stichting
mathematisch
centrum

AFDELING ZUIVERE WISKUNDE ZW 34/74 DECEMBER

A.E. BROUWER & A. SCHRIJVER

A CHARACTERIZATION OF SUPERCOMPACTNESS WITH AN APPLICATION TO TREELIKE SPACES

2e boerhaavestraat 49 amsterdam

Printed at the Mathematical Centre, 49, 2e Boerhaavestraat, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O), by the Municipality of Amsterdam, by the University of Amsterdam, by the Free University at Amsterdam, and by industries.

AMS (MOS) subject classification scheme (1970): 54D30, 54F50

A characterization of supercompactness with an application to treelike spaces

by

A.E. Brouwer & A. Schrijver

ABSTRACT

The concept of interval structure is introduced and a characterization of supercompactness is given in terms of interval structures. This characterization is used to prove the supercompactness of a compact treelike space.

KEY WORDS AND PHRASES: *interval structure, supercompact, treelike.*

1. SUPERCOMPACTNESS

In this section we give definitions of supercompact spaces and interval structures and a criterion for supercompactness with help of interval structures.

DEFINITION. Let X be a set and $\mathcal{S}$ a subset of the powerset $\mathcal{P}(X)$. Then $\mathcal{S}$ is called *binary* if for each nonempty $\mathcal{S}' \subset \mathcal{S}$ with $\cap\mathcal{S}' = \emptyset$ there exist S_1 and S_2 in $\mathcal{S}$ such that $S_1 \cap S_2 = \emptyset$.

DEFINITION. A topological space X is called *supercompact* if there exists a binary closed subbase for X.

By ALEXANDER's lemma it can be easily seen that every supercompact space is compact.

DEFINITION. Let X be a set and $I\colon X \times X \to \mathcal{P}(X)$. Write $I(x,y) = I((x,y))$. Then I is called an *interval structure* on X if:

(i) $x,y \in I(x,y)$, $(x,y \in X)$,

(ii) $I(x,y) = I(y,x)$, $(x,y \in X)$,

(iii) if $u,v \in I(x,y)$ then $I(u,v) \subset I(x,y)$, $(u,v,x,y \in X)$,

(iv) $I(x,y) \cap I(x,z) \cap I(y,z) \neq \emptyset$, $(x,y,z \in X)$.

Axioms (i), (ii) and (iii) together can be replaced by the following axiom:

$$u,v \in I(x,y) \text{ iff } I(u,v) \subset I(y,x) \qquad (x,y,u,v \in X).$$

Examples of interval-structures:

a. if $(X,\leq)$ is a lattice, then $I(x,y) = \{z \mid x\wedge y \leq z \leq x\vee y\}$ defines an interval-structure;

b. if X is a treelike space, then $I(x,y) = \{z \mid z \text{ separates } x \text{ and } y\} \cup \{x,y\}$ defines an interval-structure (see section 3).

DEFINITION. Let I be an interval structure on the set X and $X' \subset X$. X' is I-*closed* if for each $x,y \in X'$ $I(x,y) \subset X'$.

THEOREM 1.1. *Let* X *be a topological space. Then:*
X *is supercompact if and only if* X *is compact and there exists a closed subbase* $\mathcal{S}$ *and an interval structure* I *such that every* $S \in \mathcal{S}$ *is* I-*closed.*

PROOF.

(a) Let X be a supercompact space and let $\mathcal{S}$ be a binary closed subbase for X. Define $I: X \times X \to \mathcal{P}(X)$ by

$$I(x,y) = \bigcap\{S \in \mathcal{S} \mid x,y \in S\}, \quad (x,y \in X).$$

Then I is an interval structure on X and each $S \in \mathcal{S}$ is clearly I-closed. To prove the former, we will only show that for each $x,y,z \in X$ $I(x,y) \cap I(x,z) \cap I(y,z) \neq \emptyset$. By the definition of I: $I(x,y) \cap I(x,z) \cap I(y,z) = \bigcap\{S \in \mathcal{S} \mid \{x,y,z\} \cap S$ contains two or more elements$\}$. Suppose this intersection is empty. Then, since $\mathcal{S}$ is binary, there exist S_1 and S_2 in $\mathcal{S}$ such that $\{x,y,z\} \cap S_1$ and $\{x,y,z\} \cap S_2$ both contain two or more elements and $S_1 \cap S_2 = \emptyset$, which is a contradiction.

(b) Conversely, let X be a compact space with closed subbase $\mathcal{S}$, and let I be an interval structure on X, such that each $S \in \mathcal{S}$ is I-closed. We prove that $\mathcal{S}$ is binary.
Let $\mathcal{S}' \subset \mathcal{S}$ be such that $\bigcap\mathcal{S}' = \emptyset$. Then, since X is compact, there exists a finite subset $\mathcal{S}_0' \subset \mathcal{S}'$ such that $\bigcap\mathcal{S}_0' = \emptyset$. Hence it is enough to prove the following: if $S_1,\dots,S_k \in \mathcal{S}$ and $S_1\cap\dots\cap S_k = \emptyset$ then there exist i,j $(1 \le i,j \le k)$ such that $S_i \cap S_j = \emptyset$.
We proceed by induction with respect to k. If k = 1 or 2 it is trivial. Suppose $k \ge 3$ and for each $k' < k$ the statement is true.
Define:

$$T_1 = \phantom{S_1 \cap{}} S_2 \cap S_3 \cap S_4\cap\dots\cap S_k,$$
$$T_2 = S_1 \cap \phantom{S_2\cap{}} S_3 \cap S_4\cap\dots\cap S_k,$$
$$T_3 = S_1 \cap S_2 \cap \phantom{S_3\cap{}} S_4\cap\dots\cap S_k.$$

If one of these is empty, then the induction hypothesis applies.
Suppose therefore $T_i \neq \emptyset$ $(i=1,2,3)$, and take $x \in T_1$, $y \in T_2$ and $z \in T_3$.
Then

$$x,y \in S_3 \cap S_4\cap\dots\cap S_k,$$
$$x,z \in S_2 \cap S_4\cap\dots\cap S_k,$$
$$y,z \in S_1 \cap S_4\cap\dots\cap S_k,$$

and thus

$$I(x,y) \subset S_3 \cap S_4\cap\dots\cap S_k,$$
$$I(x,z) \subset S_2 \cap S_4\cap\dots\cap S_k,$$

$I(y,z) \subset S_1 \cap S_4 \cap \ldots \cap S_k$,

But $I(x,y) \cap I(x,z) \cap I(y,z) \neq \emptyset$, so that

$(S_1 \cap S_4 \cap \ldots \cap S_k) \cap (S_2 \cap S_4 \cap \ldots \cap S_k) \cap (S_3 \cap S_4 \cap \ldots \cap S_k) =$
$= S_1 \cap S_2 \cap S_3 \cap S_4 \cap \ldots \cap S_k \neq \emptyset$.

This contradicts our hypothesis. □

For some related ideas see GILMORE [1].

2. TREELIKENESS

In this section we recall the definition of treelike spaces and mention some of their properties.

DEFINITION. A topological space X is called *treelike* if it is connected and for any two points x,y there is a point z separating x and y. Notation: $X\backslash z = \underset{x}{A} + \underset{y}{B}$ means that $X\backslash\{z\}$ can be written as the topological sum of two subspaces A and B, containing x and y respectively.

PROPOSITION 2.1. *A treelike space is Hausdorff.*

If X is treelike and $x,y \in X$ we set $E(x,y) := \{z \mid z \text{ separates } x \text{ and } y \text{ in } X\}$ and $S(x,y) := E(x,y) \cup \{x,y\}$.

PROPOSITION 2.2. *Let* X *be treelike and* $x,y \in X$. *Then* $S(x,y)$ *can be ordered in a natural way by setting* $x \leq y$ *and* $p < y$ *for* $p \in E(x,y)$ *and* $p < q$ *if* q *separates* p *and* y *for* $p,q \in E(x,y)$. *This order contains no jumps and no gaps.*

PROPOSITION 2.3. *If* X *is a treelike space and* $p \in X$ *then all components of* $X\backslash p$ *are open in* X.

PROPOSITION 2.4. *If* X *is treelike and either locally connected (cf.* WHYBURN [5]) *or locally peripherally compact (cf.* PROIZVOLOV [4]) *then for all* $x,y \in X$ $S(x,y)$ *is connected.*

The above results are well-known and can be found scattered through the literature in various forms. In many older papers separable metrizability is required. It seems that the paper of WHYBURN [5] was the first one explicitly dropping this condition. In KOK [2], a coherent account is given of the implications and interrelations of many properties of spaces, among them being treelikeness (which he calls property (*S*)). The following lemma from [1] will be needed in the next section.

LEMMA 2.5. *Let* X *be a connected topological space,* $C \subset X$ *connected,* S *a component of* $X \backslash C$. *Then* $X \backslash S$ *is connected.*

3. A COMPACT TREELIKE SPACE IS SUPERCOMPACT

In this section we first show that on each treelike space an interval structure can be defined, and next that a compact treelike space is supercompact.

PROPOSITION 3.1. *Let* X *be a treelike space. Then* $I(x,y) = S(x,y)$ *defines an interval structure on* X.

PROOF.

(i) $x,y \in S(x,y)$ by definition.

(ii) $S(x,y) = S(y,x)$ by definition.

(iii) If z separates x and y: $X\backslash z = \underset{x}{A} + \underset{y}{B}$ then $\bar{A} = A \cup \{z\}$ and $\bar{B} = B \cup \{z\}$ are both connected. Therefore if u separates x and z then $u \in A$ and B is contained in one component of $X\backslash u$, i.e. u separates x and y. This proves $z \in S(x,y) \Rightarrow S(x,z) \subset S(x,y)$.

(iv) Suppose $S(x,y) \cap S(y,z) \cap S(x,z) = \emptyset$. By definition $S(x,y) \subset S(y,z) \cup \cup S(x,z)$. Let $E := S(x,y) \cap S(y,z)$ and $F := S(x,y) \cap S(x,z)$. E and F are intervals in the order of $S(x,y)$ and $e > f$ for all $e \in E$, $f \in F$. Since $S(x,y)$ contains no gaps, either E contains a first, or F contains a last element. Suppose u is the first element of E. Now $S(y,z) \cup \{u\} = S(y,u) \cup S(u,z)$. (Because $v \in E(y,u) \Rightarrow v \in E(x,y)\backslash E(x,z) \Rightarrow v \in E(y,z)$ and conversely $v \in E(y,z)\backslash E(y,u) \Rightarrow v \in E(u,z)$.) But this would imply that $S(y,z) = S(y,u)\backslash u + S(u,z)\backslash u$ contained a gap. Contradiction. (Cf. H. KOK [2] pp.45-50). □

NOTE: We need this proposition only in the case that X is compact, in which

case a much shorter proof of (iv) can be given, namely:
Suppose $S(x,y) \cap S(y,z) \cap S(x,z) = \emptyset$. Since $S(p,q)$ is closed and connected for $p,q \in X$ we have $S(x,y) = S(x,y) \underset{y}{\cap} S(y,z) + S(x,y) \underset{x}{\cap} S(x,z)$, a contradiction.

THEOREM 3.2. *Let* X *be a compact treelike space. Then* X *is supercompact.*

PROOF. Using theorem 1.1 and proposition 3.1 it is sufficient to exhibit a closed subbase $\mathcal{S}$ consisting of connected sets.
Claim: $\mathcal{S} := \{X \setminus C \mid p \in X,\ C \text{ component of } X \setminus p\}$ is such a subbase.
First, by proposition 2.3 each $S \in \mathcal{S}$ is closed. Next, by lemma 2.5 each $S \in \mathcal{S}$ is connected. If $x,y \in X$ and p separates x and y : $X \setminus p = A + B +$ + other components, where A and B are connected, then A and B are disjoint neighbourhoods of x and y in the topology $\mathcal{T}$ generated by $\mathcal{S}$, which is therefore Hausdorff. Since this topology is weaker than the original compact topology on X, both topologies coincide. □

This last result has been proved independently (using a different method) by J. VAN MILL [3].

REFERENCES

[1] GILMORE, P.C., *Families of sets with faithful graph representations.* Report Thomas J. Watson Research Center, Yorktown Heights, 1962.

[2] KOK, H., *Connected orderable spaces.* MC Tract 49, Amsterdam, 1973.

[3] MILL, J. VAN, *A topological characterization of products of compact treelike spaces.* (in preparation).

[4] PROIZVOLOV, V.V., *On peripherally bicompact treelike spaces.* Soviet Math. Dokl. 10 (1969) no. 6 pp. 1491-1493.

[5] WHYBURN, G.T., *Cut points in general topological spaces.* Proc. Nat. Acad. Sci. USA 61 (1968) pp.380-387.