

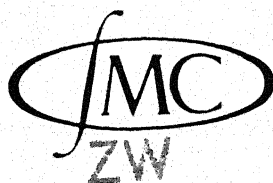
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MATHEMATISCH CENTRUM
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AMSTERDAM

1957 - 028 *A*

Embedding of a distributive lattice into a boolean algebra

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(Reprinted from Proceedings of the KNAW, series A,
60(1957), Indagationes Mathematicae, 19(1957),
p. 73 - 81)



1957

EMBEDDING OF A DISTRIBUTIVE LATTICE INTO A BOOLEAN ALGEBRA

BY

W. PEREMANS

(Communicated by Prof. J. F. KOKSMA at the meeting of October 27, 1956)

1. *Introduction.* The fact that every distributive lattice can be embedded into a Boolean algebra is a trivial consequence of the well-known theorem which states that every distributive lattice is isomorphic to a ring of sets. This method of proving the embeddability is not algebraic and makes use of the axiom of choice. One should like to have a more direct algebraic construction of the embedding. An attempt in this direction has been made by MAC NEILLE [3]. He first constructs a Boolean ring R containing the given distributive lattice D as a subset. In order to make the ring operations of R compatible with the lattice operations of D he takes an ideal I in R and forms the residue class ring R/I . It remains to prove that two different elements of D are incongruent modulo I ; this fact has not been proved correctly in the paper of MAC NEILLE. I have not been able to fill out this gap in his proof without assuming the embeddability. If one assumes that D can be embedded into a Boolean ring B , it is easy to construct a homomorphic mapping of R into B , which leaves the elements of D invariant and turns all elements of I into zero. From the induced mapping of R/I into B it follows that different elements of D are incongruent modulo I .

In section 2 of this paper I give a new proof for the embeddability, which does not make use of the concept of a Boolean ring. Some heuristic remarks will perhaps facilitate the understanding of this proof. Let us assume for a moment, that we have a Boolean algebra B , which contains D as a sublattice. Without loss of generality we may assume that B is generated by D . We denote the greatest and least elements of B by 1 and 0. It is well-known, that every element of B may be put into the form $\bigcup_{k=1}^n (a_k \cap b'_k)$, in which a_k and b_k are elements of D or 0 or 1. To start the construction we extend D to a distributive lattice D' by adjoining a new least element 0 and a new greatest element 1 to D (this is done even if D has already a greatest or least element; we return to this question in section 4). We form the set W of all finite non-empty sets of pairs (a, b) with $a, b \in D'$. It will be necessary to introduce identifications in W . The operation \cup is defined as set-theoretic union and gives

no difficulties. To see how \cap has to be defined, we remember that the pair (a, b) stands for $a \cap b'$; therefore we define $(a, b) \cap (c, d)$ to be $(a \cap c, b \cup d)$; for sets of pairs this construction is applied to all combinations of pairs of the first and the second set. In order to prove the axioms of a distributive lattice we must be able to cancel those pairs from a set of pairs, which are redundant because they represent an element of $B \leq$ another element of B , which also is represented in the set. That this is possible follows from lemma 1.1, which gives a necessary and sufficient condition for the inequality $a \cap b' \geq c \cap d'$, formulated in terms of the lattice generated by a, b, c, d .

Lemma 1.1. In a Boolean algebra $a \cap b' \geq c \cap d'$ holds if and only if $c \leq a \cup d$ and $d \geq b \cap c$.

Proof. Assume $a \cap b' \geq c \cap d'$. Then $a \cup d \geq (a \cap b') \cup d \geq (c \cap d') \cup d = c \cup d \geq c$. $d \geq b \cap c$ is proved similarly. Now assume $c \leq a \cup d$ and $d \geq b \cap c$. Then $c \cap d' \leq (a \cup d) \cap d' \cap c = a \cap d' \cap c \leq a \cap (b' \cup c') \cap c = a \cap b' \cap c \leq a \cap b'$.

The identifications i. and ii. of section 2 are defined according to lemma 1.1. With this identification all axioms of a distributive lattice can be proved. To get complements we remember that the complement of $\bigcup_{k=1}^n (a_k \cap b'_k)$ is $\bigcap_{k=1}^n (a'_k \cup b_k)$; this element may be put again in the form $\bigcup_{i=1}^m (c_i \cap d'_i)$. Identifications v. and vi. guarantee that the corresponding sets of pairs are really complementary. So we get a Boolean algebra. Finally we have to construct an isomorphic mapping of D into this Boolean algebra. We map $a \in D$ onto the pair $(a, 0)$. Identifications iii. and iv. guarantee that this mapping preserves \cup (for \cap no identifications are needed). It remains to prove that the mapping is one-to-one, i.e. that if $(a, 0)$ and $(b, 0)$ are identified, then $a = b$. The proof of this statement is inspired by the following considerations. If $(a, 0)$ and $(b, 0)$ are identified, there is a chain of primitive identifications of the types described above, beginning with $(a, 0)$ and ending with $(b, 0)$. At an intermediate stage we have a set of pairs, which represents the element a . So every pair of this set has to be $\leq a$. Now it is possible to prove formally that if a pair (e, f) is \geq all pairs of a set of pairs in the sense of lemma 1.1., this property also holds after a primitive identification, applied to this set of pairs. From this $a = b$ is easily deduced.

The Boolean algebra obtained in this way is a free extension of D in this sense, that it can be mapped homomorphically into every other Boolean extension of D . This follows easily from the fact, that all identifications made correspond to equalities in every Boolean extension of D . In section 4 we discuss the question whether this homomorphism is an isomorphism. In general this is not true, but exceptions are only caused by the greatest and least elements. If D has a greatest element g , our extension B has a greatest element 1, which is different from g , but there

exists also a Boolean extension of D with g as its greatest element. If we eliminate this exception and the corresponding exception for the least element, isomorphism can be proved.

In section 3 we discuss the relation between our result and a result of DILWORTH [1].

In section 5 we show that it is possible to decide in a finite number of steps, whether two sets of pairs must be identified or not. This proves that our method is really constructive.

2. Let D be a distributive lattice. We take two new elements 0 and 1, and take the set $D' = \{D, 0, 1\}$. By putting $0 < x$ and $x < 1$ for all $x \in D'$, D' is made into a distributive lattice. Let V be the set of all pairs (a, b) with $a, b \in D'$ and let W be the set of all non-empty finite subsets of V . The elements of W are called sets of pairs. We give a list of *elementary transformations*, which are applicable to elements of W .

- i. Let $\alpha \in W$, $(a, b) \in \alpha$, $(c, d) \in \alpha$, $(a, b) \neq (c, d)$, $c \leq a \cup d$ and $d \geq b \cap c$. We form $\alpha_1 \in W$ by cancelling (c, d) in α .
- ii. Let $\alpha \in W$, $(a, b) \in \alpha$, $(c, d) \in V$, $(c, d) \notin \alpha$, $c \leq a \cup d$ and $d \geq b \cap c$. We form $\alpha_1 \in W$ by adding (c, d) to α .
- iii. Let $\alpha \in W$, $(a, c) \in \alpha$, $(b, c) \in \alpha$. We form $\alpha_1 \in W$ by first cancelling a (possibly empty) subset of the set consisting of (a, c) and (b, c) from α and then adding (if necessary) $(a \cup b, c)$ to the obtained set of pairs.
- iv. Let $\alpha \in W$, $(a \cup b, c) \in \alpha$. We form $\alpha_1 \in W$ by first cancelling or not cancelling $(a \cup b, c)$ from α and then (if necessary) adding (a, c) and (b, c) to the obtained set of pairs.
- v. Let $\alpha \in W$, $a_1, \dots, a_n, b_1, \dots, b_n, c, d \in D' (n \geq 1)$, $(a_k \cap c, b_k \cup d) \in \alpha$ for $k = 1, \dots, n$, $(c, \bigcup_{k=1}^n a_k \cup d) \in \alpha$, $(\bigcap_{k=1}^n b_k \cap c, d) \in \alpha$, and, if $n > 1$, for every j with $1 \leq j \leq n-1$ and every set $i_1, \dots, i_j, k_1, \dots, k_{n-j}$, which is a permutation of $1, \dots, n$, $(\bigcap_{\nu=1}^j b_{i_\nu} \cap c, \bigcup_{\mu=1}^{n-j} a_{k_\mu} \cup d) \in \alpha$. We form $\alpha_1 \in W$ by first cancelling a (possibly empty) subset of the pairs mentioned in this point from α and then (if necessary) adding (c, d) to the obtained set of pairs.
- vi. Let $\alpha \in W$, $a_1, \dots, a_n, b_1, \dots, b_n, c, d \in D' (n \geq 1)$, $(c, d) \in \alpha$. Form $\alpha_1 \in W$ by first cancelling or not cancelling (c, d) from α and then adding (if necessary) $(a_k \cap c, b_k \cup d)$ for $k = 1, \dots, n$, $(c, \bigcup_{k=1}^n a_k \cup d)$, $(\bigcap_{k=1}^n b_k \cap c, d)$, and, if $n > 1$, for every j with $1 \leq j \leq n-1$ and every set $i_1, \dots, i_j, k_1, \dots, k_{n-j}$, which is a permutation of $1, \dots, n$, $(\bigcap_{\nu=1}^j b_{i_\nu} \cap c, \bigcup_{\mu=1}^{n-j} a_{k_\mu} \cup d)$ to the obtained set of pairs.

Obviously i. and ii., iii. and iv., v. and vi. are mutually inverse transformations. We define an equivalence relation on W by putting $\alpha \sim \alpha_1$

if and only if a finite (possibly empty) sequence of elementary transformations exists, which, applied successively on α , yield α_1 .

We define a binary operation \cup on W by taking for $\alpha \cup \beta$ the set-theoretic union of α and β . The following lemma is trivial.

Lemma 2.1. If $\alpha, \alpha_1, \beta, \beta_1 \in W$, $\alpha \sim \alpha_1, \beta \sim \beta_1$, then $\alpha \cup \beta \sim \alpha_1 \cup \beta_1$.

We define a binary operation \cap on W in the following way: $\alpha \cap \beta$ is the set consisting of the pairs $(a \cap c, b \cup d)$, where (a, b) runs through α and (c, d) runs through β . Obviously this operation is commutative.

Lemma 2.2. If $\alpha, \alpha_1, \beta, \beta_1 \in W$, $\alpha \sim \alpha_1, \beta \sim \beta_1$, then $\alpha \cap \beta \sim \alpha_1 \cap \beta_1$.

Proof. We may restrict ourselves to the case that $\beta_1 = \beta$ and that α_1 can be obtained from α by an elementary transformation.

i. Obviously $\alpha_1 \cap \beta \subset \alpha \cap \beta$. The only pairs, which possibly are elements of $\alpha \cap \beta$ and not of $\alpha_1 \cap \beta$ are pairs of the form $(c \cap e, d \cup f)$ with $(e, f) \in \beta$. They may be cancelled according to i., as $(a \cap e, b \cup f) \in \alpha \cap \beta$ and $c \cap e \leq (a \cap e) \cup d \cup f$ and $d \cup f \geq (b \cup f) \cap c \cap e$.

ii. α is obtained from α_1 by application of i.

iii. According to $(a \cup b) \cap e = (a \cap e) \cup (b \cap e)$, by iii. we may cancel those elements $(a \cap e, c \cup f)$ and $(b \cap e, c \cup f)$, which are not in $\alpha_1 \cap \beta$ and add, if necessary, $((a \cup b) \cap e, c \cup f)$ for every $(e, f) \in \beta$.

iv. α is obtained from α_1 by application of iii.

v. It is obvious that, for every $(e, f) \in \beta$, v. may be applied with (c, d) replaced by $(c \cap e, d \cap f)$.

vi. α is obtained from α_1 by application of v.

Let B be the set of the equivalence classes of W with respect to \sim . According to lemmas 2.1 and 2.2 the operations \cup and \cap may be defined on B with representants.

Lemma 2.3. B is a Boolean algebra.

Proof. Obviously \cup is idempotent, associative and commutative. That \cap is idempotent, follows from the fact, that if $(a, b) \in V$ and $(c, d) \in V$, and if (a, b) and $(a \cap c, b \cup d)$ are elements of a set of pairs, we may cancel $(a \cap c, b \cup d)$ by i. if this pair is different from (a, b) . Associativity and commutativity of \cap are obvious. The absorption laws $(u \cup v) \cap v = v$ and $(u \cap v) \cup v = v$ are proved in the same way as the idempotency of \cap . The distributive law $(u \cup v) \cap w = (u \cap w) \cup (v \cap w)$ is obvious. The element g of B , which contains the set of pairs consisting of the pair $(1, 0)$, is the greatest element of B . This follows from the fact, that, for every $(a, b) \in V$, $a \leq 1 \cup b$ and $b \geq 0 \cap a$; therefore $g \cup u = g$ for every $u \in B$. The element l of B , which contains the set of pairs consisting of the pair $(0, 0)$, is the least element of B . This follows from the fact, that, for every $(a, b) \in V$, $0 \leq a \cup 0$ and $0 \geq b \cap 0$; therefore $l \cup u = u$ for every $u \in B$. The complement of an element of B may be obtained in the following way. Take an $\alpha \in W$ from this element; let $(a_1, b_1), \dots, (a_n, b_n)$ be the elements of α . Form $\alpha' \in W$ consisting of the pairs $(1, \bigcup_{k=1}^n a_k), (\bigcap_{k=1}^n b_k, 0)$ and, if $n > 1$, for every j with $1 < j < n-1$ and

every set $i_1, \dots, i_j, k_1, \dots, k_{n-j}$, which is a permutation of $1, \dots, n$, of the pair $(\bigcap_{\nu=1}^j b_{i_\nu}, \bigcup_{\mu=1}^{n-j} a_{k_\mu})$. The element of B containing α' is the complement of the given element. To prove this we first consider $\alpha \cup \alpha'$. This may be transformed by v . into the set consisting of $(1, 0)$. Now $\alpha \cap \alpha'$ consists of pairs which all have the form (a, b) with $a \leq b$. It is easy to prove that such a set may be transformed by i . and ii . into the set consisting of $(0, 0)$. This completes the proof.

To prove that B contains a sublattice isomorphic to D , we need the following lemma.

Lemma 2.4. If $(e, f) \in V$, if $\alpha \in W$, if $\alpha_1 \in W$, if $\alpha \sim \alpha_1$ and if

$$(2.1) \quad \begin{cases} a \leq e \cup b \\ b \geq f \cap a \end{cases}$$

holds for all $(a, b) \in \alpha$, (2.1) also holds for all $(a, b) \in \alpha_1$.

Proof. It is sufficient to prove the lemma for the case that α_1 is obtained from α by an elementary transformation. Now all cases are trivial except case v . In that case we are given the following inequalities:

$$\begin{aligned} a_k \cap c &\leq e \cup b_k \cup d, & b_k \cup d &\geq f \cap a_k \cap c \\ c &\leq e \cup \bigcup_{k=1}^n a_k \cup d, & \bigcup_{k=1}^n a_k \cup d &\geq f \cap c \\ \bigcap_{k=1}^n b_k \cap c &\leq e \cup d, & d &\geq f \cap \bigcap_{k=1}^n b_k \cap c \\ \bigcap_{\nu=1}^j b_{i_\nu} \cap c &\leq e \cup \bigcup_{\mu=1}^{n-j} a_{k_\mu} \cup d, & \bigcup_{\mu=1}^{n-j} a_{k_\mu} \cup d &\geq f \cap \bigcap_{\nu=1}^j b_{i_\nu} \cap c. \end{aligned}$$

We have to prove $c \leq e \cup d$ and $d \geq f \cap c$. By mathematical induction with respect to j we prove $c \leq e \cup \bigcup_{\mu=1}^{n-j} a_{k_\mu} \cup d$ for every j with $0 \leq j \leq n-1$ and for every set of different indices k_1, \dots, k_{n-j} with $1 \leq k_\mu \leq n$ ($\mu=1, \dots, n-j$). For $j=0$ this inequality is given. Now take $j \geq 1$ and an index l with $1 \leq l \leq n$ and $l \neq k_\mu$ for all μ . By induction we have

$$\begin{aligned} c &\leq e \cup \bigcup_{\mu=1}^{n-j} a_{k_\mu} \cup a_l \cup d, & c &\leq e \cup \bigcup_{\mu=1}^{n-j} a_{k_\mu} \cup (a_l \cap c) \cup d \leq \\ & & &\leq e \cup \bigcup_{\mu=1}^{n-j} a_{k_\mu} \cup b_l \cup d. \end{aligned}$$

If i_1, \dots, i_j is a set of complementary indices of k_1, \dots, k_{n-j} , we get

$$\begin{aligned} c &\leq e \cup \bigcup_{\mu=1}^{n-j} a_{k_\mu} \cup \left(\bigcap_{\nu=1}^j b_{i_\nu} \right) \cup d, \\ c &\leq e \cup \bigcup_{\mu=1}^{n-j} a_{k_\mu} \cup \left(\bigcap_{\nu=1}^j b_{i_\nu} \cap c \right) \cup d \leq e \cup \bigcup_{\mu=1}^{n-j} a_{k_\mu} \cup d. \end{aligned}$$

This proves the inequality. If we take $j=n-1$, we get $c \leq e \cup a_l \cup d$

for all $l=1, \dots, n$,

$$c \leq e \cup (a_l \cap c) \cup d \leq e \cup b_l \cup d, c \leq e \cup \left(\bigcap_{k=1}^n b_k \right) \cup d,$$

$$c \leq e \cup \left(\bigcap_{k=1}^n b_k \cap c \right) \cup d \leq e \cup d.$$

The other inequality $d > f \cap c$ is proved dually.

We now define a mapping ϑ of D' into B by taking for $\vartheta(a)$ the element of B containing the set of pairs consisting of $(a, 0)$. That ϑ is one-to-one follows from

Lemma 2.5. If $\alpha \in W$ consists of the pair $(a, 0)$ and $\alpha_1 \in W$ of the pair $(b, 0)$ and if $\alpha \sim \alpha_1$, then $a=b$.

Proof. $a < a \cup 0, 0 > 0 \cap a$, so we may apply lemma 2.4 with $e=a, f=0$. This yields $b < a \cup 0 = a$. Similarly we find $a < b$, so $a=b$.

That ϑ preserves \cup and \cap is trivial (for \cup we need iii.). So ϑ is an isomorphic mapping. We now have proved our main theorem.

Theorem 2.1. If D is a distributive lattice, a Boolean algebra exists, containing D as a sublattice.

We shall denote by $B(D)$ the Boolean algebra, which is obtained from D by the construction described in this section. For the sake of simplicity we identify D' with its isomorphic image $\vartheta(D')$. Then 1 and 0 are the greatest and least elements of $B(D)$.

$B(D)$ is a free extension of D in the following sense.

Theorem 2.2., If B_1 is a Boolean algebra containing D as a sublattice, a homomorphic mapping of $B(D)$ into B_1 exists, whose restriction to D is the identical mapping.

Proof. We first map W into B_1 . If $\alpha \in W$ and if $(a_1, b_1), \dots, (a_n, b_n)$ are the elements of α , we map α onto the element $\bigcup_{k=1}^n (a_k \cap b'_k)$ of B_1 (here for 1 and 0 the greatest and least elements of B_1 have to be taken). It is easy to show that equivalent elements of W have the same image in B_1 (for i. and ii. lemma 1.1 is used). So we get an induced mapping of $B(D)$ into B_1 , which satisfies all properties required.

3. In this section we discuss a result of R. P. DILWORTH [1], which is closely related to ours. He has proved that every lattice P can be embedded into a lattice N , in which every element has a unique complement. One could guess, that our result is a special case of this theorem. This is not the case, except if P has only one element.

Theorem 3.1. If P is a distributive lattice with at least two elements, the lattice N obtained from P by the construction of Dilworth, is not distributive.

Proof. For terminology and notation we refer to [1]. In this proof references to lemma's and theorems are to [1]. If $a \in P$, then $a \in N$ (lemma 3.1). We prove $a^* \in N$. Sub-polynomials of a^* are a and a^* .

Now $a \simeq (X^*)^*$ is impossible by theorem 2.10. If $a^* \simeq (X^*)^*$, then $a \simeq X^*$ by theorem 2.5, and this again is impossible by theorem 2.10. So $a^* \in N$ and therefore $a' \equiv a^*$ for all $a \in P$. If $a, b \in P$ and $a' \supseteq b'$, then $a^* \supseteq b^*$ and, by theorem 2.5, $a \simeq b$. Theorems 2.3 and 1.3 now yield $a = b$. So we have found that, if $a \neq b$, a' and b' are incomparable. Now by assumption P has at least two elements; then P has also two elements a and b with $a > b$. If N would be distributive (and therefore a Boolean algebra), this would imply $b' > a'$. So we have got a contradiction: N is not distributive.

If P has only one element, N is the four-element Boolean algebra.

4. We now discuss the question whether the homomorphism of theorem 2.2 is an isomorphism. We may put this question also in the following form: is the least Boolean extension of D determined uniquely up to isomorphism? In general this is not true. Assume e.g. that D has a greatest element g . This element is different from the greatest element 1 of $B(D)$. We consider the sublattice B_1 of B consisting of those elements x of B satisfying $x < g$. Then B_1 is a Boolean algebra containing D as a sublattice, but it is clear that no isomorphic mapping of B onto B_1 exists, which leaves invariant all elements of D . With an eventual least element l of D we may proceed similarly. So if D has a greatest and a least element, we have found four essentially different least extensions; if D has a greatest and no least, or a least and no greatest element, we have found two essentially different least extensions and if D has no greatest and no least elements, we have found only one least extension. We prove that these are the only possibilities.

If D contains a greatest element g , we form $D'' = \{D, 0\}$ and put $0 < x$ for all $x \in D''$. We construct a Boolean algebra $B_2(D)$ as in section 2 with D' replaced by D'' . Then $B_2(D)$ has g as its greatest and 0 as its least element. Similarly we construct (if possible) $B_3(D)$ with greatest element 1 and least element l and $B_4(D)$ with greatest element g and least element l .

Theorem 4.1. Let D be a distributive lattice and B^* a Boolean algebra containing D as a sublattice and generated by D . Let 1^* and 0^* be the greatest and least elements of B^* . There exists an isomorphic mapping of B^* , which leaves invariant all elements of D , onto

$$\begin{aligned} B(D), & \text{ if } 1^* \notin D \text{ and } 0^* \notin D, \\ B_2(D), & \text{ if } 1^* \in D \text{ and } 0^* \notin D, \\ B_3(D), & \text{ if } 1^* \notin D \text{ and } 0^* \in D, \\ B_4(D), & \text{ if } 1^* \in D \text{ and } 0^* \in D. \end{aligned}$$

Proof. We introduce the symbol B_5 to denote $B(D)$, $B_2(D)$, $B_3(D)$ or $B_4(D)$ corresponding to the four cases of the theorem. In the same way as was done in the proof of theorem 2.2 we can construct a homomorphic mapping φ of B_5 into B^* leaving invariant all elements of D . As B^* is generated by D , this is a mapping onto B^* . So the only thing

we have to prove is, that the mapping is one-to-one. We take two inequivalent sets of pairs α and β and have to prove that their images are different. Let $(a_1, b_1), \dots, (a_n, b_n)$ be the elements of α and $(c_1, d_1), \dots, (c_m, d_m)$ the elements of β . We now take the finite set U consisting of the elements a_k, b_k, c_l, d_l ($k=1, \dots, n; l=1, \dots, m$), and moreover in the first case of the theorem of 1 and 0, in the second case of g and 0, in the third case of 1 and l , and in the fourth case of g and l . The sublattice D_1 of B_5 generated by U and the Boolean subalgebra B'_5 of B_5 generated by D_1 are also finite. We also form the set U^* consisting of the elements $1^*, 0^*, a_k, b_k, c_l, d_l$ ($k=1, \dots, n; l=1, \dots, m$) and the sublattice D_1^* of B^* generated by U^* and the Boolean subalgebra B'^* of B^* generated by D_1^* . Obviously φ induces an isomorphic mapping of D_1 onto D_1^* and a homomorphic mapping of B'_5 onto B'^* . Moreover the elements of B_5 which contain α and β are also elements of B'_5 . It is sufficient to prove that the mapping of B'_5 onto B'^* is isomorphic. This is implied by the following lemma, in which we have reduced the problem to finite lattices.

Lemma 4.1. Let D be a finite distributive lattice with greatest element 1 and least element 0 and let B_1 and B_2 be Boolean algebras containing D as a sublattice and generated by D . Let 1 and 0 be also the greatest and least elements of B_1 and B_2 . There exists a (uniquely determined) isomorphic mapping of B_1 onto B_2 , which leaves invariant all elements of D .

This lemma follows from some well-known theorems about finite distributive lattices. We call an isomorphic mapping φ of a finite distributive lattice A onto a ring of sets with carrier S reduced,¹ if $\varphi(0)=\phi$, $\varphi(1)=S$ and if $p \in \varphi(x) \Leftrightarrow q \in \varphi(x)$ for all $x \in A$ implies $p=q$. The well-known representation of A as a ring of sets with join-irreducible elements $\neq 0$ is reduced in this sense. Two reduced mappings of A are essentially equal: there exists a one-to-one mapping between the carriers which maps sets corresponding to the same element of A onto each other. If A is a Boolean algebra a reduced representation of A maps A onto a field of sets.

To prove our lemma we take reduced representations of B_1 and B_2 as fields of sets F_1 and F_2 with carriers S_1 and S_2 (e.g. with join-irreducible elements). These representations induce representations of D as rings of sets R_1 and R_2 with carriers S_1 and S_2 . It is easy to show that these representations are also reduced. So there exists a one-to-one mapping ψ of S_1 onto S_2 which maps elements of R_1 onto corresponding elements of R_2 . It is easy to infer from this, that ψ induces an isomorphic mapping of F_1 onto F_2 and therefore also of B_1 onto B_2 ; the latter induces the identical mapping on D . This completes the proof.

From the results of this section we see why in section 2 D was extended with elements 0 and 1 even if it had already least or greatest elements itself. To get the free extension this is necessary in any case.

5. We now discuss the following question. Is it possible to decide in a finite number of steps, whether two given sets of pairs are equivalent or not? We assume that D is completely known. We use the notation of section 4. We take two sets of pairs α and β and form U and D_1 as in the proof of theorem 4.1, first case. As D_1 is a sublattice of D' , it is known. The least Boolean extension B_1 of D_1 is uniquely determined up to isomorphism, and may be constructed in a finite number of steps. Furthermore B_1 is isomorphic to a subalgebra of $B(D)$ containing the elements, which contain α and β . Now α and β are equivalent if and only if the elements $\bigcup_{k=1}^n (a_k \cap b'_k)$ and $\bigcup_{i=1}^m (c_i \cap d'_i)$ of B_1 are equal. This may be decided in a finite number of steps.

This construction gives a method to determine equivalence of sets of pairs, which could serve as a definition. Perhaps this definition could lead to a new proof of embeddability.

Finally we remark, that it is possible, using metamathematical or topological methods, to prove the embeddability of every distributive lattice, if the embeddability of every finite distributive lattice is known (cf. [2] and [4]). These proofs, however, make use of the axiom of choice (Gödel's completeness theorem or Tychonoff's theorem).

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