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A. SCHRIJVER

LINKING SYSTEMS

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by

A. Schrijver

ABSTRACT

In this report the notion of a linking system is introduced, some examples are given and the relations with matroid theory and bipartite graphs are shown. Furthermore we give necessary and sufficient conditions for a matroid to be a binary deltoid and also for a base of a deltoid to be a so-called deltoid-base.

Keywords & Phrases: matroids (deltoids, gammoids), graphs (digraphs, bipartite graphs), linking systems, representable over a field.

0. INTRODUCTION.

Let (X,Y,E) be a (finite) bipartite graph and $X' \subset X$, $Y' \subset Y$. We call X' & Y' *linked in* E if there is a *linking* $\ell : X' \to Y'$, which is a bijection, such that for each $x \in X'$ $(x,\ell(x))$ is an edge of the bipartite graph. It was proved by PERFECT & PYM [11] that, if (X',Y') and (X'',Y'') are maximal linked pairs (maximal under co-ordinate-wise inclusion), then X' & Y'' are linked in E (and, of course, also X'' & Y' are linked in E). A similar phenomenon in directed graphs was discovered by PYM [13] and it also occurs in matrices, when we define "linked pairs" in a suitable way. It forms the base of the definition of a *linking system*, which is a triple (X,Y,Λ) with finite sets X and Y and Λ the collection of *linked pairs* $(\emptyset \neq \Lambda \subset P(X) \times P(Y))$, such that (i) if $(X',Y') \in \Lambda$ then $|X'| = |Y'|$, (ii) if $(X',Y') \in \Lambda$ and $X'' \subset X'$, then there is a $Y'' \subset Y'$ with $(X'',Y'') \in \Lambda$, (iii) if $(X',Y') \in \Lambda$ and $Y'' \subset Y'$, then there is an $X'' \subset X'$ with $(X'',Y'') \in \Lambda$, (iv) if $X' \subset X$ and $Y' \subset Y$ and (X'',Y'') and (X''',Y''') are minimal in $\Lambda \cap (P(X') \times P(Y'))$, then $(X'',Y''') \in \Lambda$. If $(X',Y') \in \Lambda$ we shall say that X' & Y' are *linked in* Λ. Of course, if (X,Y,E) is a bipartite graph and Δ_E is the collection of linked pairs of that bipartite graph, then (X,Y,Δ_E) is a linking system, namely a so-called *deltoid linking system*.

A second example is induced by a (finite) directed graph (Z,E) and sets $X, Y \subset Z$. Then (X,Y,Γ_E) is a linking system, where $(X',Y') \in \Gamma_E$ iff $|X'| = |Y'|$ and there are $|X'|$ pair-wise vertex-disjoint paths starting in X' and finishing in Y'. A linking system constructed in this way is called a *gammoid linking system* and it is easy to see that every deltoid-linking system is a gammoid-linking system. When we define a matrix over a field F as a triple (X,Y,ϕ) with finite sets X and Y and ϕ an F-valued function defined on $X \times Y$ (the relation with usually defined matrices is obvious), this gives us a third example of a linking system: given a matrix over F (X,Y,ϕ), define (X,Y,Λ_ϕ) by $(X',Y') \in \Lambda_\phi$ iff the submatrix of (X,Y,ϕ) generated by X' and Y' (again in the obvious way) is regular. It can be proved without great difficulties that this is a linking system. Linking systems induced by a matrix over F are called *representable over* F and if a linking system is representable over $GF(2)$ (the field with two elements) we call this linking system *binary*.

There is a close correspondence between linking systems and so-called based matroids (matroids with a fixed base) (see section 3) and in many proofs we make use of this relation.

In section 4 we prove that a matroid $(X,\mathcal{I})$ ($\mathcal{I}$ is the collection of independent sets) and a linking system (X,Y,Λ) induce a matroid $(Y,\mathcal{I}*\Lambda)$ if we take as independent sets of Y all sets which are linked in Λ with some independent subset of X. Further, if (X,Y,Λ_1) and (Y,Z,Λ_2) are linking systems, then we define the linking systems $(X,Z,\Lambda_1*\Lambda_2)$ by: X' & Z' are linked in $\Lambda_1 * \Lambda_2$ iff X' & Y' are linked in Λ_1 and Y' & Z' are linked in Λ_2 for some $Y' \subset Y$. The fact that this is indeed a linking system is proved in section 4.

In section 5 we define for each linking system (X,Y,Λ) a bipartite graph (X,Y,E_Λ) by: $(x,y) \in E$ iff $\{x\}$ & $\{y\}$ are linked in Λ. We prove that, if there is exactly one linking $\ell : X' \to Y'$ in E_Λ, then X' & Y' are linked in Λ, and if X' & Y' are linked in Λ, then X' & Y' are linked in E_Λ. So the linking system (X,Y,Δ_E) induced by the bipartite graph (X,Y,E) is the greatest linking system with underlying bipartite graph (X,Y,E). Using the results of section 5, we prove in section 6 that a linking system (X,Y,Λ) is a binary deltoid linking system iff the underlying bipartite graph (X,Y,E_Λ) is a forest, i.e. contains no circuits. Binary deltoid linking systems have a close relationship with binary deltoids in matroid theory.

Finally, in section 7, using methods of the preceding sections, we give a criterion for bases of a deltoid to be a deltoid-base, i.e. a base X such that there is a bipartite graph (X,Y,E) which induces that deltoid. We show that the deltoid-bases of a deltoid form again a deltoid, namely a semi-regular matroid.

1. PRELIMINARIES

In the sequel all sets (except $\mathbb{N}$ and $\mathbb{Z}$) are supposed to be finite.

If we speak of "(X',Y') is maximal with some property", then this is meant under the ordering: $(X',Y') \leq (X'',Y'')$ iff $X' \subset X''$ and $Y' \subset Y''$. Further, if X, Y, Z are sets, we shall write $X \setminus Y \cup Z$ for $(X\setminus Y) \cup Z$.

In this section we give some definitions and known facts about matrices, graphs and matroids.

a. Matrices.

A *matrix* M *over a field* F is defined as a triple (X,Y,ϕ), where X and Y are sets and $\phi : X \times Y \to F$. (X',Y',ϕ') is a *submatrix of* M if $X' \subset X$, $Y' \subset Y$ and $\phi' = \phi \mid X' \times Y'$. The *dual matrix* M^* of M is the matrix (Y,X,ϕ^*) with $\phi^*(y,x) = \phi(x,y)$ $(x \in X, y \in Y)$.
If $M_1 = (X,Y,\phi_1)$ and $M_2 = (Y,Z,\phi_2)$ are matrices over F, then $M_1M_2 = (X,Z,\phi_1\phi_2)$ is the matrix over F with $\phi_1\phi_2(x,z) = \sum_{y \in Y} \phi_1(x,y)\phi_2(y,z)$ $(x \in X, z \in Z)$.
If $M = (X,Y,\phi)$ is a matrix over F, then define $\lambda_\phi : P(X) \times P(Y) \to \mathbb{Z}$ by $\lambda_\phi(X',Y')$ = the rank of the submatrix (X',Y',ϕ') of M. X' & Y' are called *linked in* ϕ if (X',Y',ϕ') is a regular matrix, i.e. if $\lambda_\phi(X',Y') = |X'| = |Y'|$, and *maximal linked in* ϕ if (X',Y') is maximal with this property.

PROPOSITION 1.1. *Let* $M = (X,Y,\phi)$ *be a matrix over a field* F *and* $X',X'' \subset X$, $Y',Y'' \subset Y$. If X' & Y' *are maximal linked in* ϕ *and likewise* X'' & Y'', *then* X' & Y'' *are (maximal) linked in* ϕ; *particularly* $|X'| = |X''|$.

PROOF. Straightforward. □

b. Graphs.

A *bipartite graph* or *b.g.* B is defined as a triple (X,Y,E), where X and Y are sets and $E \subset X \times Y$. (X',Y',E') is a *sub-b.g. of* B if $X' \subset X$, $Y' \subset Y$ and $E' = E \cap (X' \times Y')$. The *dual b.g.* B^* of B is the b.g. (Y,X,E^*) with $(y,x) \in E^* \Leftrightarrow (x,y) \in E$ $(x \in X, y \in Y)$.
If $B_1 = (X,Y,E_1)$ and $B_2 = (Y,Z,E_2)$ are b.g.'s, then $B_1B_2 = (X,Z,E_1E_2)$ is the b.g. with: $(x,z) \in E_1E_2 \Leftrightarrow \exists y \in Y : (x,y) \in E_1$ and $(y,z) \in E_2$ $(x \in X, z \in Z)$.
In the definition of a b.g. we do not require (as usually is done) that X and Y are disjoint sets. The definitions and propositions below are derived from and can be brought back to the usual definitions and propositions, i.e. where X and Y are disjoint.

Let $B = (X,Y,E)$ be a b.g. $E_0 \subset E$ is called a *matching of* B if for every $(x,y),(x',y') \in E_0$ $[x = x'$ iff $y = y']$ is valid, and the *matching-number of* B is the maximal cardinality of the matchings of B. The pair (X',Y') is called a *covering of* B if for each $(x,y) \in E : x \in X'$ or $y \in Y'$.
Now we can state:

THEOREM 1.2. (KÖNIG) *Let* $B = (X,Y,E)$ *be a b.g.. Then the matching-number of* B *equals* $\min \{|X'| + |Y'| \mid X' \subset X,\ Y' \subset Y,\ (X',Y')$ is a covering of $B\}$.

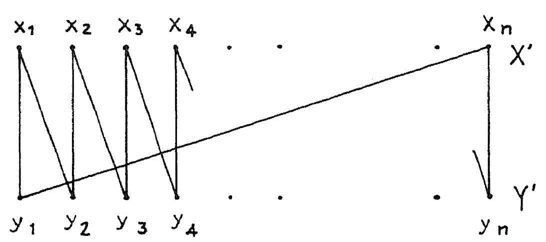
PROOF. See e.g. HARARY [3]. □

If $B = (X,Y,E)$ is a b.g., then define $\delta_E : P(X) \times P(Y) \to \mathbb{Z}$ by $\delta_E(X',Y') =$ the matching-number of the subgraph (X',Y',E') of B. X' & Y' are called *linked in* E if $\delta_E(X',Y') = |X'| = |Y'|$, i.e. if there exists a *linking* $\ell : X' \to Y'$, which is defined as a bijection such that $\forall x \in X' : (x,\ell(x)) \in E$. X' & Y' are called *maximal linked in* E if (X',Y') is maximal with this property.

PROPOSITION 1.3. *Let* $B = (X,Y,E)$ *be a b.g. and* $X',X'' \subset X$, $Y',Y'' \subset Y$. If X' & Y' *are maximal linked in* E *and likewise* X'' & Y'', *then* X' & Y'' *are (maximal) linked in* E; *particularly* $|X'| = |X''|$.

PROOF. See MIRSKY & PERFECT [8] or PERFECT & PYM [11]. □

Let $B = (X,Y,E)$ be a b.g.. Supposing that the notion of "connectedness" (of b.g.'s) is known, we call the pair (X',Y') *connected* if the sub-b.g. (X',Y',E') is a connected b.g. and (X',Y') is a *component of* B if (X',Y') is maximal with this property. B is *complete* if $E = X \times Y$ and *semi-complete* if for each component (X',Y') of B the sub-b.g. (X',Y',E') is complete.



The pair (X',Y') is called a *circuit of* B if we can write: $X' = \{x_1,\dots,x_n\}$ and $Y' = \{y_1,\dots,y_n\}$ such that $n \geq 2$, $(x_1,y_1),\dots,(x_n,y_n)$, $(x_1,y_2),(x_2,y_3),\dots,(x_{n-1},y_n),(x_n,y_1) \in E$, and B is a *forest* if B has no circuits.

A *digraph* is defined as a pair (Z,E) where Z is a set and $E \subset Z \times Z$. If $X,Y \subset Z$, we say that X & Y are *linked in* E iff $|X| = |Y|$ and there exist $|X|$ pairwise vertex-disjoint paths starting in X and finishing in Y. If $X,Y \subset Z$ and $X' \subset X$, $Y' \subset Y$, then X' & Y' are *maximal linked in* $E(X,Y)$ iff X' & Y' are linked in E and (X',Y') is maximal with this property.

Let $X, Y \subset Z$ and $Z' \subset Z$. Z' is called (X,Y)*-disconnecting* if each path from an $x \in X$ to any $y \in Y$ contains a point of Z'.
Now we can state

<u>THEOREM 1.4</u>. (MENGER) *Let* (Z,E) *be a digraph and* $X, Y \subset Z$. *Then:*

$$\max\{|X'| \mid X' \subset X,\ Y' \subset Y,\ X' \ \&\ Y' \text{ linked in } E\} =$$
$$\min\{|Z'| \mid Z' \subset Z,\ Z' \text{ is } (X,Y)\text{-disconnecting}\}.$$

<u>PROOF</u>. See e.g. HARARY [3]. □

We also have:

<u>PROPOSITION 1.5</u>. *Let* (Z,E) *be a digraph and* $X', X'' \subset X \subset Z$ *and* $Y', Y'' \subset Y \subset Z$. *If* X' & Y' *are maximal linked in* $E(X,Y)$ *and likewise* X'' & Y'', *then* X' & Y'' *are (maximal) linked in* $E(X,Y)$; *particularly* $|X'| = |X''|$.

<u>PROOF</u>. See PYM [13] or BRUALDI & PYM [2]. □

c. <u>Matroids</u>.

Matroids can be represented in several ways; see e.g. HARARY & WELSH [4] or WILSON [14]. Here we give three representations.

A *matroid* with *collection of independent sets* I is a pair (Z,I) such that:

(i) Z is a set, I is a non-empty collection of subsets of Z;

(ii) if $A \subset B \in I$, then $A \in I$;

(iii) if $Z' \subset Z$ and A and B are maximal independent subsets of Z', then $|A| = |B|$.

A *matroid* with *collection of bases* $\mathcal{B}$ is a pair $(Z,\mathcal{B})$ such that:

(i) Z is a set, $\mathcal{B}$ is a non-empty collection of subsets of Z;

(ii) if $A,B \in \mathcal{B}$ and $b \in B \setminus A$, then there exists an $a \in A \setminus B$ such that $B\setminus\{b\} \cup \{a\} \in \mathcal{B}$.

A *matroid* with *rank-function* ρ is a pair (Z,ρ) such that:

(i) Z is a set, $\rho : \mathcal{P}(Z) \to \mathbb{Z}$;

(ii) if $Z' \subset Z$, then $0 \le \rho(Z') \le |Z'|$;

(iii) if $Z' \subset Z'' \subset Z$, then $\rho(Z') \le \rho(Z'')$;

(iv) if $Z', Z'' \subset Z$, then $\rho(Z'\cap Z'') + \rho(Z'\cup Z'') \le \rho(Z') + \rho(Z'')$.

Relations between $\mathcal{I}$, $\mathcal{B}$ and ρ are:

- $\mathcal{B}$ is the collection of all maximal elements of $\mathcal{I}$;
- $\mathcal{I}$ is the collection of all sets contained in some element of $\mathcal{B}$;
- $\rho(Z')$ is the maximal cardinality of an independent subset of $Z' \subset Z$;
- $\mathcal{I}$ is the collection of all sets $Z' \subset Z$ with $\rho(Z') = |Z'|$;
- $\mathcal{B}$ is the collection of all sets $Z' \subset Z$ with $\rho(Z') = |Z'| = \rho(Z)$;
- $\rho(Z') = \max. \{|B\cap Z'| \mid B\in\mathcal{B}\}$.

These relations represent canonical isomorphisms between matroids with collection of independent sets, and matroids with collection of bases and matroids with rank-function. Therefore we identify these three structures and call them indiscriminately *matroids*.

Note: if a matroid is given as $(Z,\mathcal{I})$ without further specification, then it is understood that $\mathcal{I}$ is its collection of independent sets. Likewise the symbols $\mathcal{B}$ and ρ will be reserved for bases-collections resp. rankfunctions. Also, given a matroid with collection $\mathcal{I}$ (without or with a super- or subscript) we take silently for granted that the corresponding basescollection and rankfunctions are resp. $\mathcal{B}$ and ρ (without or with the same super- of subscript), and vice versa. Furthermore, in subscripts the symbols $\mathcal{B}$, $\mathcal{I}$ and ρ are interchangeable, e.g. $X_{\mathcal{B}_1}$ will mean the same as $X_{\mathcal{I}_1}$ and X_{ρ_1}.

A subset of Z which is not independent is called *dependent* and a minimal dependent subset of Z is a *circuit*. It can be proved that, if $B \in \mathcal{B}$ and $x \notin B$, then there is exactly one circuit C with $x \in C \subset B \cup \{x\}$.

If $A \subset Z$ then the *closure* $[A]$ of A is the set $\{x \in Z \mid \rho(A \cup \{x\}) = \rho(A)\}$. It can be seen that $[A] = \cup\{X \mid A \subset X \text{ and } \rho(X) = \rho(A)\}$ and $\rho([A]) = \rho(A)$.

Let $M = (Z,\mathcal{I})$ be a matroid (with bases-collection $\mathcal{B}$ and rank-function ρ). The *dual matroid* M^* of M is the matroid $(Z,\mathcal{B}^*)$ with bases-collection $\mathcal{B}^* = \{Z \backslash B \mid B \in \mathcal{B}\}$. The rank-function ρ^* is then $\rho^*(Z') = |Z'| + \rho(Z \backslash Z') - \rho(Z), (Z' \subset Z)$. Of course, $M = M^{**}$.

Let $Z' \subset Z$. The *restriction* $M \times Z'$ of M to Z' is the matroid $(Z',\mathcal{I} \times Z')$ with $\mathcal{I} \times Z' = \mathcal{I} \cap \mathcal{P}(Z')$. $M \times Z'$ has as rank-function $\rho \times Z'$ with $\forall Z'' \subset Z'$: $(\rho \times Z')(Z'') = \rho(Z'')$.
The *contraction* $M \cdot Z'$ of M to Z' is the matroid $(Z',\mathcal{I} \cdot Z')$ where $\mathcal{I} \cdot Z' = \{Z'' \subset Z' \mid \text{for each independent set } Z''' \subset Z \backslash Z' \text{ is } Z'' \cup Z''' \text{ independent}\}$.
It can be proved that $M \cdot Z' = (M^* \times Z')^*$. If $Z \backslash Z'$ is independent in M, then the collection of independent sets of $M \cdot Z'$ is: $\{Z'' \subset Z' \mid Z'' \cup (Z \backslash Z') \text{ is independent in } M\}$.
The rank-function $\rho \cdot Z'$ is given by:

$$(\rho \cdot Z')(Z'') = \rho((Z \backslash Z') \cup Z'') - \rho(Z \backslash Z') \ , \ (Z'' \subset Z').$$

Let $M_1 = (Z_1,\mathcal{I}_1)$ and $M_2 = (Z_2,\mathcal{I}_2)$ be two matroids (where Z_1 and Z_2 are not necessarily disjoint), with rank-functions resp. ρ_1 and ρ_2.
The *union* $M_1 \vee M_2$ of M_1 and M_2 is the matroid $(Z_1 \cup Z_2, \mathcal{I}_1 \vee \mathcal{I}_2)$ with collection of independent sets $\mathcal{I}_1 \vee \mathcal{I}_2 = \{I_1 \cup I_2 \mid I_1 \in \mathcal{I}_1, I_2 \in \mathcal{I}_2\}$. The rank-function $\rho_1 \vee \rho_2$ of $M_1 \vee M_2$ is $(\rho_1 \vee \rho_2)(Z') = \min_{Z'' \subset Z'} (\rho_1(Z'' \cap Z_1) + \rho_2(Z'' \cap Z_2) + |Z' \backslash Z''|)$, $(Z' \subset Z_1 \cup Z_2)$.

Let $M = (Z,\mathcal{B})$ be a matroid. M is called *uniform* if there exists some $k \geq 0$ such that $\mathcal{B} = \{Z' \subset Z \mid |Z'| = k\}$ and *semi-uniform* if there exists a partition $Z_1,\dots,Z_n$ of Z and $k_1,\dots,k_n \geq 0$ such that $\mathcal{B} = \{Z' \subset Z \mid \forall i=1,\dots n : |Z' \cap Z_i| = k_i\}$.

In the sequel we need often a matroid with some fixed base, so we shall speak of a *based matroid*, which is a triple (X,Y,ρ), resp. $(X,Y,\mathcal{B})$, resp. $(X,Y,\mathcal{I})$ with disjoint sets X and Y, such that (i) $(X \cup Y,\rho)$, resp. $(X,Y,\mathcal{B})$, resp. $(X,Y,\mathcal{I})$ is a matroid and (ii) X is a base of this matroid. The *dual based matroid* of (X,Y,ρ) is then (Y,X,ρ^*), which is again a based matroid.

EXAMPLES OF MATROIDS

a. Deltoids.

Let (X,Y,E) be a b.g. with disjoint X and Y. Let $\mathcal{B}_E = \{(X\backslash X') \cup Y' \mid X' \subset Y,\ Y' \subset Y$ and X' & Y' linked in E$\}$. Then $(X\cup Y,\mathcal{B}_E)$ is a matroid and $(X,Y,\mathcal{B}_E)$ a based matroid (see e.g. INGLETON & PIFF [5]). Matroids M constructed in this way are called *deltoids*; X is called a *deltoid-base* for M and $(X,Y,\mathcal{B}_E)$ a *deltoid-based deltoid*. In general it is not true that every base of a deltoid is a deltoid-base (we give a criterion for a base to be a deltoid-base in section 7). If M is a deltoid, then also M^* is a deltoid. Also we have $(\mathcal{B}_E)^* = \mathcal{B}_{E^*}$. Further we can state

PROPOSITION 1.6. *Let* $M = (Z,\mathcal{B})$ *be a matroid. Then:*

(i) M *is uniform iff there exists a complete b.g.* (X,Y,E) *such that* $\mathcal{B} = \mathcal{B}_E$;

(ii) M *is semi-uniform iff there exists a semi-complete b.g.* (X,Y,E) *such that* $\mathcal{B} = \mathcal{B}_E$.

PROOF. Obvious. □

b. Gammoids.

Let (Z,E) be a digraph and X and Y disjoint subsets of Z. Let $\mathcal{B}_E = \{(X\backslash X') \cup Y' \mid X'$ & Y' linked in E$\}$. Then $(X\cup Y,\mathcal{B}_E)$ is a matroid and $(X,Y,\mathcal{B}_E)$ is a based matroid (see PERFECT [9], INGLETON & PIFF [5], MASON [7]). Matroids M constructed in this way are called *gammoids*; X is called a *gammoid-base* for M. In section 7 we show that every base of a *gammoid* is a gammoid-base. If M is a gammoid then also M^* is a gammoid.
Of course every deltoid is a gammoid; in fact INGLETON & PIFF [5] proved that a matroid M is a gammoid iff M is a contraction of a restriction of a deltoid.

c. Matroids representable over a field.

If there is a linear space L over a field F, and a mapping $\phi : Z \to L$ and if $\mathcal{I} = \{Z'\subset Z \mid \phi|Z'$ is one-to-one and $\phi[Z']$ is linearly independent$\}$, then $M = (Z,\mathcal{I})$ is a matroid and M is called *representable over* F. If M is representable over GF(2) then we call M *binary*. The rank-function ρ of this matroid is $\rho(Z') = \mathrm{rank}(\phi[Z'])$, $(Z'\subset Z)$, where rank is used in its algebraic meaning.

Let (X,Y,ϕ) be a matrix over a field F and X and Y disjoint sets. Let $\mathcal{B}_\phi = \{(X\backslash X') \cup Y' \mid X' \,\&\, Y' \text{ linked in } \phi\}$. Then $(X\cup Y,\mathcal{B}_\phi)$ is a matroid and $(X,Y,\mathcal{B}_\phi)$ is a based matroid. Matroids constructed in this way are exactly the matroids representable over F. Then, of course, M is representable over F iff M^* is representable over F, and $(\mathcal{B}_\phi)^* = \mathcal{B}_{\phi^*}$. If M is representable over F, then every restriction or contraction of M is representable over F. Also we have:

PROPOSITION 1.7.

(i) (PIFF & WELSH) *Let* M_1 *and* M_2 *be matroids. Then there exists a natural number* N *such that: if* M_1 *and* M_2 *are representable over a field* F *and* $|F| \geq N$, *then* $M_1 \vee M_2$ *is representable over* F.

(ii) (INGLETON & PIFF) *Let* M *be a gammoid. Then there exists a natural number* N *such that: if* F *is a field with* $|F| \geq N$ *then* M *is representable over* F.

PROOF. (i) See PIFF & WELSH [12]

(ii) See INGLETON & PIFF [5]. □

2. LINKING SYSTEMS

In this section we give two definitions of a "linking system" and show the correspondence between these two definitions. Also we give some examples.

In this and in the sequel we need the following notation. Let X and Y be sets and $\Lambda \subset \mathcal{P}(X) \times \mathcal{P}(y)$. Then denote:

$X' \overset{\Lambda}{\longleftrightarrow} Y'$ iff $(X',Y') \in \Lambda$;

$X' \overset{\Lambda}{\longrightarrow} Y'$ iff $X' \overset{\Lambda}{\longleftrightarrow} Y''$ for some $Y'' \subset Y'$;

$X' \overset{\Lambda}{\longleftarrow} Y'$ iff $X'' \overset{\Lambda}{\longleftrightarrow} Y'$ for some $X'' \subset X'$;

$X'' \overset{\Lambda}{\longleftrightarrow} Y''$ (maximal in (X',Y')) iff

(1) $X'' \overset{\Lambda}{\longleftrightarrow} Y''$ and $X'' \subset X'$, $Y'' \subset Y'$, and

(2) (X'',Y'') is maximal with property (1).

First we define a linking system by means of "linked pairs".

<u>DEFINITION</u>. A *linking system* or *l.s.* is a triple (X,Y,Λ) such that

(i) X and Y are sets and $\emptyset \neq \Lambda \subset P(X) \times P(Y)$; Λ is called the set of *linked pairs*;

(ii) if $X' \overset{\Lambda}{\longleftrightarrow} Y'$ then $|X'| = |Y'|$;

(iii) if $X' \overset{\Lambda}{\longleftrightarrow} Y'$ and $X'' \subset X'$ and $Y'' \subset Y'$, then $X'' \overset{\Lambda}{\longrightarrow} Y'$ and $X' \overset{\Lambda}{\longleftarrow} Y''$;

(iv) if $X'' \overset{\Lambda}{\longleftrightarrow} Y''$ (maximal in (X',Y')) and $X''' \overset{\Lambda}{\longleftrightarrow} Y'''$ (maximal in (X',Y')) then $X'' \overset{\Lambda}{\longleftrightarrow} Y'''$.

<u>DEFINITION</u>. Let (X,Y,Λ) be a l.s. and $X' \subset X$, $Y' \subset Y$. Then X' & Y' are called *linked in* Λ if $X' \overset{\Lambda}{\longleftrightarrow} Y'$.

Our alternative definition uses the notion of a "linking function".

<u>DEFINITION</u>. A *linking system* or *l.s.* is a triple (X,Y,λ) such that:

(i) X and Y are sets and $\lambda : P(X) \times P(y) \to \mathbb{Z}$; λ is called the *linking function*;

(ii) $0 \leq \lambda(X',Y') \leq \min\{|X'|, |Y'|\}$, $\quad (X' \subset X; Y' \subset Y)$;

(iii) if $X' \subset X''$ and $Y' \subset Y''$ then $\lambda(X',Y') \leq \lambda(X'',Y'')$, $\quad (X',X'' \subset X; Y',Y'' \subset Y)$;

(iv) $\lambda(X' \cap X'', Y' \cup Y'') + \lambda(X' \cup X'', Y' \cap Y'') \leq \lambda(X',Y') + \lambda(X'',Y'')$, $\quad (X',X'' \subset X; Y',Y'' \subset Y)$.

If (X,Y,λ) resp. (X,Y,Λ) is an l.s. and $X' \subset X$, $Y' \subset Y$, then of course (X',Y',λ') resp. (X',Y',Λ') is again an l.s., where $\lambda' = \lambda \mid P(X) \times P(Y)$, resp. $\Lambda' = \Lambda \cap P(X) \times P(Y)$. (X',Y',λ'), resp. (X',Y',Λ') is called a *sub linking system* of (X,Y,λ), resp. (X,Y,Λ).

The relation between both definitions is:

(i) given an l.s. (X,Y,λ) with linking function λ, define as set of linked pairs: $\Lambda = (X',Y') \mid X' \subset X, Y' \subset Y, \lambda(X',Y') = |X'| = |Y'|\}$;

(ii) given an l.s. (X,Y,Λ) with set of linked pairs Λ, define as linking functions: $\lambda(X',Y') = \max\{|X''| \mid X'' \subset X', Y'' \subset Y', (X'',Y'') \in \Lambda\}$.

The fact that this is indeed a one-to-one correspondence between the two representations, is a consequence of the following

<u>PROPOSITION 2.1</u>.

(a) *Let* (X,Y,λ) *be an l.s. with linking function* λ.

(i) *If* $\lambda(X',Y') = |X'| = |Y'|$ *and* $X'' \subset X'$, *then there exists a* $Y'' \subset Y'$ *such that* $\lambda(X'',Y'') = |X''| = |Y''|$.

(ii) *If* $\lambda(X',Y') = |X'| = |Y'|$ *and* $Y'' \subset Y'$, *then there exists an* $X'' \subset X'$ *such that* $\lambda(X'',Y'') = |X''| = |Y''|$.

(iii) *If* $\lambda(X',Y') = |X'|$, *then there is a* $Y'' \subset Y'$ *with* $\lambda(X',Y'') = |X'| = |Y''|$.

(iv) *If* $\lambda(X',Y') = |Y'|$, *then there is an* $X'' \subset X'$ *with* $\lambda(X'',Y') = |X''| = |Y'|$.

(v) *If* (1) $\lambda(X',Y') = |X'| = |Y'|$ *and* (2) (X',Y') *maximal with property* (1), *and* (3) $\lambda(X'',Y'') = |X''| = |Y''|$ *and* (4) (X'',Y'') *maximal with property* (3), *then* $\lambda(X',Y'') = |X'| = |Y''|$.

(vi) $\lambda(X',Y') = \max\{|X''| \mid X'' \subset X',\ Y'' \subset Y',\ \lambda(X'',Y'') = |X''| = |Y''|\}$.

(b) *Let* (X,Y,Λ) *be an l.s. with set of linked pairs* Λ.

Let for each $X' \subset X$, $Y' \subset Y$:

$$\lambda(X',Y') = \max\{|X''| \mid X'' \subset X',\ Y'' \subset Y',\ X'' \overset{\Lambda}{\longleftrightarrow} Y''\}.$$

Then for each $X', X'' \subset X$, $Y', Y'' \subset Y$:

$$\lambda(X'\cap X'',Y'\cup Y'') + \lambda(X'\cup X'',Y'\cap Y'') \le \lambda(X',Y') + \lambda(X'',Y'').$$

PROOF.

(a) (i) We may suppose that $|X'\backslash X''| = 1$ and $X'' = X'\ \{x\}$. We have to prove the existence of a $y \in Y'$ with $\lambda(X'\backslash\{x\}, Y'\backslash\{y\}) = |X'| - 1$. Assume therefore that for each $y \in Y'$: $\lambda(X'\backslash\{x\}, Y'\backslash\{y\}) \le |X'| - 2$.
Then we can prove by induction that for each $y_1,\ldots,y_k \in Y'$ $\lambda(X'\backslash\{x\}, Y'\backslash\{y_1,\ldots,y_k\}) \le |X'| -k-1$. For $k = 1$, this is our assumption. If $k > 1$ and $\lambda(X'\backslash\{x\}, Y'\backslash\{y_1,\ldots,y_{k-1}\}) \le |X'|-(k-1)-1$, then $\lambda(X'\backslash\{x\}, Y'\backslash\{y_1,\ldots,y_k\}) \le \lambda(X'\backslash\{x\},Y'\backslash\{y_1,\ldots,y_{k-1}\}) +$
$\lambda(X'\backslash\{x\},Y'\backslash\{y_k\}) - \lambda(X'\backslash\{x\},Y') \le$
$|X'| - (k-1) - 1 + |X'| -2 - \lambda(X',Y') - \lambda(\emptyset,Y') + \lambda(\{x\},Y') \le$
$|X'| - k - 1$.
But now we have $\lambda(X'\backslash\{x\},Y'\backslash Y') \le |X'| - |Y'| -1 = -1$.
This contradicts the definition of a linking function.

(a) (ii) Interchange the roles of X' and Y' in (a) (i).

(a) (iii) Let $|Y'| > |X'|$. It is sufficient to prove that there is a $y \in Y'$ with $\lambda(X',Y'\backslash\{y\}) = |X'|$. Suppose that for each $y \in Y'$ $\lambda(X',Y'\backslash\{y\}) \le |X'| -1$. Then it is easy to prove by induction that for each $y_1,\ldots,y_k \in Y'$: $\lambda(X',Y'\backslash\{y_1,\ldots,y_k\}) \le |X'| -k$.

But then $\lambda(X',Y'\backslash Y') \leq |X'| - |Y'| < 0$. This contradicts the definition of a linking function.

(a) (iv) Interchange the roles of X' and Y' in (a)(iii).

(a) (v) First we prove that $|X'| = |X''|$ $(= |Y'| = |Y''|)$.
Suppose $|X'| = |Y'| < |X''| = |Y''|$. Since (X',Y') is maximal under property (1) we have: for each $x \in X''$, $y \in Y''$:
$\lambda(X' \cup \{x\}, Y' \cup \{y\}) = |X'| = |Y'|$. Then, by induction, we have for each $x \in X'$, $y_1,\ldots,y_k \in Y''$: $\lambda(X' \cup \{x\}, Y' \cup \{y_1,\ldots,y_K\}) = |X'| = |Y'|$.
For k = 1, this is known.

If $k > 1$ and $\lambda(X' \cup \{x\}, Y' \cup \{y_1,\ldots,y_{k-1}\}) = |X'| = |Y'|$, then
$\lambda(X' \cup \{x\}, Y' \cup \{y_1,\ldots,y_k\}) \leq \lambda(X' \cup \{x\}, Y' \cup \{y_1,\ldots,y_{k-1}\}) +$
$\lambda(X' \cup \{x\}, Y' \cup \{y_k\}) - \lambda(X' \cup \{x\}, Y') = |X'| + |X'| - |X'| = |X'|$.
So we know: for each $x \in X''$: $\lambda(X' \cup \{x\}, Y' \cup Y'') = |X'| = |Y'|$.
By a similar argument: $\lambda(X' \cup X'', Y' \cup Y'') = |X'| = |Y'|$.
But $\lambda(X' \cup X'', Y' \cup Y'') \geq \lambda(X'',Y'') = |X''| = |Y''|$, and therefore $|X'| \geq |X''|$. This contradicts our assumption.

Second we prove that $\lambda(X',Y'') = |X'|$. Let $C \subset Y$, such that (1) $\lambda(X',C) = |X'| = |C|$ and (2) $|C \cap Y''|$ maximal with property (1).
We have to prove that $C = Y''$. Suppose, therefore, that $y \in Y'' \setminus C$.
Then: $\lambda(X',(C \cap Y'') \cup \{y\}) \geq \lambda(X', C \cup \{y\}) + \lambda(X' \cup X'', (C \cap Y'') \cup \{y\}) -$
$- \lambda(X' \cup X'', C \cup \{y\}) \geq |X'| + |C \cap Y''| + 1 - |X'| = |C \cap Y''| + 1$.
Then also by (a) (iv) there exists some $D \subset X'$ with
$\lambda(D,(C \cap Y'') \cup \{y\}) = |C \cap Y''| + 1$ and some maximal pair (D',C') with
$\lambda(D',C') = |D'| = |C'|$, $D \subset D' \subset X'$ and $(C \cap Y'') \cup \{y\} \subset C' \subset C \cup \{y\}$.
But then, by the first part of this proof (a)(v), $D' = X'$. Also:
$|C' \cap Y''| > |C \cap Y''|$. This contradicts the maximality of $|C \cap Y''|$ and thus $C = Y''$.

(a) (vi) Of course $\lambda(X',Y') \geq \max\{|X''| \mid X'' \subset X', Y'' \subset Y', \lambda(X'',Y'') = |X''| = |Y''|\}$.
It is sufficient to prove that if $\lambda(X',Y') < |X'|$, then there exists an $x \in X'$ with $\lambda(X' \backslash \{x\}, Y') = \lambda(X',Y')$. For then there exists an $X'' \subset X'$ with $\lambda(X'',Y') = |X''| = \lambda(X',Y')$, and then by (a)(iii) there exists a $Y'' \subset Y'$ with $\lambda(X'',Y'') = |X''| = |Y''| = \lambda(X',Y')$.

Suppose that for each $x \in X'$: $\lambda(X'\setminus\{x\},Y') < \lambda(X',Y')$. Then, by induction, for each $x_1,\ldots,x_k \in X'$: $\lambda(X'\setminus\{x_1,\ldots,x_k\},Y') \leq \lambda(X',Y') - k$ and then: $\lambda(X'\setminus X',Y') \leq \lambda(X',Y') - |X'| < 0$. This is not true.

(b) Let X_1, Y_1, X_2, Y_2 be such that: $X_1 \overset{\Lambda}{\longleftrightarrow} Y_1$ (maximal in $(X'\cap X'',Y'\cup Y'')$) and $X_2 \overset{\Lambda}{\longleftrightarrow} Y_2$ (maximal in $(X'\cup X'',Y'\cap Y'')$).

Then, of course, $|X_1| = |Y_1| = \lambda(X'\cap X'',Y'\cup Y'')$ and $|X_2| = |Y_2| = \lambda(X'\cup X'',Y'\cap Y'')$.

Thus there exist X_3, Y_3, X_4, Y_4 with: $X_1 \subset X_3 \subset X' \cup X''$, $Y_1 \subset Y_3 \subset Y' \cup Y''$, $X_3 \overset{\Lambda}{\longleftrightarrow} Y_3$ (maximal in $(X'\cup X'',Y'\cup Y'')$), $X_2 \subset X_4 \subset X' \cup X''$, $Y_2 \subset Y_4 \subset Y' \cup Y''$ and $X_4 \overset{\Lambda}{\longleftrightarrow} Y_4$ (maximal in $(X'\cup X'',Y'\cup Y'')$).

Therefore, by condition (iv) of the definition of a set of linked pairs, $X_3 \overset{\Lambda}{\longleftrightarrow} Y_4$ and $|X_3| = |Y_4|$.

Then, by condition (iii):

$X_3 \cap X' \overset{\Lambda}{\longrightarrow} Y_4$,

say $X_3 \cap X' \overset{\Lambda}{\longleftrightarrow} Y_5 \subset Y_4$,

also: $X_3 \cap X' \overset{\Lambda}{\longleftarrow} Y_5 \cap Y'$,

say $X_3 \cap X' \supset X_5 \overset{\Lambda}{\longleftrightarrow} Y_5 \cap Y'$

i.e. $X_5 \overset{\Lambda}{\longrightarrow} Y'$ with $|X_5| = |Y_5\cap Y'| = |Y_5| - |Y_5\setminus Y'| = |X_3\cap X'| - |Y_5\setminus Y'| \leq |X_3\cap X'| - |Y_4\setminus Y'|$.

Similarly, there exists an $X_6 \subset X''$ with $X_6 \overset{\Lambda}{\longrightarrow} Y''$ and $|X_6| \geq |X_3\cap X''| - |Y_4\setminus Y''|$.

Now we have:

$\lambda(X',Y') + \lambda(X'',Y'') \geq |X_5| + |X_6| \geq |X_3\cap X'| - |Y_4\setminus Y'| + |X_3\cap X''| - |Y_4\setminus Y''| = |X_3| + |X_3\cap X'\cap X''| - |Y_4| + |Y_4\cap Y'\cap Y''| \geq |X_1| + |Y_2| = \lambda(X'\cap X'',Y'\cup Y'') + \lambda(X'\cup X'',Y'\cap Y'')$. □

Note: if a linking system is given as (X,Y,λ) without further specification it is understood that λ is its linking function. Likewise the symbol Λ will be reserved for the set of linked pairs. Also, given an l.s. with linking functions λ (without or with a subscript) we take silently for granted that the corresponding set of linked pairs is Λ (without or with the same subscript), and vice versa. Furthermore, in subscripts we use the symbols Λ and λ promiscuously, e.g. X_Λ will mean the same as X_λ.

PROPOSITION 2.2. *Let* (X,Y,λ) *be an l.s. Then:*

(i) $X' \overset{\Lambda}{\longleftarrow} Y'$ *iff* $\lambda(X',Y') = |Y'|$;

(ii) $X' \overset{\Lambda}{\longrightarrow} Y'$ *iff* $\lambda(X',Y') = |X'|$;

(iii) $X' \overset{\Lambda}{\longleftrightarrow} Y'$ *iff* $X' \overset{\Lambda}{\longrightarrow} Y'$ *and* $X' \overset{\Lambda}{\longleftarrow} Y'$;

(iv) *if* $X' \overset{\Lambda}{\longrightarrow} Y'$ *and* $X'' \subset X'$ *then* $X'' \overset{\Lambda}{\longrightarrow} Y'$;

(v) *if* $X' \overset{\Lambda}{\longleftarrow} Y'$ *and* $Y'' \subset Y'$ *then* $X' \overset{\Lambda}{\longleftarrow} Y''$;

(vi) *if* $X'' \overset{\Lambda}{\longleftrightarrow} Y''$ (maximal in (X',Y')) *and* $X''' \overset{\Lambda}{\longleftrightarrow} Y'''$ (maximal in (X',Y')), *then* $|X''| = |Y'''|$.

PROOF. Obvious. □

If (X,Y,λ) is an l.s. then the *dual linking system* is the l.s. (Y,X,λ^*) with for each $X' \subset X$, $Y' \subset Y$: $\lambda^*(Y',X') = \lambda(X',Y')$. Then of course $\Lambda^* = \{(Y',X') \mid X' \overset{\Lambda}{\longleftrightarrow} Y'\}$.

EXAMPLES OF LINKING SYSTEMS

a. Deltoid linking systems.

Let (X,Y,E) be a b.g. Then the set $\Delta_E = \{(X',Y') \mid X' \text{ \& } Y' \text{ linked in } E\}$ forms the set of linked pairs of an l.s. (X,Y,Δ_E) (see proposition 1.3). Its linking function δ_E is then, of course, given by: $\delta_E(X',Y') = \max\{|X''| \mid X'' \subset X', Y'' \subset Y', X'' \text{ \& } Y'' \text{ linked in } E\}$, and by KÖNIG's theorem 1.2 this equals: $\min\{|X''| + |Y''| \mid X'' \subset X', Y'' \subset Y', (X'',Y'')$ is a covering of $(X',Y',E')\}$. Linking systems constructed in this way are called *deltoid linking systems*. It can be seen easily that $(\Delta_E)^* = \Delta_{E^*}$, so the dual of a deltoid linking system is again a deltoid linking system.

b. Gammoid linking systems.

Let (Z,E) be a digraph, and $X, Y \subset Z$. Then the set $\Gamma_E = \{(X',Y') \mid X' \text{ \& } Y' \text{ linked in } E\}$ forms the set of linked pairs of an l.s. (X,Y,Γ_E) (see proposition 1.5). Its linking function γ_E is then, of course, given by: $\gamma_E(X',Y') = \max\{|X''| \mid X'' \subset X', Y'' \subset Y', X'' \text{ \& } Y'' \text{ linked in } E\}$, and by MENGER's theorem 1.4. this equals: $\min\{|Z'| \mid Z' \subset Z, Z'$ is (X',Y')-disconnecting$\}$. Linking systems constructed in this way are called *gammoid linking systems*. Of course, every deltoid l.s. is a gammoid l.s. and the dual of a gammoid l.s. is itself a gammoid l.s.

c. Linking systems representable over a field.

Let (X,Y,ϕ) be a matrix over a field F. Then the set $\Lambda_\phi = \{(X',Y') \mid X' \,\&\, Y' \text{ linked in } \phi\}$ forms the set of linked pairs of an l.s. (X,Y,Λ_ϕ) (see proposition 1.1) and its linking function λ_ϕ is then given by: $\lambda_\phi(X',Y')$ = the rank of the submatrix (X',Y',ϕ') of (X,Y,ϕ). Linking systems constructed in this way are called *representable over* F; an l.s. representable over GF(2) is called *binary*.
Again, $(\Lambda_\phi)^* = \Lambda_{\phi^*}$ and the dual of an l.s. representable over F is itself representable over F.

3. MATROIDS AND LINKING SYSTEMS

There exist close relations between the notions of matroids and linking systems and in this section we give some of these relations. First we notice

PROPOSITION 3.1. *Let* (X,Y,Λ) *be an l.s. and* $Y' \subset Y$. *Let* I *be the set of all* $X' \subset X$ *with* $X' \xrightarrow{\Lambda} Y'$. *Then* I *forms the collection of independent subsets of a matroid* (X,I).

PROOF. Straightforward. □

If the l.s. of proposition 3.1 is a deltoid l.s. then the matroid formed in proposition 3.1 is a so-called *transversal matroid*; if the l.s. is a gammoid l.s. then it can be proved that the matroid is a gammoid and if the l.s. is representable over a field F, then also the matroid is representable over F.
Second we give a one-to-one correspondence between based matroid (see section 1) and linking systems.

Let X and Y be disjoint sets. The one-to-one relations will be between based matroids (X,Y,B), resp. (X,Y,I), resp. (X,Y,ρ) and linking systems (X,Y,Λ), resp. (X,Y,λ). The correspondence will be denoted by subscripts, i.e. with B, I, ρ will correspond Λ_B, Λ_I or Λ_ρ, resp. λ_B, λ_I or λ_ρ, and with λ, resp. Λ will correspond B_λ or B_Λ, resp. I_λ or I_Λ, resp. ρ_Λ or ρ_λ.

Let $(X,Y,\mathcal{B})$ be a based matroid, with rank-function ρ.
Define: $\Lambda_{\mathcal{B}} = \{(X',Y') \in \mathcal{P}(X) \times \mathcal{P}(Y) \mid (X\backslash X') \cup Y' \in \mathcal{B}\}$
and $\lambda_{\mathcal{B}}(X',Y') = \rho((X\backslash X')\cup Y') - |X\backslash X'| \quad (X'\subset X, Y'\subset Y)$.

PROPOSITION 3.2. $(X,Y,\Lambda_{\mathcal{B}})$ *is a linking system, with linking function* $\lambda_{\mathcal{B}}$.

PROOF. First we prove that $(X,Y,\lambda_{\mathcal{B}})$ is an l.s. and second that $\Lambda_{\mathcal{B}}$ is the corresponding set of linked pairs.
(1) Conditions (i) and (iv) of the definition of a linking function can be checked straightforwardly.
The proof of condition (ii) runs as follows.
Since we know that $X \setminus X'$ is an independent set of the matroid, it is true that $|X\backslash X'| \le \rho((X\backslash X')\cup Y')$, i.e. $0 \le \lambda_{\mathcal{B}}(X',Y')$.
Further, since X is a base, $\rho((X\backslash X')\cup Y') \le |X|$, and so
$\rho((X\backslash X')\cup Y') - |X\backslash X'| \le |X'|$, i.e. $\lambda_{\mathcal{B}}(X',Y') \le |X'|$.
Also, of course, $\rho((X\backslash X')\cup Y') \le |(X\backslash X')\cup Y'| = |X\backslash X'| + |Y'|$, i.e.
$\lambda_{\mathcal{B}}(X',Y') \le |Y'|$.
Finally we prove condition (iii).
If $X' \subset X'' \subset X$ and $Y' \subset Y'' \subset Y$, then

$$\lambda_{\mathcal{B}}(X',Y') = \rho((X\backslash X')\cup Y') - |X\backslash X'| \le \rho((X\backslash X'')\cup Y') + \rho(X''\backslash X') - |X\backslash X'|$$
$$\le \rho((X\backslash X'')\cup Y'') + |X''\backslash X'| - |X\backslash X'| = \lambda_{\mathcal{B}}(X'',Y'').$$

(2) Second we have to prove: $\lambda_{\mathcal{B}}(X',Y') = |X'| = |Y'|$ iff $(X',Y') \in \Lambda_{\mathcal{B}}$; $(X'\subset X, Y'\subset Y)$.
This goes straightforwardly: $\lambda_{\mathcal{B}}(X',Y') = |X'| = |Y'|$ iff
$\rho((X\backslash X')\cup Y') - |X\backslash X'| = |X'| = |Y'|$ iff $\rho((X\backslash X')\cup Y') = |X| = |X\backslash X'| + |Y'|$
iff $(X\backslash X') \cup Y'$ is a base iff $(X',Y') \in \Lambda_{\mathcal{B}}$. □

Let (X,Y,Λ) be a l.s. (with disjoint X and Y), with linking function λ.
Define: $\mathcal{B}_\Lambda = \{(X\backslash X')\cup Y' \mid X' \overset{\Lambda}{\longleftrightarrow} Y'\}$,
$\mathcal{I}_\Lambda = \{(X\backslash X')\cup Y' \mid X' \overset{\Lambda}{\longleftarrow} Y'\}$,
and $\rho_\Lambda(X'\cup Y') = \lambda(X\backslash X',Y') + |X'| \qquad (X'\subset X, Y'\subset Y)$.

PROPOSITION 3.3. $(X,Y,\mathcal{B}_\Lambda)$ *is a based matroid, with set of independent sets* $\mathcal{I}_\Lambda$ *and rank-function* ρ_Λ.

PROOF. First we prove that (X,Y,ρ_Λ) is a based matroid, secondly that $\mathcal{B}_\Lambda$ is the corresponding bases-collection and thirdly that $\mathcal{I}_\Lambda$ is the corresponding collection of independent sets.

(1) Conditions (i), (ii) and (iv) can be proved straightforwardly.

The proof of condition (iii) runs as follows.

If $X' \subset X'' \subset X$, $Y' \subset Y'' \subset Y$, then:

$\rho_\Lambda(X' \cup Y') = \lambda(X \backslash X', Y') + |X'| \le \lambda(X \backslash X'', Y') + \lambda(X'' \backslash X', Y') + |X'| \le$
$\lambda(X \backslash X'', Y'') + |X'' \backslash X'| + |X'| = \rho_\Lambda(X'', Y'')$.

Also, since $\rho_\Lambda(X \cup \emptyset) = \lambda(\emptyset, \emptyset) + |X| = \rho_\Lambda(X \cup Y)$, it is true that X is a base.

(2) Second we have to prove:

$X' \cup Y' \in \mathcal{B}_\Lambda$ iff $\rho_\Lambda(X' \cup Y') = |X' \cup Y'| = |X|$ $\qquad (X' \subset X, Y' \subset Y)$.

This follows from:

$\rho_\Lambda(X' \cup Y') = |X' \cup Y'| = |X|$ iff $\lambda(X \backslash X', Y') + |X'| = |X' \cup Y'| = |X|$ iff
$\lambda(X \backslash X', Y') = |X \backslash X'| = |Y'|$ iff $(X \backslash X') \overset{\Lambda}{\longleftrightarrow} Y'$ iff $X' \cup Y' \in \mathcal{B}_\Lambda$.

(3) Finally, let $X' \subset X$, $Y' \subset Y$.

Then $X' \cup Y' \in \mathcal{I}_\Lambda$ iff $(X \backslash X') \overset{\Lambda}{\longleftarrow} Y'$ iff $(X \backslash X'') \overset{\Lambda}{\longleftrightarrow} Y''$ for some $X'' \supset X', Y'' \supset Y'$ iff $X'' \cup Y'' \in \mathcal{B}_\Lambda$ for some $X'' \supset X'$, $Y'' \supset Y'$. □

PROPOSITION 3.4.

(i) *If* (X,Y,Λ) *is a linking system, then* $\Lambda = \Lambda_{\mathcal{B}_\Lambda}$ *and* $(\mathcal{B}_\Lambda)^* = \mathcal{B}_{\Lambda^*}$.

(ii) *If* $(X,Y,\mathcal{B})$ *is a based matroid, then* $\mathcal{B} = \mathcal{B}_{\Lambda_\mathcal{B}}$ *and* $(\Lambda_\mathcal{B})^* = \Lambda_{\mathcal{B}^*}$.

PROOF. Obvious. □

PROPOSITION 3.5. *Let* (X,Y,Λ) *and* $(X,Y,\mathcal{B})$ *be a corresponding pair of an and a based matroid. Let* F *be a field. Then:*

(i) (X,Y,Λ) *is a deltoid l.s. iff* $(X,Y,\mathcal{B})$ *is a deltoid-based deltoid.*

(ii) (X,Y,Λ) *is a gammoid l.s. iff* $(X,Y,\mathcal{B})$ *is a based gammoid.*

(iii) (X,Y,Λ) *is representable over* F *iff* $(X,Y,\mathcal{B})$ *is representable over* F.

PROOF. see the examples of section 1 and 2. □

PROPOSITION 3.6. *Let* (X,Y,Λ) *be a gammoid l.s. Then there exists a natural number* N *such that* (X,Y,Λ) *is representable over every field* F *with* $|F| \ge N$.

PROOF. Since, by proposition 3.5, $(X,Y,\mathcal{B}_\Lambda)$ is a based gammoid, there exists, by proposition 1.7 (ii), a natural number N such that $(X,Y,\mathcal{B}_\Lambda)$ is representable over every field F with $|F| \geq N$, but then, again by proposition 3.5, also (X,Y,Λ) is representable over every field F with $|F| \geq N$. □

4. LINKING OF MATROIDS AND LINKING SYSTEMS

In this section we show how a matroid $(X,\mathcal{I})$ and an l.s. (X,Y,Λ) can be linked and form a new matroid $(Y,\mathcal{I}*\Lambda)$, and how two linking systems (X,Y,Λ_1) and (Y,Z,Λ_2) can be linked and form a new l.s. $(X,Z,\Lambda_1*\Lambda_2)$.

THEOREM 4.1. *Let* $(X,\mathcal{I})$ *be a matroid and* (X,Y,Λ) *an l.s.*
Set $\mathcal{I} * \Lambda = \{Y' \subset Y \mid \text{there is an } X' \in \mathcal{I} \text{ such that } X' \overset{\Lambda}{\longleftrightarrow} Y'\}$. *Then* $(Y,\mathcal{I}*\Lambda)$ *is a matroid with collection of independent sets* $\mathcal{I} * \Lambda$ *and with rank-function given by:*

$$\rho * \lambda(Y') = \min_{X'\subset X}[\lambda(X',Y') + \rho(X\backslash X')] \qquad (X'\subset X,\ Y'\subset Y).$$

PROOF. We may suppose that X and Y are disjoint (otherwise take disjoint copies of X and Y). We know that for each $X' \subset X$, $Y' \subset Y$:
$X' \cup Y' \in \mathcal{I}_\Lambda$ iff $(X\backslash X') \overset{\Lambda}{\longleftarrow} Y'$.
Now, form the matroid $M = (X\cup Y,\mathcal{I}\vee\mathcal{I}_\Lambda)$ (see section 1.c), and take $Y' \subset Y$.
Then, of course, $Y' \in \mathcal{I} * \Lambda$ iff $X \cup Y'$ is independent in $\mathcal{I} \vee \mathcal{I}_\Lambda$, i.e., since X is independent in $(X\cup Y,\mathcal{I}\vee\mathcal{I}_\Lambda)$, iff Y' is independent in $(\mathcal{I}\vee\mathcal{I}_\Lambda) \cdot Y$;
thus $\mathcal{I} * \Lambda = (\mathcal{I}\vee\mathcal{I}_\Lambda) \cdot Y$.
The rank-function $\rho * \lambda$ is then, of course, $(\rho\vee\rho_\lambda) \cdot Y$,
so for each $Y' \subset Y$ we have:

$$(\rho*\lambda)(Y') = ((\rho\vee\rho_\lambda)\cdot Y)(Y') = (\rho\vee\rho_\lambda)(X\cup Y') - (\rho\vee\rho_\lambda)(X) =$$

$$\min_{\substack{X'\subset X\\ Y''\subset Y'}} [\rho(X') + \rho_\lambda(X'\cup Y'') + |X\backslash X'| + |Y'\backslash Y''|] - |X| =$$

$$\min_{\substack{X'\subset X\\ Y''\subset Y'}} [\rho(X') + \lambda(X\backslash X',Y'') + |X'| + |X\backslash X'| + |Y'\backslash Y''| - |X|] =$$

$$\min_{\substack{X'\subset X\\ Y''\subset Y'}} [\rho(X') + \lambda(X\backslash X',Y'') + |Y'\backslash Y''|] = \min_{X'\subset X}[\rho(X') + \lambda(X\backslash X',Y')] =$$

$$\min_{X'\subset X}[\lambda(X',Y') + \rho(X\backslash X')].$$

<u>DEFINITION</u>. $I * \Lambda$ is called the *product* of I and Λ.

<u>COROLLARY 4.2</u>. (PERFECT [10]). *Let* (X,Y,E) *be a b.g. and* (X,I) *a matroid. Then the set* J = {Y' ⊂ Y | Y' linked in E with some independent subset of X} *forms the collection of independent subsets of a matroid* (Y,J).

<u>PROOF</u>. Straightforward from theorem 4.1. □

<u>COROLLARY 4.3</u>. (BRUALDI [1]). *Let* (Z,E) *be a digraph and* X, Y ⊂ Z. *Let further* (X,I) *be a matroid. Then the set* J = {Y' ⊂ Y | Y' linked in E with some independent subset of X} *forms the collection of independent subsets of a matroid* (Y,J).

<u>PROOF</u>. Straightforward from theorem 4.1. □

<u>COROLLARY 4.4</u>. (see theorem 3.1.) *Let* (X,Y,Λ) *be an l.s. and* Y' ⊂ Y. *Let I be the set of all* X' ⊂ X *with* $X' \xrightarrow{\Lambda} Y'$. *Then I forms the collection of independent subsets of a matroid* (X,I).

<u>PROOF</u>. Set $J = P(Y')$. Then (Y,J) is a matroid and $I = J * \Lambda^*$. □

<u>THEOREM 4.5</u>. *Let* (X,Y,Λ) *be an l.s. and* (X,I) *a matroid. Let* B *be a base of* (Y,$I*\Lambda$) *and* $A \in I$ *such that* $A \xleftrightarrow{\Lambda} B$. *Let* [A] *be the closure of* A *in the matroid*, ([A],Y,Λ') *the new l.s. with* $\Lambda' = \Lambda \cap (P([A]) \times P(Y))$ *and* ([A],I') *the new matroid with* $I' = I \cap P([A])$. *Then*:

(i) $Y' \in I * \Lambda$ *iff* $[A] \supset X' \xleftrightarrow{\Lambda} Y'$ *for some* $X' \in I$;

(ii) $I * \Lambda = I' * \Lambda'$.

<u>PROOF</u>. It is sufficient to prove (i), for (ii) is a corollary of (i). Let D be a base of (Y,$I*\Lambda$). We have to prove the existence of an independent subset C of [A] such that $C \xleftrightarrow{\Lambda} D$.

Suppose C ⊂ X is such that:

(1) C is independent in (X,I) and $C \xleftrightarrow{\Lambda} D$;

(2) $\rho(C \cup A)$ is minimal with property (1);

(3) $|C \cup A|$ is minimal with properties (1) and (2).

If $\rho(C \cup A) = |A|$ then $C \cup A \subset [A]$, which was to be proved.

Therefore suppose $\rho(C \cup A) > |A| = |B| = |C| = |D|$.

Then there exists an $x \in A \setminus C$ such that $C \cup \{x\}$ is independent.
Then: $\lambda((A\cap C)\cup\{x\},D) \geq \lambda(C\cup\{x\},D) + \lambda((A\cap C)\cup\{x\},B\cup D) - \lambda(C\cup\{x\},B\cup D) =$
$= |D| + |(A\cap C)\cup\{x\}| - |C| = |(A\cap C)\cup\{x\}|$.
Therefore: $(A\cap C) \cup \{x\} \xrightarrow{\Lambda} D$,

also: $C \xleftrightarrow{\Lambda} D$,

and so there exists a $C' \subset C \cup \{x\}$ such that $C' \xleftrightarrow{\Lambda} D$ and $(A\cap C) \cup \{x\} \subset C'$.
But then is C' independent, $\rho(C'\cup A) \leq \rho(C\cup A)$ and $|C'\cap A| > |C\cap A|$ i.e. $|C'\cup A| < |C\cup A|$, contradicting conditions (1), (2), (3) above. □

COROLLARY 4.6. (MASON [6]). *Let* (X,Y,E) *be a b.g. and* $(X,\mathcal{I})$ *a matroid. Let* $(Y,\mathcal{J})$ *be the product of* $\mathcal{I}$ *and* Δ_E *and* B *a base of this matroid, which is linked to an independent set* $A \subset X$. *Then* $(Y,\mathcal{J})$ *is induced by* $([A],Y,E\cap([A]\times Y))$ *and* $([A],\mathcal{I}\cap P([A]))$.

PROOF. Straightforward from theorem 4.5. □

THEOREM 4.7. *Let* (X,Y,Λ_1) *and* (Y,Z,Λ_2) *be linking systems. Define* $\Lambda_1 * \Lambda_2 = \{(X',Z') \mid X' \subset X,\ Z' \subset Z$ and for some $Y' \subset Y$: $X' \xleftrightarrow{\Lambda_1} Y'$ and $Y' \xleftrightarrow{\Lambda_2} Z'\}$, *(the product of* Λ_1 *and* Λ_2*). Then* $(X,Z,\Lambda_1*\Lambda_2)$ *is a linking system, with linking function given by* $(\lambda_1*\lambda_2)(X',Z') = \min_{Y'\subset Y}[\lambda_1(X',Y') + \lambda_2(Y\setminus Y',Z)]$.

PROOF. Without restrictions on the generality we may suppose that X, Y and Z are pairwise disjoint sets.
Now we have: $(X',Z') \in \Lambda_1 * \Lambda_2$ iff $(X\setminus X') \cup Y \cup Z'$ is a base of the matroid $(X\cup Y\cup Z,\mathcal{B}_{\Lambda_1}\vee\mathcal{B}_{\Lambda_2})$, i.e., since Y is independent in $(X\cup Y\cup Z,\mathcal{B}_{\Lambda_1}\vee\mathcal{B}_{\Lambda_2})$, iff $(X\setminus X') \cup Z'$ is a base of the contraction of $(X\cup Y\cup Z,\mathcal{B}_{\Lambda_1}\vee\mathcal{B}_{\Lambda_2})$ to $X \cup Z$. Since X is also a base of this last matroid, we have:

$$\Lambda_1 * \Lambda_2 = \Lambda_{((\mathcal{B}_{\Lambda_1}\vee\mathcal{B}_{\Lambda_2})\cdot(X\cup Z))}.$$

Therefore, $\Lambda_1 * \Lambda_2$ is an l.s. Its linking function $\lambda_1 * \lambda_2$ equals then, of course, $\lambda_{((\rho_{\lambda_1}\vee\rho_{\lambda_2})\cdot(X\cup Z))}$ and so, for each $X'\subset X$, $Z' \subset Z$:

$$(\lambda_1*\lambda_2)(X',Z') = \lambda_{((\rho_{\lambda_1}\vee\rho_{\lambda_2})\cdot(X\cup Z))}(X',Z') \overset{\text{prop.3.2.}}{=\!=\!=}$$

$((\rho_{\lambda_1} \vee \rho_{\lambda_2}) \cdot (X \cup Z))\ ((X\backslash X') \cup Z') - |X\backslash X'| \overset{\text{section 1.c.}}{=}$

$(\rho_{\lambda_1} \vee \rho_{\lambda_2})((X\backslash X') \cup Y \cup Z') - (\rho_{\lambda_1} \vee \rho_{\lambda_2})(Y) - |X\backslash X'| =$

$$\min_{\substack{X'' \subset X\backslash X' \\ Y'' \subset Y \\ Z'' \subset Z'}} [\rho_{\lambda_1}(X'' \cup Y'') + \rho_{\lambda_2}(Y'' \cup Z'') + |X\backslash (X' \cup X'')| + |Y\backslash Y''| + |Z'\backslash Z''| - |Y| - |X\backslash X'| \overset{\text{prop.3.3.}}{=}$$

$$\min_{\substack{X'' \subset X\backslash X' \\ Y'' \subset Y \\ Z'' \subset Z'}} [\lambda_1(X\backslash X'', Y'') + \lambda_2(Y\backslash Y'', Z'') + |Z'\backslash Z''|] =$$

$$\min_{\substack{X'' \subset X\backslash X' \\ Y'' \subset Y}} [\lambda_1(X\backslash X'', Y'') + \min_{Z'' \subset Z'} [\lambda_2(Y\backslash Y'', Z'') + |Z'\backslash Z''|]] =$$

$$\min_{\substack{X'' \subset X\backslash X' \\ Y'' \subset Y}} [\lambda_1(X\backslash X'', Y'') + \lambda_2(Y\backslash Y'', Z')] = \min_{Y'' \subset Y} [\lambda_1(X', Y'') + \lambda_2(Y\backslash Y'', Z')]. \square$$

PROPOSITION 4.8. *Let* (X,I) *be a matroid and* (X,Y,Λ_1), (Y,Z,Λ_2) *and* (Z,U,Λ_3) *linking systems. Then:*

(i) $(I*\Lambda_1) * \Lambda_2 = I * (\Lambda_1*\Lambda_2)$;

(ii) $(\Lambda_1*\Lambda_2) * \Lambda_3 = \Lambda_1 * (\Lambda_2*\Lambda_3)$.

PROOF. Obvious. □

Finally, we show how some properties of matroids and linking systems behave, when we link them with each other.

THEOREM 4.9.

(i) *Let* (X,I) *be a matroid and* (X,Y,Λ_1) *an l.s. Then there exists a natural number* N *such that: if* (X,I) *and* (X,Y,Λ_1) *are representable over a field* F *with* $|F| \geq N$, *then* $(Y,I*\Lambda_1)$ *is representable over* F.

(ii) *Let* (X,Y,Λ_1) *and* (Y,Z,Λ_2) *be linking systems. Then there exists a natural number* N *such that: if* (X,Y,Λ_1) *and* (Y,Z,Λ_2) *are representable over a field* F *with* $|F| \geq N$, *then* $(X,Z,\Lambda_1*\Lambda_2)$ *is representable over* F.

PROOF. By proposition 1.7.(i) and the proof of proposition 4.1. resp. 4.7., the theorem can be easily proved by constructing the corresponding based matroids $(X,Y,\mathcal{B}_{\Lambda_1})$ and $(Y,Z,\mathcal{B}_{\Lambda_2})$. □

REMARK. In general it is not valid that, if (X,Y,Λ_1) and (Y,Z,Λ_2) are representable over a field F, then also $(X,Z,\Lambda_1 * \Lambda_2)$ is representable over F. For instance, let $X = Z = \{a,b\}$ and $Y = \{c,d\}$ and (X,Y,ϕ) the matrix over GF(2) with $\phi(a,c) = \phi(a,d) = \phi(b,d) = 1$ and $\phi(b,c) = 0$.
Set $\Lambda_1 = \Lambda_\phi$ and $\Lambda_2 = (\Lambda_\phi)^*$. Then Λ_1 and Λ_2 are binary, but $\Lambda_1 * \Lambda_2$ is not binary.

THEOREM 4.10. *Let* (X,Y,ϕ_1) *and* (Y,Z,ϕ_2) *be matrices over the field* F. *Then* $\Lambda_{\phi_1} * \Lambda_{\phi_2} \supset \Lambda_{\phi_1\phi_2}$, *i.e.* $\lambda_{\phi_1} * \lambda_{\phi_2} * \lambda_{\phi_1\phi_2}$.

PROOF: We have to prove: if X' & Z' are linked in $\Lambda_{\phi_1\phi_2}$, then there is a $Y' \subset Y$, such that X' & Y' are linked in Λ_{ϕ_1} and Y' & Z' are linked in Λ_{ϕ_2}. In fact it is sufficient to prove: if $|X| = |Z|$ and (X,Y,ϕ_1) and (Y,Z,ϕ_2) are matrices over F and furthermore $(X,Z,\phi_1\phi_2)$ is a regular matrix, then there exists a $Y' \subset Y$, such that (X,Y',ϕ_1) and (Y',Z,ϕ_2) are regular matrices. The latter is left to the reader. □

REMARK. In general it is not valid that $\Lambda_{\phi_1} * \Lambda_{\phi_2} = \Lambda_{\phi_1\phi_2}$, as the preceding remark shows.

THEOREM 4.11. *Let* (X,Y,E_1) *and* (Y,Z,E_2) *be bipartite graphs. Then* $\Delta_{E_1} * \Delta_{E_2} \subset \Delta_{E_1E_2}$, *i.e.* $\delta_{E_1} * \delta_{E_2} \le \delta_{E_1E_2}$.

PROOF. If X' & Z' are linked in $\Delta_{E_1} * \Delta_{E_2}$, then there is some $Y' \subset Y$ with: X' & Y' linked in E_1 and Y' & Z' linked in E_2. But then X' & Z' linked in E_1E_2, i.e. in $\Delta_{E_1E_2}$. □

REMARK. In general it is not valid that the product of two deltoid linking systems, is again a deltoid linking system, as shows the following example: Let $|X| = |Z| = 2$ and $|Y| = 1$; further let $E_1 = X \times Y$ and $E_2 = Y \times Z$. Then $\Delta_{E_1} * \Delta_{E_2}$ is not a deltoid linking system.

THEOREM 4.12. *The product of two gammoid systems is again a gammoid linking system.*

PROOF. Left to the reader. □

5. LINKING SYSTEMS AND BIPARTITE GRAPHS

In this section we give a relation between linking systems and bipartite graphs. We have already defined for each b.g. (X,Y,E) a corresponding l.s. (X,Y,Δ_E) with: $(X',Y') \in \Delta_E$ iff X' & Y' linked in E.
We now define for each l.s. (X,Y,Λ) a b.g. $(X,Y,E\ )$ by:
$(x,y) \in E_\Lambda$ iff $(\{x\},\{y\}) \in \Lambda$, $(x \in X,\ y \in Y)$.
The mapping $E \to \Delta_E$ gives, of course, a one-to-one relation between bipartite graphs and deltoid linking systems, and also $E = E_{\Delta_E}$. It is evidently in general not true, that the mapping $\Lambda \to E_\Lambda$ is a one-to-one function.

Furthermore we can state:

PROPOSITION 5.1. *Let* (X,Y,Λ) *be an l.s. Then:* (X,Y,Λ) *is a deltoid linking system iff* $\Lambda = \Delta_{E_\Lambda}$.

PROOF. Obvious. □

Now we show that the deltoid linking system corresponding to a b.g. is, in an obvious way, the greatest l.s. corresponding to that b.g.

THEOREM 5.2. *Let* (X,Y,Λ) *be an l.s. Then* $\Lambda \subset \Delta_{E_\Lambda}$, *i.e.* $\lambda \le \delta_{E_\lambda}$.

PROOF. We have to prove: if $(X',Y') \in \Lambda$, then X' & Y' are linked in E_Λ.
Suppose therefore that X' & Y' are linked in Λ, but not in E_Λ. By König's theorem 1.2. there exist $X'' \subset X'$, $Y'' \subset Y'$ with $|X''| + |Y''| < |X'| = |Y'|$ and $X'' \cup Y''$ is a covering of $(X',Y',E_\Lambda \cap (X' \times Y'))$.
Then set $X''' = X' \setminus X''$ and $Y''' = Y' \setminus Y''$. Then for each $x \in X'''$ and $y \in Y'''$ $(x,y) \notin E_\Lambda$. Therefore $\lambda(X''',Y''') = 0$. But then:
$\lambda(X',Y') \le \lambda(X''',Y') + \lambda(X'',Y') \le \lambda(X''',Y''') + \lambda(X''',Y'') + \lambda(X'',Y') \le 0 +$
$+ |Y''| + |X''| < |X'|$.
Thus X' & Y' are not linked in Λ and this contradicts our assumption. □

<u>COROLLARY 5.3</u>. *Let* (X,Y,E) *be a b.g. Then* Δ_E *is the maximum (under inclusion) of all linking systems* Λ *with* $E_\Lambda = E$.

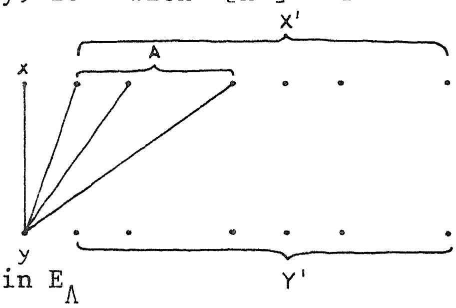
<u>PROOF</u>. Obvious. □

So we have proved:
if $X' \overset{\Lambda}{\longleftrightarrow} Y'$, then there is a linking of X' & Y' in E_Λ.

In the following theorem we prove:
if there is exactly one linking of X' & Y' in E_Λ, then $X' \overset{\Lambda}{\longleftrightarrow} Y'$.

<u>THEOREM 5.4</u>. *Let* (X,Y,Λ) *be an l.s. and* $X' \subset X$, $Y' \subset Y$, $|X'| = |Y'|$. *If there is precisely one linking in* E_Λ *between* X' *and* Y' *then* X' & Y' *are linked in* Λ.

<u>PROOF</u>. We prove this with induction to $|X'|$.
If $X' = \emptyset$ then X' & Y' are linked in Λ.
Suppose we have proved the statement for all $X' \subset X$, $Y' \subset Y$ with $|X'| = |Y'| \leq k$ and take $X' \subset X$, $Y' \subset Y$, $x \in X$, $y \in Y$ such that:
(i) $|X'| = |Y'| = k$, $x \in X \setminus X'$, $y \in Y \setminus Y'$,
(ii) the only linking between $X' \cup \{x\}$ and $Y' \cup \{y\}$ is ℓ with $\ell[X'] = Y'$ and $\ell(x) = y$.



Set $A = \{a \in X' \mid (a,y) \in E_\lambda\}$.
There exists precisely one linking ℓ' between X' and Y' (namely $\ell' = \ell \mid X'$), therefore (by induction) $\lambda(X',Y') = |X'| = |Y'|$.
Also $\forall a \in A$: $X' \setminus \{a\} \cup \{x\}$ & Y' are not linked in E_Λ (otherwise there would be another linking between $X' \cup \{x\}$ & $Y' \cup \{y\}$), and therefore, by theorem 5.2, $X' \setminus \{a\} \cup \{x\}$ & Y' are not linked in Λ, thus:
$\forall a \in A$: $\lambda(X'\setminus\{a\}\cup\{x\},Y') \leq |X'| - 1$.
Now, by another induction, we can prove $\forall a_1,\ldots,a_k$:
$\lambda(X'\setminus\{a_1,\ldots,a_k\}\cup\{x\},Y') \leq |X'| - k$. (If k =1 this is true, and if the proposition is proved for k, then

$$\lambda(X'\setminus\{a_1,\ldots,a_{k+1}\}\cup\{x\},Y') \leq \lambda(X'\setminus\{a_1,\ldots,a_k\}\cup\{x\},Y') + \lambda(X'\setminus\{a_{k+1}\}\cup\{x\},Y') - $$
$$-\lambda(X'\cup\{x\},Y') \leq |X'| - k + |X'| - 1 - |X'| = |X'| - (k+1).)$$

Then it follows: $\lambda(X'\setminus A\cup\{x\},Y') \leq |X'| - |A| = |X'\setminus A|$.

Therefore: $\lambda(X'\cup\{x\},Y'\cup\{y\}) \geq \lambda(X'\backslash A\cup\{x\},Y'\cup\{y\}) + \lambda(X'\cup\{x\},Y') - \lambda(X'\backslash A\cup\{x\},Y') \geq \lambda(X'\backslash A\cup\{x\},\{y\}) + \lambda(X'\backslash A,Y'\cup\{y\}) - \lambda(X'\backslash A,\{y\}) + |Y'| - |X'\backslash A| = 1 + |X'\backslash A| - 0 + |Y'| - |X'\backslash A| = |Y'| + 1 = |X'\cup\{x\}|$.

Thus $X'\cup\{x\}$ & $Y'\cup\{y\}$ are linked in Λ. □

There is a relation between the b.g. corresponding with an l.s. and the circuits (see section 1.c) of the corresponding based matroid.

PROPOSITION 5.5. *Let* $(X,Y,\mathcal{B})$ *be a based matroid and* $E = E_{\Lambda_{\mathcal{B}}}$.
Then:

(i) *for each* $y \in Y$ *the unique circuit of the matroid* $(X\cup Y,\mathcal{B})$ *contained in* $X \cup \{y\}$ *is the set* $\{y\} \cup \{x \in X \mid (x,y) \in E\}$;

(ii) *for each* $x \in X$ *the unique circuit of the matroid* $(X\cup Y,\mathcal{B}^*)$ *contained in* $Y \cup \{x\}$ *is the set* $\{x\} \cup \{y \in Y \mid (x,y) \in E\}$.

PROOF. Left to the reader. □

To conclude this section, we show that an l.s. (X,Y,Λ) is completely determined by the sub linking systems $(X_1,Y_1,\Lambda\cap(P(X_1)\times P(Y_1))),\ldots,(X_n,Y_n,\Lambda\cap(P(X_n)\times P(Y_n)))$, where $(X_1,Y_1),\ldots,(X_n,Y_n)$ are the components of (X,Y,E_Λ).

THEOREM 5.6. *Let* (X,Y,λ) *be a linking system*, $X', X'' \subset X$, $Y', Y'' \subset Y$, $X' \cap X'' = Y' \cap Y'' = \emptyset$, $|X'| = |Y''|$, $|X''| = |Y'|$. *If* $X' \cup Y''$ *is independent in* E_λ, *i.e. if* $E_\lambda \cap (X'\times Y'') = \emptyset$, *then:* $X' \cup X''$ & $Y' \cup Y'$ *are linked in* $\lambda \Leftrightarrow X'$ & Y' *and* X'' & Y'' *are linked in* λ.

PROOF. $\Rightarrow$ $\lambda(X',Y') \geq \lambda(X',Y'\cup Y'') + \lambda(X',\emptyset) - \lambda(X',Y'') = |X'| + 0 - 0 = |Y'|$;
$\lambda(X'',Y'') \geq \lambda(X'\cup X'',Y'') + \lambda(\emptyset,Y'') - \lambda(X',Y'') = |Y''| + 0 - 0 = |X''|$.
$\Leftarrow$ $\lambda(X'\cup X'',Y'\cup Y'') \geq \lambda(X',Y'\cup Y'') + \lambda(X'\cup X'',Y'') - \lambda(X',Y'') = |X'| + |Y''| - 0 = |X'\cup X''|$. □

COROLLARY 5.7. *Let* (X,Y,Λ) *be an l.s. and* $(X_1,Y_1),\ldots,(X_n,Y_n)$ *the components of* (X,Y,E_Λ). *Let for each* $i = 1,\ldots,n$: $X_i' \subset X_i$ *and* $Y_i' \subset Y_i$.
Then: $\bigcup_{i=1}^{n} X_i'$ & $\bigcup_{i=1}^{n} Y_i'$ *are linked in* Λ *iff for each* $i = 1,\ldots,n$: X' & Y' *are linked in* Λ.

PROOF. It is easy to prove, by induction to k and using theorem 5.6, that for each $k = 1,\ldots,n$: $\bigcup_{i=1}^{k} X'_i$ & $\bigcup_{i=1}^{k} Y'_i$ are linked in Λ iff $\forall i = 1,\ldots,k$: X'_i & Y'_i are linked in Λ. □

6. BINARY DELTOIDS

In this chapter we give a characterization of binary deltoids in terms of their underlying bipartite graphs.

In the preceding sections we have introduced two mappings:

$E \to \Delta_E$, from bipartite graphs to deltoid linking systems;

$\Lambda \to E_\Lambda$, from linking systems to bipartite graphs.

In this section we need a third relation:

$E \to B_E$, between bipartite graphs and binary linking systems.

This is defined as follows. Let (X,Y,E) be a b.g. and (X,Y,ϕ) the matrix over GF(2) such that: $\phi(x,y) = 1$ iff $(x,y) \in E$, $(x \in X, y \in Y)$.

Then $B_E = \Lambda_\phi$, the linking system corresponding with the matrix (X,Y,ϕ) (see section 2). The corresponding linking function is denoted by $\beta_E (\equiv \lambda_\phi)$.

The mapping $E \to B_E$ gives, of course, a one-to-one relation between bipartite graphs and binary linking systems, and a binary l.s. is completely determined by its b.g. Also $E = E_{B_E}$.

Furthermore we can state:

PROPOSITION 6.1. *Let* (X,Y,Λ) *be an l.s. Then:* (X,Y,Λ) *is binary iff* $\Lambda = B_{E_\Lambda}$.

PROOF. Obvious. □

THEOREM 6.2. *If* (X,Y,E) *is a b.g., then the following statements are equivalent:*

(i) (X,Y,δ_E) *is a binary linking system*;

(ii) $\beta_E = \delta_E$, *or* $B_E = \Delta_E$;

(iii) *if* (X,Y,Λ) *is an l.s. and* $E = E_\Lambda$ *then* $\Lambda = \Delta_E$ *(i.e. there is only one linking system correponding to* E);

(iv) *if* X' & Y' *are linked in* E, *then the number of linkings of* X' & Y' *is odd*;

(v) *if* X' & Y' *are linked in* E, *then the number of linkings of* X' & Y' *is one*;

(vi) (X,Y,E) *is a forest, i.e. it contains no circuits.*

<u>PROOF</u>.

(i) ↔ (ii) By proposition 6.1. we have: Δ_E is binary iff $\Delta_E = B_{E_{\Delta_E}} = B_E$.

(ii) ⟶ (iv) If X' & Y' are linked in E, then X' & Y' are linked in Δ_E and so (by (ii) in B_E; therefore (X',Y',ϕ_E') is regular in GF(2), and thus the number of linkings in E of X' and Y' is odd ($\equiv 1$ (mod.2)).

(iv) ⟶ (vi) Suppose there are circuits. Let then $x_1,\dots,x_n \in X$ and $y_1,\dots,y_n \in Y$ such that:

(1) $(x_1,y_1), (x_2,y_1), (x_2,y_2),\dots, (x_n,y_n), (x_1,y_n) \in E, \quad n \geq 2$,

(2) $|\{x_1,\dots,x_n\}| = n$ minimal under property (1).

Then the only two linkings are:

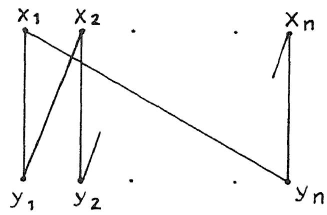
ℓ_1 with $\forall i = 1,\dots,n$: $\ell_1(x_i) = y_i$,

and

ℓ_2 with $\forall i = 2,\dots,n$: $\ell_2(x_i) = y_{i-1}$,

and

$\ell_2(x_1) = y_n$.



Suppose there is a third linking ℓ_3 with $\ell_3(x_1) = y_j$ (without loss of generality) and $2 \leq j < n$.

Then $(\{x_1,x_{j+1},\dots,x_n\},\{y_j,\dots,y_n\})$ is a circuit with length $2(n+1-j)$, and $2 \leq n+1-j < n$, contradicting property (2).

Thus then the number of linkings of X' & Y' is 2, contradicting (iv).

(vi) ⟶ (v) Suppose X' & Y' are linked and the number of linkings is more than one.

Then the pair (X',Y') contains a circuit, thus our assumption contradicts (vi).

(v) ⟶ (iii) If (X,Y,Λ) is an l.s. with $E_\Lambda = E$, then we have to prove $\Delta_E \subset \Lambda$. This is true, since: if X' & Y' are linked in Δ_E, then X' & Y' are linked in E, thus, by (v), there is exactly one linking in E_Λ of X' & Y'; therefore, by theorem 5.4, X' & Y' are linked in Λ.

(iii) ⟶ (ii) Since $E_{B_E} = E$ it follows from (iii) that $B_E = \Delta_E$. □

<u>COROLLARY 6.3</u>. *Let* (X,Y,Λ) *be a linking system. Then the following statements are equivalent.*

(i) (X,Y,Λ) *is a binary deltoid linking system;*

(ii) *if* (X,Y,Λ_1) *is a linking system and* $E_{\Lambda_1} = E_\Lambda$, *then* $\Lambda_1 = \Lambda$*;*

(iii) *if* X' & Y' *are linked in* E_Λ, *then the number of linkings is odd;*

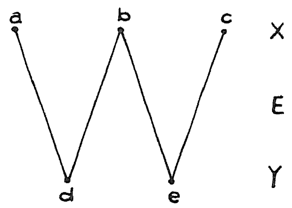
(iv) *if* X' & Y' *are linked in* E_Λ, *then the number of linkings is one;*

(v) (X,Y,E_Λ) *is a forest.*

<u>PROOF</u>. Straightforward from theorem 6.2. □

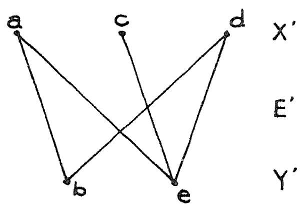
7. DELTOID-BASES

If (X,Y,$\mathcal{B}$) is a deltoid-based deltoid, then not every base of (X∪Y,$\mathcal{B}$) is a deltoid-base of the deltoid. For example, set X = {a,b,c}, Y = {d,e}, E = {(a,d),(b,d),(b,e),(c,e)} and $\mathcal{B} = \mathcal{B}_E$, i.e. (X,Y,$\mathcal{B}$) is the deltoid-based deltoid generated by the b.g. (X,Y,E). Take X' = {a,b,c} and Y' = {b,e}.



Then X' is another base of (X∪Y,$\mathcal{B}$), but not a deltoid-base; so (X',Y',$\mathcal{B}$) is a based, but not a deltoid-based matroid. If X' were a deltoid-base, there would be a b.g. (X',Y',E') with $\mathcal{B} = \mathcal{B}_{E'}$. But then:

(a,b) ∈ E', since {a,c,d} is a base of $\mathcal{B}_E$,
(a,e) ∈ E', since {b,c,d} is a base of $\mathcal{B}_E$,
(c,e) ∈ E', since {a,d,e} is a base of $\mathcal{B}_E$,
(d,b) ∈ E', since {a,b,c} is a base of $\mathcal{B}_E$,
(d,e) ∈ E', since {a,c,e} is a base of $\mathcal{B}_E$.



Then also: {a,d} & {b,e} linked in E' and so {b,c,e} ∈ $\mathcal{B}_{E'}$, but {b,c,e} ∉ $\mathcal{B}_E$. Therefore $\mathcal{B} = \mathcal{B}_E \neq \mathcal{B}_{E'}$.

Below we characterize the deltoid-bases of a deltoid, given one deltoid-base and the corresponding b.g.

<u>DEFINITION</u>. Let (X,Y,E) be a b.g. with disjoint X and Y, Let further X' ⊂ X, Y' ⊂ Y and X' & Y' linked in E with linking $\ell : X' \to Y'$. Then define: $\ell' : X \cup Y \to X \cup Y$,

by: $\ell'(x) = \ell(x)$, if $x \in X'$;

$\ell'(y) = \ell^{-1}(y)$, if $y \in Y'$;

$\ell'(z) = z$, if $z \notin X' \cup Y'$.

Of course, ℓ' is a bijection and for each $z \in X \cup Y$: $\ell'(\ell'(z)) = z$.

In the sequel when we speak of the b.g. corresponding with a based matroid (or with a base of a matroid), we shall mean the b.g. corresponding with the l.s. (see section 5), corresponding with that based matroid (or with the based matroid generated by that base; see section 3).
First, we prove that, when we replace a deltoid-base by another base, the corresponding b.g. will become greater (or remains equal) in the following sense.

<u>THEOREM 7.1</u>. *Let* $M = (X,Y,\mathcal{B})$ *be a deltoid-based deltoid with corresponding* b.g. (X,Y,E), *let* $X' \subset X$, $Y' \subset Y$, X' & Y' *linked in* E *and let* $\bar{X} = X \setminus X' \cup Y'$, $\bar{Y} = Y \setminus Y' \cup X'$ *and* $(\bar{X},\bar{Y},\bar{E})$ *the b.g. corresponding with the based deltoid* $(\bar{X},\bar{Y},\mathcal{B})$.
Then for each linking ℓ *of* X' & Y' *holds*:
$\forall x \in X, y \in Y: (x,y) \in E \Rightarrow (\ell'(x),\ell'(y)) \in \bar{E}$.

<u>PROOF</u>. Take $x \in X$, $y \in Y$ such that $(x,y) \in E$, and let ℓ be a linking of X' & Y'. Then:

a. if $x \notin X'$, $y \notin Y'$: $(x,y) \in E \Rightarrow X' \cup \{x\}$ & $Y' \cup \{y\}$ linked in E
$\Rightarrow X \setminus (X' \cup \{x\}) \cup (Y' \cup \{y\}) = \bar{X} \setminus \{x\} \cup \{y\}$ is a base of M
$\Rightarrow (x,y) = (\ell'(x),\ell'(y)) \in \bar{E}$;

b. if $x \in X'$, $y \notin Y'$: $(x,y) \in E \Rightarrow X'$ & $Y' \setminus \{\ell'(x)\} \cup \{y\}$ linked in E
$\Rightarrow X \setminus X' \cup [Y' \setminus \{\ell'(x)\} \cup \{y\}] = \bar{X} \setminus \{\ell'(x)\} \cup \{y\}$ is a base of $M \Rightarrow (\ell'(x),y) = (\ell'(x),\ell'(y)) \in \bar{E}$;

c. if $x \notin X'$, $y \notin Y'$: $(x,y) \in E \Rightarrow X' \setminus \{\ell'(y)\} \cup \{x\}$ & Y' linked in E
$\Rightarrow X \setminus [X' \setminus \{\ell'(y)\} \cup \{x\}] \cup Y' = \bar{X} \setminus \{x\} \cup \{\ell'(y)\}$ is a base of $M \Rightarrow (x,\ell'(y)) = (\ell'(x),\ell'(y)) \in \bar{E}$;

d. if $x \in X'$, $y \in Y'$: $(x,y) \in E \Rightarrow X' \setminus \{\ell'(y)\}$ & $Y' \setminus \{\ell'(x)\}$ linked in E
$\Rightarrow X \setminus [X' \setminus \{\ell'(y)\}] \cup [Y' \setminus \{\ell'(x)\}] =$
$\bar{X} \setminus \{\ell'(x)\} \cup \{\ell'(y)\}$ is a base of $M \Rightarrow (\ell'(x),\ell'(y)) \in \bar{E}$. □

In what follows we need the notion of a normal edge in a b.g.

DEFINITION. Let (X,Y,E) be a b.g. and $e = (x,y) \in E$. Then e is called a *normal edge* of E if for each $x' \in X$ and $y' \in Y$ it is true that: if (x',y) and $(x,y') \in E$ then $(x',y') \in E$.
Define further $E_N = \{e \in E \mid e \text{ is a normal edge of } E\}$. A linking $\ell : X' \to Y'$ is called *normal* if for each $x \in X'$: $(x,\ell(x)) \in E_N$.

PROPOSITION 7.2. *If* (X,Y,E) *is a b.g. then* (X,Y,E_N) *is a semi-complete b.g.*

PROOF. It is sufficient to prove that for each $x, x' \in X$, $y, y' \in Y$ holds: $(x',y), (x,y), (x,y') \in E_N \Rightarrow (x',y') \in E_N$.
Take $x, x' \in X$, $y, y' \in Y$ and suppose (x',y), (x,y) and (x,y') are normal edges. Then, since (x,y) is a normal edge, $(x',y') \in E$.
Let further $x'' \in X$, $y'' \in Y$ such that (x'',y') and (x',y'') belong to E. We have to prove: $(x'',y'') \in E$. This can be done straightforwardly; since (x',y) is normal, $(x,y'') \in E$ and then, since (x,y') is normal, $(x'',y'') \in E$. □

Second we show that a necessary and sufficient condition for a new base to be a deltoid-base is, that the new b.g. (corresponding with the new base) is not properly greater then the former b.g. but equals this b.g. in a sense to be made more precise. Also we prove that every linking, lying under the transition of the two bases, is normal.

THEOREM 7.3. *Let* $M = (X,Y,\mathcal{B})$ *be a deltoid-based deltoid with corresponding* b.g. (X,Y,E), *let* $X' \subset X$, $Y' \subset Y$, X' & Y' *linked in* E *and let* $\bar{X} = X \setminus X' \cup Y'$ *and* $\bar{Y} = Y \setminus Y' \cup X'$ *and* $(\bar{X},\bar{Y},\bar{E})$ *the b.g. corresponding with the based deltoid* $(\bar{X},\bar{Y},\mathcal{B})$.

Then the following statements are equivalent:

(i) $(\bar{X},\bar{Y},\mathcal{B})$ *is a deltoid-based deltoid* (i.e. $\bar{X}$ *is a deltoid-base*);
(ii) *for each linking* ℓ *of* X' & Y' *in* E *holds*: $(x,y) \in E \Leftrightarrow (\ell'(x),\ell'(y)) \in \bar{E}$;
(iii) *for some linking* ℓ *of* X' & Y' *in* E *holds*: $(x,y) \in E \Leftrightarrow (\ell'(x),\ell'(y)) \in \bar{E}$;
(iv) *every linking* ℓ *of* X' & Y' *is normal*;
(v) *there exists a normal linking* ℓ *of* X' & Y'.

PROOF.

(i) → (ii) Let ℓ be a linking of X' & Y' in E. Then ℓ^{-1} is a linking of Y' & X' in $\bar{E}$, for if $y \in Y'$ then $(\ell^{-1}(y),y) \in E$, therefore by theorem 7.1, $(\ell'(\ell^{-1}(y)),\ell'(y)) = (y,\ell^{-1}(y)) \in \bar{E}$.Also $(\ell^{-1})' = \ell'$. Then $\forall x \in X'$, $\forall y \in Y'$: $(\ell'(x),\ell'(y)) \in \bar{E} \Rightarrow$ (by (i) and theorem 7.1) $((\ell^{-1})'(\ell'(x)),(\ell^{-1})'(\ell'(y))) = (\ell'\ell'(x),\ell'\ell'(y)) = (x,y) \in E$. The converse is theorem 7.1.

(ii) → (iv) Let ℓ be a linking of X' & Y' in E, let $x \in X'$, $x' \in X$, $y' \in Y$ and $(x',\ell(x)),(x,\ell(x)),\ (x,y') \in E$. We have to prove: $(x',y') \in E$.

a. if $x' \notin X'$, $y' \notin Y'$, then: $X' \cup \{x'\}$ & $Y' \cup \{y'\}$ linked in E
$\Rightarrow X \setminus (X' \cup \{x'\}) \in (Y' \cup \{y'\}) = \bar{X} \setminus \{x'\} \cup \{y'\}$ is a base of M
$\Rightarrow (x',y') \in \bar{E} \Rightarrow (\ell'(x'),\ell'(y')) = (x',y') \in E$;

b. if $x' \in X'$, $y' \notin Y'$, then: X' & $Y' \setminus \{\ell(x')\} \cup \{y'\}$ linked in E
$\Rightarrow X \setminus X' \cup [Y' \setminus \{\ell(x')\} \cup \{y'\}] = \bar{X} \setminus \{\ell(x')\} \cup \{y'\}$ is a base of M
$\Rightarrow (\ell(x'),y') \in \bar{E} \Rightarrow (\ell'(\ell(x')),\ell'(y')) = (x',y') \in E$;

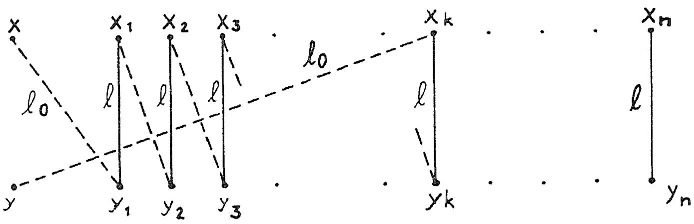
c. if $x' \notin X'$, $y' \in Y'$, then: $X' \setminus \{\ell(y') \cup \{x'\}$ & Y' linked in E
$\Rightarrow X \setminus [X' \setminus \{\ell'(y')\} \cup \{x'\}] \cup Y' = \bar{X} \setminus \{x'\} \cup \{\ell'(y')\}$ is a base of M
$\Rightarrow (x',\ell'(y')) \in \bar{E} \Rightarrow (\ell'(x'),\ell'\ell'(y')) = (x',y') \in E$;

d. if $x' \in X'$, $y' \in Y'$ then $(X' \setminus \{\ell'(y')\})$ & $(Y' \setminus \{\ell'(x')\})$ linked in E
$\Rightarrow X \setminus (X' \setminus \{\ell'(y')\}) \cup (Y' \setminus \{\ell'(x')\}) =$
$\bar{X} \setminus \{\ell'(x')\} \cup \{\ell'(y')\}$ is a base of M
$\Rightarrow (\ell'(x'),\ell'(y')) \in \bar{E} \Rightarrow (\ell'\ell'(x'),\ell'\ell'(y')) = (x',y') \in E$.

(iv) → (v) X' & Y' are linked, thus there exists a linking, which is then normal.

(v) → (iii) Take the normal linking ℓ from (v) and $x \in X$, $y \in Y$, such that $(\ell'(x),\ell'(y)) \in \bar{E}$. We have to prove: $(x,y) \in E$.

a. If $x \notin X'$, $y \notin Y'$, then:
$\bar{X} \setminus \{\ell'(x)\} \cup \{\ell'(y)\} = (X \setminus X' \cup Y') \setminus \{\ell'(x)\} \cup \{\ell'(y)\} =$
$X \setminus (X' \cup \{\ell'(x)\}) \cup (Y' \cup \{\ell'(y)\})$ is a base of $M \Rightarrow X' \cup \{x\}$ & $Y' \cup \{y\}$ linked in E, say with linking ℓ_0.



We may suppose that:

$X' = \{x_1,\dots,x_n\}$, $Y' = \{y_1,\dots,y_n\}$,

$\forall i = 1,\dots,n$: $\ell(x_i) = y_i$,

$\ell_0(x) = y_1$; $\forall i = 1,\dots,k-1$: $\ell_0(x_i) = y_{i+1}$;

$\ell_0(x_k) = y$.

Since ℓ is a normal linking it follows, respectively, that:

$(x,y_2) \in E$, $(x,y_3) \in E,\dots,$ $(x,y_k) \in E$, $(x,y) \in E$.

b. If $x \in X'$, $y \notin Y'$, then:

$\bar{X} \setminus \{\ell'(x)\} \cup \{\ell'(y)\} = (X\setminus X' \cup Y') \setminus \{\ell'(x)\} \cup \{\ell'(y)\} =$

$X \setminus X' \cup (Y'\setminus\{\ell'(x)\}\cup\{\ell'(y)\})$ is a base of $M \Rightarrow$

X' & $Y' \setminus \{\ell(x)\} \cup \{y\}$ are linked in E, say with linking ℓ_0.

We may suppose that:

$X' = \{x_1,\dots,x_n\}$,

$Y' = \{y_1,\dots,y_n\}$,

$\forall i = 1,\dots,n$:

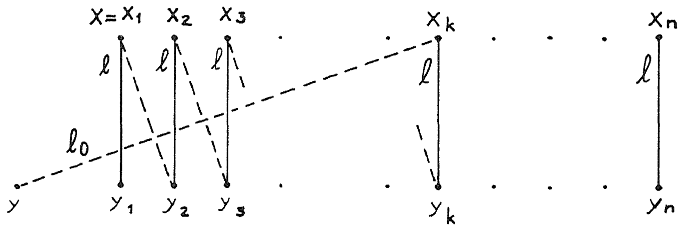
$\ell(x_i) = y_i$,

$x = x_1$;

$\forall i = 1,\dots,k-1$:

$\ell_0(x_i) = y_{i+1}$;

$\ell_0(x_k) = y$.



Since ℓ is a normal linking it follows, respectively, that:

$(x,y_3) \in E$, $(x,y_4) \in E$, ..., $(x,y_k) \in E$, $(x,y) \in E$.

c. The case $x \notin X'$ and $y \in Y'$ is treated similarly to b.

d. If $x \in X'$, $y \in Y'$ and $\ell(x) \neq y$, then:

$\bar{X} \setminus \{\ell'(x)\} \cup \{\ell'(y)\} = (X\setminus X' \cup Y') \setminus \{\ell'(x)\} \cup \{\ell'(y)\} =$

$X \setminus (X'\setminus\{\ell'(y)\}) \cup (Y'\setminus\{\ell'(x)\})$ is a base of $M \Rightarrow$

$X' \setminus \{\ell'(y)\}$ & $Y' \setminus \{\ell'(x)\}$ are linked in E, say with linking ℓ_0.

We may suppose that:

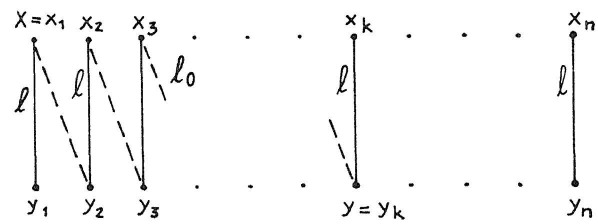
$X' = \{x_1,\dots,x_n\}$,

$Y' = \{y_1,\dots,y_n\}$,

$x = x_1$, $y = y_k$,

$\forall i = 1,\dots,n$:

$\ell(x_i) = y_i$,



and

$\forall i = 1,\ldots,k-1$: $\ell_0(x_i) = y_{i+1}$.
Since ℓ is a normal linking it follows, respectively, that:
$(x,y_3) \in E$, $(x,y_4) \in E,\ldots,(x,y_k) = (x,y) \in E$.
If $\ell(x) = y$, then also $(x,y) = (x,\ell(x)) \in E$.

(iii) $\rightarrow$ (i) Let $(\bar{X},\bar{Y},\bar{\Lambda})$ be the l.s. correponding to the based deltoid $(\bar{X},\bar{Y},\mathcal{B})$.
Then $|\mathcal{B}|$ = # pairs of linked sets in (X,Y,E) = [by (iii)]
pairs of linked sets in $(\bar{X},\bar{Y},\bar{E}) = |\Delta_{\bar{E}}| \leq$ [by thm.5.2]
$|\bar{\Lambda}| = |\mathcal{B}|$.
Thus: $|\bar{\Lambda}| = |\Delta_{\bar{E}}|$.
Also, by thm. 5.2.: $\bar{\Lambda} \subset \Delta_{\bar{E}}$
Therefore: $\bar{\Lambda} = \Delta_{\bar{E}}$, $(\bar{X},\bar{Y},\bar{\Lambda}) = (\bar{X},\bar{Y},\Delta_{\bar{E}})$ and (by prop. 3.5.)
$(\bar{X},\bar{Y},\mathcal{B})$ is a deltoid-based deltoid. □

COROLLARY 7.4.

(i) *Let* (X,Y,E) *be a b.g. with disjoint* X *and* Y. *Then the deltoid-bases of* $(X\cup Y,\mathcal{B}_E)$ *are precisely the bases of* $(X\cup Y,\mathcal{B}_{E_N})$.

(ii) *Let* $M = (Z,\mathcal{B})$ *be a deltoid and* $\mathcal{B}' = \{B \in \mathcal{B} \mid B$ *a deltoid-base of* $M\}$. *Then* $(Z,\mathcal{B}')$ *is a semi-uniform deltoid.*

PROOF.

(i) Directly from theorem 7.3.(i) and (iv).

(ii) Directly from (i) and propositions 1.6. and 7.2. □

For the sake of completeness we state

THEOREM 7.5. *Every base of a gammoid is a gammoid-base.*

PROOF. Using methods, designed by INGLETON and PIFF [5], which are too complicated to give here, one can straightforwardly prove that every base of a gammoid is a gammoid-base. □

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