

**stichting
mathematisch
centrum**



AFDELING ZUIVERE WISKUNDE

ZW 33/74

NOVEMBER

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A COMPACT TREELIKE SPACE IS THE CONTINUOUS IMAGE OF
AN ORDERED CONTINUUM

STICHTING MATHEMATISCH CENTRUM
AMSTERDAM

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Printed at the Mathematical Centre, 49, 2e Boerhaavestraat, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O), by the Municipality of Amsterdam, by the University of Amsterdam, by the Free University at Amsterdam, and by industries.

A compact treelike space is the continuous image of an ordered continuum.

by

A.E. Brouwer.

ABSTRACT

The theorem stated in the title is proved.

1. DEFINITION AND BASIC PROPERTIES OF TREELIKE SPACES.

Two points of a topological space are called *separated* when they do not belong to the same quasicomponent.

A connected topological space X is called *treelike* when given any two points $p, q \in X$ there is a point $z \in X$ separating p and q (that is, such that p and q are separated in $X \setminus \{z\}$).

If p, q are points of a treelike space X then we define

$$E(p, q) := \{z \mid z \text{ separates } p \text{ and } q \text{ in } X\}$$

$$S(p, q) := \{p\} \cup E(p, q) \cup \{q\}.$$

$E(p, q)$ is empty if and only if $p = q$.

We will write $X \setminus z = A + B$ meaning that $X \setminus \{z\}$ is the topological sum of its subspaces A and B , while $p \in A$ and $\{q, r\} \subset B$.

LEMMA 1.1. *Let X be connected, C a connected subspace of X . If $X \setminus C = A + B$ then both $A \cup C$ and $B \cup C$ are connected.*

LEMMA 1.2. *Let X be connected, C a connected subspace of X and S a component of $X \setminus C$. Then $X \setminus S$ is connected.*

Slightly stronger versions of these lemma's are given in KOK [3].

Using these lemma's it is not too difficult to prove the propositions below. Most of them can be found in KOK [3].

PROPOSITION 1.3. *A treelike space is Hausdorff.*

PROPOSITION 1.4. *Let X be treelike. Then for all $p \in X$ each component of $X \setminus p$ is open.*

PROPOSITION 1.5. *Let X be treelike and $p, q, r \in X$. Then $S(p, q) \cap S(p, r) \cap S(q, r)$ is a singleton.*

PROOF: In BROUWER & SCHRIJVER [1] it is shown that this intersection is nonempty (see also [3] pp.45-50). On the other hand it is easily seen that it cannot contain more than one point:

Suppose $y, z \in S(p, q) \cap S(p, r) \cap S(q, r)$, $y \notin \{p, q, r, z\}$.

Then e.g. $X \setminus y = A + B + C$, but now $B \cup \{y\} \cup C$ is connected in $X \setminus z$ so that z doesn't separate q and r . The case that some or all of the points y, z, p, q, r coincide is handled similarly. \square

Let X be a treelike space and $p, q \in X$. $S(p, q)$ can be ordered in a natural way (separation order):

PROPOSITION 1.6. *The relation \leq on $S(p, q)$ defined by $x \leq y$ iff $x \in S(p, y)$ defines a continuous ordering on $S(p, q)$ (i.e. an ordering without jumps or gaps).*

PROPOSITION 1.7.

- (i) $S(p, q)$ is closed.
- (ii) If $S(p, q)$ is connected and locally connected, then it is compact.

In general $S(p, q)$ is not connected, but we have:

PROPOSITION 1.8. *Let X be treelike and either locally connected [WHYBURN] or locally peripherally compact [PROIZVOLOV]. Then for all $p, q \in X$, $S(p, q)$ is an ordered continuum, the intersection of all connected sets containing both p and q .*

Intuitively clear (and an easy consequence of proposition 1.5) is the following

PROPOSITION 1.9. [H. KOK, p.44]. *Let X be treelike and suppose that for each $p \in X$, $X \setminus p$ has at most two components. Then X is (weakly) orderable.*

A based treelike space is a triple (X, a, b) where X is a treelike space and a and b are (not necessarily different) points of X . As usual we identify a based treelike space with its underlying treelike space.

Let (X, a, b) be a based treelike space. The subset $S(a, b)$ of (X, a, b) is called its base. In a based treelike space we can define a canonical projection

$$\pi : X \rightarrow S(a, b)$$

by

$$\{\pi(x)\} = S(a, b) \cap S(a, x) \cap S(x, b).$$

(Well defined because of proposition 1.5)

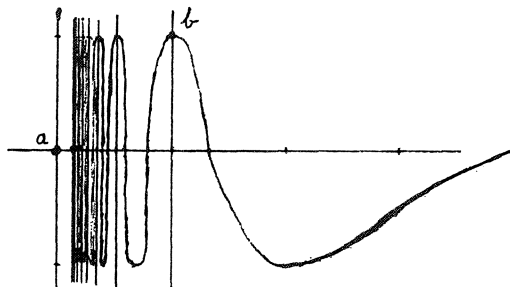
If $S(a, b)$ is not connected, then certainly π cannot be continuous, but we have:

PROPOSITION 1.10.

- (i) $\pi \circ \pi = \pi$
- (ii) If $c \in S(a, b)$ and $X \setminus c = A + B + C$ where A and B are connected (and empty when $a = c$ resp. $b = c$), then $\pi^{-1}(c) = \bar{C} = C \cup \{c\}$.
That is, $\pi^{-1}(c)$ is a closed connected set (called the stalk at c).
- (iii) If $S(a, b)$ is connected and locally connected then π is continuous. In particular this is the case when X is locally connected or locally peripherally compact.

PROOF OF (iii): Let $U \subset S(a, b)$ be open in $S(a, b)$. Since $S(a, b)$ is locally connected we may suppose U to be connected: $U = E(p, q)$ for some p, q with $a \leq p < q \leq b$. Now $\partial \pi^{-1}U = \{p, q\}$, so $\pi^{-1}U$ is open. \square

Note: connectedness of $S(a, b)$ alone is not sufficient:



Take $X = \{(x, y) \in \mathbb{R}^2 \mid (x > 0 \wedge y = \cos \frac{2\pi}{x}) \vee (\exists k \in \mathbb{N} : x = \frac{1}{k}) \vee x = y = 0\}$
with subspace topology of \mathbb{R}^2

$$a = (0, 0)$$

$$b = (1, 1)$$

Here π is not continuous.

2. THE CONSTRUCTION

THEOREM *Let X be a compact treelike space. Then X is the continuous image of an ordered continuum.*

This answers the question raised by V.V. PROIZVOLOV in [4].

PROOF. We will construct compact treelike spaces X_α and maps $p_{\alpha\beta} : X_\alpha \twoheadrightarrow X_\beta$ ($0 \leq \beta \leq \alpha \leq \alpha_0$) forming an inverse system, such that

$$X_{\alpha_0} = \lim_{\leftarrow} X_\alpha \text{ is an ordered continuum and } p_{\alpha_0 0} : X_{\alpha_0} \twoheadrightarrow X_0 = X$$

is the required mapping.

More in particular we will have based treelike spaces $(X_\alpha, a_\alpha, b_\alpha)$ and for all $\beta \leq \alpha : p_{\alpha\beta}^{-1}(a_\beta) = \{a_\alpha\}$, $p_{\alpha\beta}^{-1}(b_\beta) = \{b_\alpha\}$.

The idea is to construct the spaces X_α by transfinite induction, constantly enlarging the base, until finally $X_{\alpha_0} = S(a_{\alpha_0}, b_{\alpha_0})$. Assume that X contains more than one point.

(i) Step 0.

First set $X_0 = X$, $p_{00} = 1_X$ and choose two endpoints a_0 and b_0 from X . (This is possible because of the lemma: every nondegenerate compact connected T_1 space has at least two end points (see [5])).

(ii) A successor step.

Suppose X_β and $p_{\gamma\beta}$ defined for all ordinals $\gamma \leq \beta < \alpha$.

If α is a successor: $\alpha = \delta + 1$, and X_δ is orderable then we're done:

set $\alpha_0 = \delta$. Therefore suppose X_δ not orderable.

By proposition 1.9 we can find a ramification point p and by projecting it we may assume it to lie on $S(a_\delta, b_\delta)$, i.e. a point p such that $X_\delta \setminus p = A +_a B + C$ where A and B are clopen in $X_\delta \setminus p$ and C is non-empty, open and connected (use proposition 1.4).

$\bar{C} = C \cup \{p\}$ is compact and thus has an endpoint q different from p .

Define $Y_\alpha := (A \cup \{p\} \cup C) \times \{0\}$, $Z_\alpha := (S(q, p) \cup B) \times \{1\}$,

$$X_\alpha := (Y_\alpha + Z_\alpha) / R$$

where R identifies $(q, 0) \in Y_\alpha$ with $(q, 1) \in Z_\alpha$.

Define $p_{\alpha\alpha} = 1_{X_\alpha}$,

and $p_{\alpha\delta}((r,i)) = r \quad (r \in X_\delta, i=0,1)$;

and $p_{\alpha\beta} = p_{\delta\beta} \circ p_{\alpha\delta} \quad (\beta < \delta)$.

Now X_α is compact and treelike as is easily verified.

Note that the endpoints of X_α are exactly those of the form $p_{\alpha\delta}^{-1}(r)$ with r endpoint of X_δ different from q .

If we define $e_\alpha := p_{\alpha 0}((q,i)) = p_{\delta 0}(q) \in X$ then we can say that in this α -th step the endpoint e_α of X is removed, while the status of the other endpoints remains unchanged.

(iii) A limit step.

If α a limit ordinal, set $X_\alpha = \lim_{\leftarrow} \{X_\beta \mid \beta < \alpha\}$ and $p_{\alpha\beta} : X_\alpha \rightarrow X_\beta$ the canonical projection.

X_α is compact and connected (see e.g. Dugundji [2] p.435).

Claim: X_α is treelike.

Let $x, y \in X_\alpha$, $x \neq y$ and define $x_\beta := p_{\alpha\beta}(x)$, $y_\beta := p_{\alpha\beta}(y)$.

If for some $\beta < \alpha$ and $r \in X_\beta$ we have $p_{\alpha\beta}^{-1}(r) = \{s\}$ is a singleton, and r separates x_β and y_β , then s separates x and y :

$$X_\beta \setminus r = \begin{matrix} A & + & B \\ x_\beta & & y_\beta \end{matrix} \Rightarrow X_\alpha \setminus s = p_{\alpha\beta}^{-1}(X_\beta \setminus r) = \begin{matrix} p_{\alpha\beta}^{-1}(A) & + & p_{\alpha\beta}^{-1}(B) \\ x & & y \end{matrix} .$$

Therefore assume that if for some β r separates x_β and y_β then $|p_{\alpha\beta}^{-1}(r)| > 1$.

If for some $\beta < \alpha$ $\pi x_\beta \neq \pi y_\beta$ then choose $w \in E(\pi x_\beta, \pi y_\beta)$.

Let γ be the first ordinal $> \beta$ such that $|p_{\gamma\beta}^{-1}(w)| > 1$.

Then γ is a successor, and $|p_{\gamma\beta}^{-1}(w)| = 2$.

At the γ -th step of the construction some interval $S(q,w)$ is inserted in the base somewhere between πx_γ and πy_γ (Here $\{q\} = p_{\gamma-1}^{-1}(e_\gamma)$).

But this interval (or at least $E(q,w)$) is left untouched during all following steps since it doesn't contain any ramification points; that is, if $r \in E(q,w)$ then $|p_{\alpha\gamma}^{-1}(r)| = 1$, a contradiction.

Therefore for all $\beta < \alpha$ we have $\pi x_\beta = \pi y_\beta$.

Since $x \neq y$ for some $\beta < \alpha$ we have $x_\beta \neq y_\beta$.

Let w separate x_β and y_β . Let γ be the first ordinal $> \beta$ such that

$$|p_{\gamma\beta}^{-1}(w)| > 1.$$

Then $p_{\gamma\beta}^{-1}(w) = \{w_1, w_2\}$, and at least one of w_1 and w_2 separates x_γ and y_γ .

But both w_1 and w_2 lie in the base of X_γ , so we might as well assume

$$w \in S(a_\beta, b_\beta).$$

Now we have $w = \pi x_\beta = \pi y_\beta$.

Let $X_\beta \setminus w = P + Q + R$ where P and Q are connected.
 $\quad \quad \quad x_\beta \quad y_\beta \quad a_\beta, b_\beta$

Claim: For each $r \in P \cup Q$ $|p_{\alpha\beta}^{-1}(r)| = 1$.

For, let γ be the first ordinal $> \beta$ such that for some $r \in P$ $|p_{\gamma\beta}^{-1}(r)| > 1$.

Then at step γ in the construction we had w (or rather w' with

$\{w'\} = p_{\gamma-1, \beta}^{-1}(w)$) as a ramification point, while the endpoint q chosen lies in P .

Let $\{z\} = S(w, q) \cap S(q, x_{\gamma-1}) \cap S(x_{\gamma-1}, w) \subset X_{\gamma-1}$.

Then (with obvious identifications) $\pi x_\gamma = z \neq w'' = \pi y_\gamma$, contradiction.

From this claim it follows that $P' = p_{\alpha\beta}^{-1}[P] \subset X_\alpha$ is homeomorphic to P while $\partial P' = \{w'\}$ for some $w' \in p_{\alpha\beta}^{-1}(w)$.

Likewise $Q' = p_{\alpha\beta}^{-1}[Q]$ is mapped homeomorphically onto Q by $p_{\alpha\beta}$ while $\partial Q' = \{w''\}$. Since always $\pi x_\gamma = \pi y_\gamma$ we have $w' = w''$. But this means that

$X_\alpha \setminus w' = P' + Q' + R'$, which proves that X_α is indeed treelike.
 $\quad \quad \quad x \quad y \quad a_\alpha, b_\alpha$

Furthermore, if e is an endpoint of X_α then for all $\beta < \alpha$ $p_{\alpha\beta}(e)$ is an endpoint of X_β .

Since each successor step eliminates another of the endpoints of X , and a limit step doesn't introduce any new endpoints, there must be a last space X_{α_0} not containing any ramification points.

But this means that this X_{α_0} is orderable (proposition 1.9). \square

3. AN EXAMPLE

The foregoing seems to be a rather long proof for such an intuitively obvious construction. One has however to be careful since it is not generally true that the inverse limit of treelike spaces is again treelike:

Take an interval : $X_0 = C\{\frac{1}{2}\}$, the cone over $\frac{1}{2}$

split it into two : $X_1 = C\{\frac{1}{4}, \frac{3}{4}\}$

then split each half into two intervals:

$$X_2 = C\{\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}\}$$

etc.

Formally:

$$\text{Let } X_n = C\{\frac{k}{2^{n+1}} \mid k \text{ odd}, 1 \leq k \leq 2^{n+1}\} = \{\frac{k}{2^{n+1}}\} \times I / R$$

where R identifies all points $(x, 1)$.

$$\text{Let } p_{nm}(\frac{k}{2^{n+1}}, y) = (\frac{1}{2^{m+1}}, y) \text{ where } |\frac{k}{2^{n+1}} - \frac{1}{2^{m+1}}| < \frac{1}{2^{m+1}} \quad (m \leq n) .$$

Then $\varprojlim X_n$ is the cone over a Cantor set, and certainly not treelike (compact but not locally connected).

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