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~~On~~ ~~C~~anonical forms

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MATHEMATICS

ON CANONICAL FORMS

BY

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§ 1. In this paper we deduce in a simple way for several forms well-known canonical representations by making use of the theorem of LASKER–WAKEFORD on canonical forms. Some of these results were already given by WAKEFORD.

Further we give an extension of that theorem for simultaneous canonical representations of sets of similar forms and some applications of that extension.

§ 2. First we have to give some definitions and auxiliary theorems.

Forms $f(x) = \Sigma a_{ij} \dots x_i x_j \dots$ of degree n in m homogeneous cogredient variables will in symbolic notation be denoted by $f(x) = (a'x)^n$.

Two forms of the same degree in the same variables will be called similar.

A form $\varphi(u) = \Sigma a_{ij} \dots u_i u_j \dots$ of degree n in m homogeneous contragredient variables will in symbolic notation be denoted by $\varphi(u) = (au')^n$.

Two forms of the same degree, the one in cogredient the other in the same number of contragredient variables, will be called dual similar.

Two forms $f(x) = (a'x)^n$ and $\varphi(u) = (au')^{n'}$ are called apolar if

$$(a'a)^n = 0 \text{ in the case } n = n';$$
$$(a'a)^{n'}(a'x)^{n-n'} = 0 \text{ identical in } x, \text{ if } n > n';$$
$$(a'a)^n(au')^{n'-n} = 0 \text{ identical in } u', \text{ if } n' > n.$$

Obviously apolarity of forms is independant for linear transformations.

In all three cases apolarity of two dual forms is expressed by the vanishing of one or more expressions which are bilinear in the coefficients of the forms.

§ 3. Theorem 1. *If a form $f(x) = (a'x)^n$ is apolar to the $(n-k)^{th}$ power of a linear form (au'), then the point a is a $(k+1)$ple point of $f(x) = 0$.*

In fact the apolarity of $f(x)$ and the non-symbolical expression $(au')^{n-k}$ leads to $(a'a)^{n-k}(a'x)^k = 0$, identical in x, which is identical to the fact that a is a $(k+1)$ple point of $f(x) = 0$.

§ 4. We now formulate the theorem, first found by LASKER [1]), later extended by WAKEFORD [2]), by means of which one can decide whether a given form possesses a prescribed canonical representation [3]).

Theorem 2. *A form $f(x) = (a'x)^n$ possessing N coefficients, has a canonical representation $f(x) = F(X, c)$, where in $F(X, c)$ occur s parameters $c_1, \ldots, c_s$ and r variables $X_1, \ldots, X_r$ with*

(1) $$X_\varrho = X_\varrho(x, e) = \Sigma e_{\varrho, ij} \ldots x_i x_j \ldots \quad (\varrho = 1, \ldots, r)$$

and where the total number t of the coefficients $e_{\varrho, ij} \ldots$ satisfies the relation

(2) $$t + s \geqq N,$$

if not always (i.e. not for all choices of the $t + s$ parameters $c_1, \ldots, c_s, e_1, \ldots, e_t$) a form $\varphi(u)$, dual similar to $f(x)$, exists which is apolar to all $r + s$ forms

$$\frac{\partial F}{\partial X_\varrho}, \frac{\partial F}{\partial c_\sigma} \qquad (\varrho = 1, \ldots, r; \sigma = 1, \ldots, s).$$

If however $F(X, c)$ is not a legitimate canonical representation of $f(x)$, the dual form $\varphi(u)$ always exists.

§ 5. As a first application we prove the following theorem.

Theorem 3. *Let $d = d_m(n)$ denote the greatest number of points through which a m-ary manifold of degree n can be found, having double points in each of these d points. Then the m-ary form $f = (a'x)^n$ can be expressed as a sum of $d + 1$ and of not less n^{th} powers of linear forms.*

To prove this theorem we consider the expression

$$F = X_1^n + \ldots + X_{d+1}^n,$$

where each of the forms $X_1, \ldots, X_{d+1}$ is linear in $x_1, \ldots, x_m$.

We investigate the existence of a form $\varphi = (au')^n$ dual similar to f, which is apolar to each of the $d + 1$ forms

$\frac{\partial F}{\partial X_\varrho}$, i.e. to each of the $(n-1)^{\text{th}}$ powers X_ϱ^{n-1} $(\varrho = 1, \ldots, d + 1)$.

From the definition of d follows that no manifold of degree n exists having $d + 1$ given double points, so from the dual property of theorem 1 we know that no form φ exists apolar to $d + 1$ powers $X_1^{n-1}, \ldots, X_{d+1}^{n-1}$.

Since on the other hand a manifold of degree n exists having d given double points, again from the dual property of theorem 1 we conclude that a dual form φ exists apolar to d powers $X_1^{n-1}, \ldots, X_d^{n-1}$.

From these results by the theorem 2 follows the theorem.

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[1]) E. LASKER, Zur Theorie der kanonischen Formen, Mathematische Annalen 58 434—440 (1904).

[2]) E. K. WAKEFORD, On canonical forms, Proc. Lond. Math. Soc. 2, 18, 403—410 (1918).

[3]) For a proof of the theorem confer H. W. TURNBULL, Theory of determinants, matrices and invariants, (267—269, 2nd ed. Glasgow, 1945). The theorem however can be found from the below generalisation in § 11 by taking $k = 1$.

§ 6. The value of $d_2(n)$ is easily determined. Obviously a binary form of degree $2k$ exists having k given double points (i.e. a polynomial of degree $2k$ exists having k given double roots), but no such form exists with $k+1$ given double points. Further a binary form of degree $2k+1$ exists with k given double points but no such form exists with $k+1$ given double points. These two facts show that $d_2(n) = [n/2]$, so from theorem 3 follows:

Theorem 4. *Every binary form of degree n can be written as a sum of $[n/2]+1$ and in general not as a sum of less n^{th} powers of linear forms.*

§ 7. We further consider the ternary field. Obviously we have the trivial result $d_3(1) = 0$. Since a conic exists with two given double points (namely the straight line through those points, counted twice) and since no conic exists with three given double points, we have $d_3(2) = 2$.

Further we have $d_3(3) = 3$, because a cubic exists with 3 given double points (namely the cubic of the three sides of the triangle through those 3 points), but no cubic exists with 4 given double points.

To determine $d_3(4)$ we remark that the conic through 5 given points, counted twice, is a quartic with 5 given double points. No quartic exists however with 6 given double points for any conic through 5 of the 6 points would have $2.5 = 10 > 8$ points in common with the quartic and therefore belong to it, which is impossible. Hence $d_3(4) = 5$.

Also $d_3(5)$ can be found by elementary methods. Consider 6 given points. Then the cubic through these points having a double point in one of them together with the conic through the 5 other points compose a quintic with double points in the 6 given points. Since however, 7 points being given, any of the 7 cubics having a double point in one of them and passing through the 6 other points has $1.4 + 6.2 = 16 > 15$ points in common with a quintic with double points in the 7 given points and therefore belongs to the quintic, in general no quintic exists with 7 given double points. Hence $d_3(5) = 6$.

In general the relation (2) from § 4 gives for a ternary n-ic, which from theorem 3 can be written as a sum of $d_3(n)+1$ and of not less n^{th} powers, the relation

$$3(d_3(n)+1) \geqq N = \tfrac{1}{2}(n+1)(n+2), \text{ hence } d_3(n) \geqq \tfrac{1}{6}(n+4)(n-1).$$

Herefrom we find $d_3(6) \geqq 8\tfrac{1}{3}$, hence $d_3(6) \geqq 9$; this result is obvious since a cubic through 9 given points, counted twice, is a sextic with 9 given double points.

In general through $\frac{1}{2}h(h+3)$ points a h-ic can be brought, so the last argument shows $d_3(2h) \geqq \frac{1}{2}h(h+3)$ but this result is only better than the above result $d_3(2h) \geqq \frac{1}{6}(2h+4)(2h-1)$ if $\frac{1}{2}h(h+3) > \frac{1}{6}(2h+4)(2h-1)$ hence $h < 4$, i.e. in cases we have considered already.

§ 8. In the quaterny field we have $d_4(1) = 0$ and $d_4(2) = 3$, for a plane through 3 given points, counted twice, gives a quadric with three given double points but any line connecting two of 4 given points belongs to a quadric with double points in those 4 points, so no quadric exists with 4 given double points.

In order to determine $d_4(3)$ we remark that a cubic surface exists with double points in 4 given points (if these points have coordinates $(1, 0, 0, 0), \ldots, (0, 0, 0, 1)$ the surface $x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4 = 0$ satisfies). No cubic surface exists with 5 given double points $P_1, \ldots, P_5$. For the surface would have in common with the plane $P_1P_2P_3$ the cubic P_1P_2, P_2P_3, P_3P_1 and the point of intersection of $P_1P_2P_3$ and P_4P_5, which is impossible. Hence $d_4(3) = 4$ and from § 5 we then know that every quaternary cubic form $f = (a'x)^3$ admits of a canonical representation $F = X_1^3 + \ldots + X_5^3$, a result already given by SYLVESTER.

Obviously for every m we have $d_m(1) = 0$.

§ 9. In order to determine $d_m(2)$ we remark that through $m - 1$ points in the m-ary field a hyperplane can be brought which, counted twice, gives a hyperquadric with $m - 1$ given double points. However a hyperquadric which has double points in m given points must degenerate in all m hyperplanes through any $m - 1$ of the m points, which is impossible. Hence $d_m(2) = m - 1$ and by theorem 3 any m-ary quadratic form can be written as a sum of m and in general not of less than m squares [4]).

§ 10. We now give some other applications of the theorem 2 of LASKER–WAKEFORD.

Every plane biquadratic has the two representations

$$F = Q_1^2 + Q_2^2 + Q_3^2 \quad ; \quad F_1 = R_1R_2 - R_3^2,$$

where the 6 expressions $Q_1, \ldots, R_3$ are quadratic polynomials. Since the second expression is obtained from the first by taking

$$R_1 = Q_1 + iQ_2,\ R_2 = Q_1 - iQ_2,\ R_3 = iQ_3,$$

we only consider the first representation. Using the theorem of LASKER–WAKEFORD we have to show that not always a dual biquadratic form $\varphi = (\alpha u')^4$ exists, which is apolar to each of the three forms

$$\frac{\partial F}{\partial Q_i}, \text{ i.e. to each of the three forms } Q_i \qquad (i = 1, 2, 3).$$

Now in the case $Q_i = x_i^2$ the apolarity of φ and Q_i leads to $\alpha_{hkii} = 0$ for all h, k, i. Since in any coefficient $\alpha_{i_1i_2i_3i_4}$ of φ two of the four indices i_1, i_2, i_3, i_4 which only assume the values 1, 2, 3 must be equal, the apolar form φ does not always exist, which proves the representation is canonical.

§ 11. We now give the following generalisation of the theorem of LASKER–WAKEFIELD.

[4]) For some of these results deduced in another way cf. E. BODEWIG, Darstellung von Formen durch Potenzsummen, Giorn. di Mat., 81—100 (1926).

Theorem 5. *If k similar forms $f_1(x), \ldots, f_k(x)$, each possessing N coefficients, have the simultaneous canonical representation*

$$F_1(X, c), \ldots, F_k(X, c)$$

involving altogether s parameters $c_1, \ldots, c_s$ and r variables $X_1, \ldots, X_r$ satisfying (1), *where the above number t satisfies the relation*

$$t + s \geqq kN, \tag{3}$$

then there exists a set of numbers $\lambda_1, \ldots, \lambda_k$ (not all zero) such that a dual similar form $\varphi(u)$ of the forms f which is apolar to all $r + s$ forms $\partial F_\lambda/\partial X_\varrho$, $\partial F_\lambda/\partial c_\sigma$ ($\varrho = 1, \ldots, r$; $\sigma = 1, \ldots, s$) with

$$F_\lambda = \sum_{\varkappa=1}^{k} \lambda_\varkappa F_\varkappa(X, c)$$

does not always exist.

If however the set of forms $F_1, \ldots, F_k$ is not a legitimate simultaneous representation of the given forms $f_1, \ldots, f_k$, then there exists a set $\lambda_1, \ldots, \lambda_k$ (not all zero) for which always the dual similar form $\varphi(u)$ which is apolar to all $r + s$ forms $\partial F_\lambda/\partial X_\varrho$, $\partial F_\lambda/\partial c_\sigma$ ($\varrho = 1, \ldots, r$; $\sigma = 1, \ldots, s$) exists.

To prove this theorem we consider the identity in $\lambda_1, \ldots, \lambda_k$ and $x_1, \ldots, x_m$

$$\sum_{\varkappa=1}^{k} \lambda_\varkappa f_\varkappa(x) = \sum_{\varkappa=1}^{k} \lambda_\varkappa F_\varkappa(X(x, e), c). \tag{4}$$

Let $a_{1\varkappa}, \ldots, a_{N\varkappa}$ denote the N coefficients of $f_\varkappa(x)$ ($\varkappa = 1, \ldots, k$). From (4) by comparing coefficients we obtain the N equations

$$\sum_{\varkappa=1}^{k} \lambda_\varkappa a_{\nu\varkappa} = q_\nu(\lambda, e, c) \qquad (\nu = 1, \ldots, N).$$

From these equations the $t + s$ unknowns $e_1, \ldots, e_t$; $c_1, \ldots, c_s$ can be solved for all $\lambda_1, \ldots, \lambda_k$ and all coefficients $a_{\nu\varkappa}$($\nu = 1, \ldots, N$; $\varkappa = 1, \ldots, k$) if and only if no relation $\psi(q_1, \ldots, q_N) = 0$ holds identical in λ, e, c.

Let us first suppose that such a relation holds. Then not all sets of k forms $f_1(x), \ldots, f_k(x)$ possess the desired simultaneous canonical form. For the moment consider a special set $f_{01}(x), \ldots, f_{0k}(x)$, for which the representation is possible, i.e. a set for which the relations (4) hold with $f_{0\varkappa}(x)$ instead of $f_\varkappa(x)$ ($\varkappa = 1, \ldots, k$). Obviously such a set can be found. We then deduce from $\psi(q_1, \ldots, q_n) = 0$ the identities in λ, e and c.

$$\sum_{\nu=1}^{N} \frac{\partial\psi}{\partial q_\nu}\frac{\partial q_\nu}{\partial e_\tau} = 0; \sum_{\nu=1}^{N} \frac{\partial\psi}{\partial q_\nu}\frac{\partial q_\nu}{\partial c_\sigma} = 0 \qquad (\tau = 1, \ldots, t; \sigma = 1, \ldots, s),$$

which by their bilinear character (confer § 2) express that a form $\varphi(u)$ dual similar to the k given forms exists (the N coefficients of which are equal to $\partial\psi/\partial q_1, \ldots, \partial\psi/\partial q_N$), which for all $\lambda_1, \ldots, \lambda_n$ (not all zero) is apolar to the $t + s$ forms with coefficients

$$\frac{\partial q_1}{\partial e_\tau}, \ldots, \frac{\partial q_N}{\partial e_\tau}, \text{ resp. } \frac{\partial q_1}{\partial c_\sigma}, \ldots, \frac{\partial q_N}{\partial c_\sigma},$$

i.e. to the $t + s$ forms

$$\sum_{\varkappa=1}^{k} \lambda_\varkappa \frac{\partial f_{0\varkappa}(x)}{\partial e_\tau}; \sum_{\varkappa=1}^{k} \lambda_\varkappa \frac{\partial f_{0\varkappa}(x)}{\partial c_\sigma},$$

i.e. to the $t + s$ forms

$$\frac{\partial F_\lambda}{\partial e_\tau}, \frac{\partial F_\lambda}{\partial c_\sigma} \qquad (\tau = 1, \ldots, t; \sigma = 1, \ldots, s).$$

Now if a form $\varphi(u)$ is apolar to all $\partial F_\lambda/\partial e_\tau$ $(\tau = 1, \ldots, t)$ it is apolar to all r forms $\partial F_\lambda/\partial X_\varrho$ $(\varrho = 1, \ldots, r)$, for consider all $e_\tau = e_{\varrho,\, ij} \ldots$ with the same first index ϱ. Since φ is apolar to all $\partial F_\lambda/\partial e_{\varrho,\, ij\ldots}$ (ϱ fixed; $i, j, \ldots$ arbitrary) the form φ is apolar to all

$$\frac{\partial F_\lambda}{\partial X_\varrho} \frac{\partial X_\varrho}{\partial e_{\varrho,ij\ldots}} = \frac{\partial F_\lambda}{\partial X_\varrho} x_i x_j \ldots$$

hence to $\partial F_\lambda/\partial X_\varrho$. Conversely if φ is apolar to all $\partial F_\lambda/\partial X_\varrho$ $(\varrho = 1, \ldots, r)$, then φ is apolar to all $\partial F_\lambda/\partial e_\tau$ for from

$$\frac{\partial F_\lambda}{\partial e_\tau} = \frac{\partial F_\lambda}{\partial e_{\varrho,ij\ldots}} = \frac{\partial F_\lambda}{\partial X_\varrho} \frac{\partial X_\varrho}{\partial e_{\varrho,ij\ldots}}$$

we see that apolarity of φ and all $\partial F_\lambda/\partial X_\varrho$ leads to apolarity of φ and all $\partial F_\lambda/\partial e_\tau$.

We now may formulate our result as follows. If the given set of k similar forms does not possess the above simultaneous canonical representation, a relation $\psi(q_1, \ldots, q_N) = 0$ holds; this is equivalent to the fact that for all $\lambda_1, \ldots, \lambda_k$ (not all zero) a dual similar form $\varphi(u)$ of the given forms always exists which is apolar to each of the $r + s$ forms

$$\frac{\partial F_\lambda}{\partial X_\varrho}, \frac{\partial F_\lambda}{\partial c_\sigma} \qquad (\varrho = 1, \ldots, r; \sigma = 1, \ldots, s).$$

Conversely if the given forms do possess the simultaneous canonical representation, there is a set $\lambda_1, \ldots, \lambda_k$ such that a dual form $\varphi(u)$ apolar to all $r+s$ forms $\partial F_\lambda/\partial X_\varrho$, $\partial F_\lambda/\partial c_\sigma$ $(\varrho = 1, \ldots, r;\ \sigma = 1, \ldots, s)$ does not always (i.e. for all choices of the quantities e and c) exist.

§ 12. As an application we give the following well-known theorem.

Theorem 6. *Two m-ary quadrics*

$$f = (a'x)^2 \text{ and } g = (b'x)^2$$

possess the simultaneous representation

$$F = X_1^2 + \ldots + X_m^2 \text{ resp. } G = c_1 X_1^2 + \ldots + c_m X_m^2.$$

By theorem 5 we only have to investigate the existence of a dual form $\varphi = (au')^2$ which is apolar to all $2m$ forms

$$\frac{\partial F_\lambda}{\partial X_\varrho} = (\lambda_1 + \lambda_2 c_\varrho) X_\varrho \text{ and } \frac{\partial F_\lambda}{\partial c_\sigma} = \lambda_2 X_\sigma^2 \qquad (\varrho, \sigma = 1, \ldots, m).$$

We show that this form φ for all λ_1, λ_2 (not both zero) does not exist for all e and c. First suppose that none of the m factors $\lambda_1 + \lambda_2 c_\varrho$ vanishes. Then the apolarity of φ and the m forms $\partial F_\lambda/\partial X_\varrho$ learns that φ is identical equal to zero. Next suppose that exactly one of the m factors $\lambda_1 + \lambda_2 c_\varrho$, say $\lambda_1 + \lambda_2 c_h$, is equal to zero. The apolarity of φ and the m forms $\partial F_\lambda/\partial X_\varrho$ now learns that $a_{ij} = 0$ for all i, j apart from a_{hh}. Since from $\lambda_1 + \lambda_2 c_h = 0$ follows $\lambda_2 \neq 0$ (otherwise we had $\lambda_1 = \lambda_2 = 0$), apolarity of φ and $\lambda_2 X_h^2$ shows $a_{hh} = 0$. So in neither case φ exists, hence the representation is legitimate.

The case where two of the factors $\lambda_1 + \lambda_2 c_h$ and $\lambda_1 + \lambda_2 c_k$ for $h \neq k$ are equal to zero leads to $c_h = c_k$. In this case a dual form φ exists so then the representation is not canonical, which already can be found by considering the case $m = 2$; $h = 1$; $k = 2$.

§ 13. As a second example we consider two binary cubics f and g and show that they possess the simultaneous representation

$$F = X_1^3 + X_2^3 + X_3^3;\ G = c_1 X_1^3 + c_2 X_2^3,$$

where X_1, X_2, X_3 are linear forms in the variables x_1 and x_2 and $c_1 c_2 \neq 0$; $c_1 \neq c_2$.

For a proof we investigate the existence of a dual form $\varphi = (au')^3$, which is apolar to the 5 forms

$$\frac{\partial F_\lambda}{\partial X_i} \quad (i = 1, 2, 3) \quad \text{and} \quad \frac{\partial F_\lambda}{\partial c_j} \quad (j = 1, 2),$$

hence to the 5 forms

$$(\lambda_1 + c_1\lambda_2)\, X_1^2;\ (\lambda_1 + c_2\lambda_2)\, X_2^2;\ \lambda_1 X_3^2;\ \lambda_2 X_1^3;\ \lambda_2 X_2^3.$$

First suppose $\lambda_1 + c_j\lambda_2 \neq 0$ for $j = 1, 2$. Then apolarity of φ and either of the first two of these 5 forms learns $\varphi = 0$. Further assume that at least one of the two expressions $\lambda_1 + c_j\lambda_2 = 0$ $(j = 1, 2)$, say $\lambda_1 + c_1\lambda_2 = 0$. Then since $c_1 \neq c_2$ and not both λ_1 and λ_2 vanish, we have $\lambda_1 + c_2\lambda_2 \neq 0$ so apolarity of φ and $(\lambda_1 + c_2\lambda_2)X_2^2$ learns $a_{j22} = 0$ $(j = 1, 2)$. Since $\lambda_2 \neq 0$ (for if $\lambda_2 = 0$ from $\lambda_1 + c_1\lambda_2 = 0$ would follow $\lambda_1 = 0$), the apolarity of φ and $\lambda_2 X_1^3$ learns $a_{111} = 0$, so $\varphi = a_{112}u_1^2u_2$. Finally from $\lambda_1 \neq 0$ (for if $\lambda_1 = 0$ from $\lambda_1 + c_1\lambda_2 = 0$ and $c_1 \neq 0$ would follow $\lambda_2 = 0$) apolarity of φ and $\lambda_1 X_3^2$ leads to $a_{112} = 0$ hence $\varphi = 0$. So in all cases φ vanishes and the representation is canonical.

We further prove the following theorem.

Theorem 7. *Two binary n-ics f and g possess the simultaneous canonical representation*

$$F = \sum_{i=1}^{D} X_i^n;\ G = \sum_{i=1}^{D} c_i X_i^n \qquad (c_i \neq 0;\ c_i \neq c_j;\ \text{if } i \neq j), \tag{5}$$

where D denotes the smallest integer $\geqq [n/2] + 2$ and $\geqq \frac{2}{3}(n+1)$.

Since

$$D \geqq \tfrac{2}{3}(n+1),$$

the relation (3) is satisfied, for $t + s = 2D + D = 3D \geqq 2(n+1) = kN$. So theorem 5 can applied. We then investigate the existence of a dual form $\varphi = (au')^n$ apolar to each of the $2D$ forms

$$\frac{\partial F_\lambda}{\partial X_i} = \frac{\partial(\lambda_1 F + \lambda_2 G)}{\partial X_i} = n(\lambda_1 + c_i\lambda_2)\, X_i^{n-1};\ \frac{\partial F_\lambda}{\partial c_i} = \lambda_2 X_i^n \qquad (i = 1, \ldots, D).$$

Now since only one of the D expressions $\lambda_1 + c_i\lambda_2$ $(i = 1, \ldots, D)$ is equal to zero (for if $\lambda_1 + c_i\lambda_2 = \lambda_1 + c_j\lambda_2 = 0$ with $i \neq j$, we would deduce $\lambda_1 = \lambda_2 = 0$ which is impossible), the dual form φ must be apolar to at least $D-1$ i.e. at least $[n/2] + 1$ $(n-1)^{\text{th}}$ powers of linear forms. Now from § 6 such a form φ must vanish identically, so our representation is canonical.

Herefrom follows for two binary cubics a representation with $D = 3$, but our above result expressed more, namely that in the case $n = D = 3$ even a representation (5) with $c_3 = 0$ is possible.

For two binary biquadratics a representation (5) with $n = 4$, hence $D = 4$, is possible.

Since the general binary quadratic form can be written as a product of two binary quadratic forms and since these two forms possess the representation $X_1^2 + X_2^2$, $c_1X_1^2 + c_2X_2^2$, the binary biquadratic form possesses the canonical representation

$$(X_1^2 + X_2^2)(c_1X_1^2 + c_2X_2^2).$$

Similarly the above result shows that every binary sextic has the canonical representation

$$(X_1^3 + X_2^3 + X_3^3)(c_1X_1^3 + c_2X_2^3).$$

§ 14. Theorem 8. *Two ternary n-ics f and g possess the simultaneous canonic representation*

$$F = \sum_{i=1}^{D} X_i^n;\ G = \sum_{i=1}^{D} c_i X_i^n \qquad (c_i \neq 0;\ c_i \neq c_j \text{ if } i \neq j),$$

where D is the smallest integer

$$\geqq d_3(n) + 2 \text{ and } \geqq \frac{(n+1)(n+2)}{4}.$$

For a proof we can apply theorem 5 since the relation (3) is satisfied, for we have

$$t + s = 3D + D = 4D \geqq (n+1)(n+2) = kN.$$

We then have to investigate the existence of a dual form $\varphi(u) = (au')^n$ apolar to each of the $2D$ forms

$$\frac{\partial F_\lambda}{\partial X} = \frac{\partial(\lambda_1 F + \lambda_2 G)}{\partial X_i} = n(\lambda_1 + c_i\lambda_2)\, X_i^{n-1};\ \frac{\partial F_\lambda}{\partial c_i} = \lambda_2 X_i^n \qquad (i = 1, \ldots, D).$$

Now since at most one of the D expressions $\lambda_1 + c_i\lambda_2$ ($i = 1, \ldots, D$) can vanish (for otherwise from $c_i \neq c_j$ and from $c_i \neq 0$ would follow $\lambda_1 = \lambda_2 = 0$ which is impossible) the form has to be apolar to at least $D - 1$ forms which are $(n-1)^{\text{th}}$ powers of linear forms. From the definition of $d_3(n)$ follows (confer § 7) by theorem 1 that a form φ apolar to $d_3(n) + 1$ or more $(n-1)^{\text{th}}$ powers of linear forms vanishes. From theorem 5 then follows that the above representation is canonical.

As might follow from the case $n = 2$ the number D is not the best (i.e. the least) number possible, for from theorem 6 we know that two conics possess a representation $X_1^2 + X_2^2 + X_3^2$ resp. $c_1X_1^2 + c_2X_2^2 + c_3X_3^2$ while our formula gives $D = 4$. In the case $n = 3$ our formula is the best possible.

Theorem 9. *Two m-ary n-ics f and g possess the simultaneous canonical representation*

$$F = \sum_{i=1}^{D} X_i^n;\ G = \sum_{i=1}^{D} c_i X_i^n \qquad (c_i \neq 0;\ c_i \neq c_j, \text{ if } i \neq j),$$

where D is the smallest integer

$$\geqq d_m(n) + 2 \text{ and } \geqq \frac{2}{m+1}\binom{n+m-1}{n}.$$

The proof goes entirely similar to that of theorem 8.