma}d\sigma \ge \int_0^\infty (1-\sigma)\sigma^{\alpha-1}e^{-s\sigma} \ge \Gamma(\alpha)s^{-\alpha}(1-\alpha/s);$$

since the first integral is decreasing in s it is also larger than $\Gamma(\alpha)(1+\alpha)^{-1-\alpha}$ for $0 \le s \le 1+\alpha$. So we have for $\alpha > 0$, $0 < \gamma-\alpha \le 2$ and $s \ge 0$

(14c) $${}_1F_1(\alpha,\gamma,-s) \ge (\Gamma(\gamma)/\Gamma(\gamma-\alpha))\cdot\min\{(1+\alpha)^{-1-\alpha}, s^{-\alpha}/(1+\alpha)\}.$$

This inequality can be extended to every combination of α and γ satisfying $\gamma > \alpha > 0$. For negative values of α we can use the same device as used for (14b) provided $\gamma > 0$:

$$\frac{d}{ds}\, {}_1F_1(\alpha;\gamma;-s) = \frac{-\alpha}{\gamma}\, {}_1F_1(\alpha+1;\gamma+1;-s) \geq \frac{-\alpha}{\gamma} C_{\alpha,\gamma} R_{-\alpha-1}(s)$$

and by integration we get

(14d) $\quad {}_1F_1(\alpha,\gamma,-s) \geq C_{\alpha,\gamma} R_{-\alpha}(s) \quad$ for every $s \geq 0$, $\gamma > 0$ and $\alpha < \gamma$,

where $C_{\alpha,\gamma}$ are positive constants depending on α and γ.

When α and γ are positive and $0 \leq s \leq \gamma/\alpha$, the power series expansion of ${}_1F_1$,

$${}_1F_1(\alpha,\gamma,-s) = \sum_{k=0}^{\infty} \frac{(\alpha)_k}{(\gamma)_k k!} (-s)^k,$$

is the sum of an alternating and absolutely decreasing series, hence

(14e) $\quad 1 - \alpha s/\gamma \leq {}_1F_1(\alpha,\gamma,-s) \leq 1$ if $\gamma > 0$, $\alpha > 0$ and $0 \leq s \leq \gamma/\alpha$.

From (13) we see

(15a) $\quad F_a^+(t) = 2^{\frac{1}{2}a}\pi^{-\frac{1}{2}}\Gamma(\tfrac{1}{2}a+\tfrac{1}{2})\,{}_1F_1(-\tfrac{1}{2}a;\tfrac{1}{2};-\tfrac{1}{2}t^2)$, $\quad$ (if $a \neq -1,-3,-5$,etc),

(15b) $\quad F_a^-(t) = 2^{\frac{1}{2}a+\frac{1}{2}}\pi^{-\frac{1}{2}}\,(\tfrac{1}{2}a+1)\,t\,{}_1F_1(-\tfrac{1}{2}a+\tfrac{1}{2};3/2;-\tfrac{1}{2}t^2)$, (if $a \neq -2,-4,-6$,etc),

with the required asymptotic behaviour

(16) $\quad F_a^+(t) = |t|^a(1 + \mathcal{O}(t^{-2}))$ and $F_a^-(t) = t|t|^{a-1}(1 + \mathcal{O}(t^{-2}))$.

In particular we have $F_0^-(t) = \mathrm{erf}(t/\sqrt{2})$ and $F_0^+ = 1$; F_0^+, F_1^-, F_2^+, F_3^- etc. are polynomials of degree 0,1,2, etc.

By differentiation of (10) we find that $(F_a^{\pm})'$ satisfy $y'' + ty' +$ $-\,(a-1)y = 0$ and by (16) and a symmetry argument we find

(17) $$(F_a^+)' = aF_{a-1}^- \text{ (if } a \neq -1,-3,\text{etc.) and}$$

$$(F_a^-)' = aF_{a-1}^+ \text{ (if } a \neq 0,-2,-4,\text{etc.)}$$

The inequalities (14) provide positive constants $C_{a,k}$ (k=1,2,3,etc.) depending on a such that

(18a) $$0 < C_{a,1}R_a(t) \le F_a^+(t) \le C_{a,2}R_a(t) \quad \text{if} \quad a > -1,$$

(18b) $$|F^-(t)| \le C_{a,3}|t|R_{a-1}(t) \quad \text{if} \quad a > -2;$$

by differentiation we find

$$(F_a^-(t))' = t^{-1}F_a^-(t) - 2^{\frac{1}{2}a+\frac{1}{2}}3^{-1}\pi^{-\frac{1}{2}}t^2(1-a)\Gamma(\tfrac{1}{2}a+1)\,{}_1F_1(-\tfrac{1}{2}a+\tfrac{3}{2};\tfrac{5}{2};-\tfrac{1}{2}t^2),$$

hence it satisfies (by 14a-b)

(19a) $$|(F_a^-(t))'| \le C_{a,4}R_{a-1}(t) \quad \text{if} \quad a > -2$$

and proceeding in the same way we find in general

(19b) $$|(F_a^-(t))^{(n)}| \le D_{a,n}R_{a-n}(t) \quad \text{if} \quad a > -2,$$

(20) $$|(F_a^+(t))^{(n)}| \le D_{a,n}R_{a-n}(t) \quad \text{if} \quad a > -1,$$

in which $D_{a,n}$ are constants depending on a and on the order of differentiation.

c. The function χ is the solution of the equation

(21) $$y'' + (\zeta + \frac{1}{\zeta})y' = -1$$

which is regular at $\zeta = 0$ and takes the value 0 at $\zeta = 1/\sqrt{\varepsilon}$; hence

(22) $$\chi(\zeta) := \int_\zeta^{1/\sqrt{\varepsilon}} (1-\exp-\tfrac{1}{2}t^2)\frac{dt}{t} = \int_r^1 (1-\exp\frac{-s^2}{2\varepsilon})\frac{ds}{s} \qquad (r = \zeta\sqrt{\varepsilon}).$$

For $0 < r \le 1$ and $\varepsilon \downarrow 0$ it has the asymptotic behaviour

$$\chi(r/\sqrt{\varepsilon}) = -\log r + \mathcal{O}(\frac{1}{r}\exp -\frac{r^2}{2\varepsilon}) = -\log r + o(\varepsilon^n r^{-2n-1}) \tag{23}$$

for all $n \in \mathbb{N}$,

as is easily seen from the last integral of (22). From the first integral in (22) follows

$$0 \le \chi(\zeta) \le \chi(0) \le -\tfrac{1}{2}\log\varepsilon + \int_0^1 (1-\exp-\tfrac{1}{2}t^2)\frac{dt}{t} \le \tag{24}$$

$$\le 1 - \tfrac{1}{2}\log\varepsilon \qquad (0 \le \zeta \le 1/\sqrt{\varepsilon}).$$

Its derivatives satisfy

$$\chi'(\zeta) = \frac{1}{\zeta}(1-\exp-\zeta^2) \le \zeta R_{-2}(\zeta), \tag{25}$$

$$\chi''(\zeta) = \exp(-\zeta^2) - \zeta^{-2}(1-\exp-\zeta^2) \le R_{-2}(\zeta). \tag{26}$$

d. The equation

$$y'' + (\zeta + \zeta^{-1})y' - (k^2\zeta^{-2} + \mu)y = 0, \tag{27}$$

arising from $(\Delta + r\frac{\partial}{\partial r} - \mu)\Phi = 0$ by separation of variables, is transformed in the confluent hypergeometric equation

$$tw'' + (k+1-t)w' - \tfrac{1}{2}(k-\mu)w = 0, \qquad (k \ge 0),$$

by substitution $y(\zeta) = \zeta^k w(-\frac{1}{2}\zeta^2)$. There is only one solution, regular at the origin,

$$y_k(\zeta) := \zeta^k {}_1F_1(\tfrac{1}{2}k-\tfrac{1}{2}\mu;k+1;-\tfrac{1}{2}\zeta^2) \tag{28}$$

and it has the asymptotic behaviour (cf. 13) for $\zeta \to \infty$

$$y_k(\zeta) = \{\frac{2^{\frac{1}{2}k-\frac{1}{2}\mu}k!}{\Gamma(\frac{1}{2}k+\frac{1}{2}\mu+1)}\zeta^{\mu} - e^{\frac{1}{2}i\pi(k+\mu)}\frac{2^{\frac{1}{2}k+\frac{1}{2}\mu}}{\Gamma(\frac{1}{2}k-\frac{1}{2}\mu)}\zeta^{-2-\mu}e^{-\frac{1}{2}\zeta^2}\}(1+\mathcal{O}(k^2\zeta^{-2})), \tag{29}$$

whose second term is important only when $\mu = -2,-4$, etc.

Now we derive a number of inequalities on y_k and its derivatives. Without further notice we will use in the sequel the positivity of ${}_1F_1(\alpha;\gamma;-x)$ for $x > 0$, $\gamma > 0$ and $\alpha < \gamma$ as proved in (14c-d). We assume also $k \geq 0$, $\zeta \geq 0$, $\mu > -1$. From (14e) we find for $0 \leq \zeta \leq 1$ and $k \geq \mu$

(30) $$\zeta^k \geq y_k(\zeta) \geq \zeta^k\left(1 - \frac{k-\mu}{k+1}\cdot\frac{\zeta^2}{4}\right) \quad \text{and especially} \quad \frac{3}{4} \leq y_k(1) \leq 1.$$

For the derivative we have, using well-known relations (cf. [10] ch. 6.2).

(31a) $$y_k'(\zeta) = k\zeta^{k-1}\,{}_1F_1(\tfrac{1}{2}k-\tfrac{1}{2}\mu;k+1;-\tfrac{1}{2}\zeta^2) - \tfrac{1}{2}\zeta^{k+1}\,\frac{k-\mu}{k+1}\,{}_1F_1(\tfrac{1}{2}k-\tfrac{1}{2}\mu+1;k+2;-\tfrac{1}{2}\zeta^2) =$$

(31b) $$= \zeta^{k-1}\{\mu\,{}_1F_1(\tfrac{1}{2}k-\tfrac{1}{2}\mu;k+1;-\tfrac{1}{2}\zeta^2) + (k-\mu)\,{}_1F_1(\tfrac{1}{2}k-\tfrac{1}{2}\mu+1;k+1;-\tfrac{1}{2}\zeta^2)\}.$$

Hence from (31a)

(32a) $$y_k'(\zeta) \leq k\zeta^{-1}y_k(\zeta) \quad \text{if } k > \mu$$

and from (31b)

(32b) $$y_k'(\zeta) \geq \mu\zeta^{-1}y_k(\zeta) \quad \text{if } k > |\mu|$$

Since $(\zeta/a)^\mu$ satisfies $u' = \mu\zeta^{-1}u$ and $\left.(\frac{\zeta}{a})^\mu\right|_{\zeta=1} = \left.\frac{y_k(\zeta)}{y_k(a)}\right|_{\zeta=1}$,

we conclude from (32b)

(33) $$0 < \frac{y_k(\zeta)}{y_k(a)} \leq (\frac{\zeta}{a})^\mu \quad \text{if } 0 < \zeta \leq a \text{ and } k > |\mu|.$$

Formulae (30), (32) and (33) result in

(34a) $$\left|\frac{y_k'(\zeta)}{y_k(a)}\right| \leq \frac{ky_k(\zeta)}{\zeta y_k(a)} \begin{cases} \leq k\zeta^{\mu-1}a^{-\mu}, \text{ if } 1 \leq \zeta \leq a \text{ and } k > |\mu|, \\[2ex] \leq \dfrac{ky_k(\zeta)}{\zeta y_k(1)}\cdot\dfrac{y_k(1)}{y_k(a)} \leq \dfrac{4}{3}k\zeta^{k-1}a^{-\mu}, \\ \qquad \text{if } 0 < \zeta \leq 1 \leq a \text{ and } k > |\mu|. \end{cases}$$

For the function z, defined by (cf. 31a)

$$z(\zeta) := \zeta y_k'(\zeta) - k y_k(\zeta) = -\tfrac{1}{2}\zeta^{k+2}\,\frac{k-\mu}{k+1}\;{}_1F_1(\tfrac{1}{2}k-\tfrac{1}{2}\mu+1, k+2, -\tfrac{1}{2}\zeta^2),$$

we can do the analogue of (31) and (32), resulting in

$$(k+2)\zeta^{-1} z(\zeta) \le z'(\zeta) \le \mu\zeta^{-1} z(\zeta), \qquad \text{if } k > |\mu|,$$

and we get from this as in (34a)

$$\text{(34b)} \qquad \left|\frac{\zeta y_k''(\zeta)}{y_k(a)}\right| \le \frac{k^2 y_k(\zeta)}{\zeta y_k(a)} \le \frac{4k^2 R_{\mu-1}(\zeta)}{3R_\mu(a)} .$$

When $\mu \ge 0$ and $k \in [0,\mu]$ we estimate $y_k(\zeta)$ and its derivatives from above with aid of (14b) and $y_k(a)$ from below for sufficiently large a with aid of (14c-d). Hence also in this case we can find constants C_k such that at least for sufficiently large values of a and $k \in [0,\mu]$ we also have

$$\text{(34c)} \qquad \left|\frac{y_k(\zeta)}{y_k(a)}\right| \le C_k a^{-\mu} R_\mu(\zeta), \quad \left|\frac{y_k'(\zeta)}{y_k(a)}\right| \le C_k a^{-\mu} R_{\mu-1}(\zeta) \text{ and}$$

$$\left|\frac{\zeta y''_k(\zeta)}{y_k(a)}\right| \le C_k a^{-\mu} R_{\mu-1}(\zeta).$$

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