stichting
mathematisch
centrum

DEPARTMENT OF PURE MATHEMATICS ZW 47/76 MAY

J. DE VRIES

ON THE EXISTENCE OF G-COMPACTIFICATIONS

Prepublication

2e boerhaavestraat 49 amsterdam

Printed at the Mathematical Centre, 49, 2e Boerhaavestraat, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O), by the Municipality of Amsterdam, by the University of Amsterdam, by the Free University at Amsterdam, and by industries.

AMS(MOS) subject classification scheme (1970): 54H15, 54D35

On the existence of G-compactifications*)

by

J. de Vries

ABSTRACT

The main result of this paper is that every Tychonoff G-space has a G-compactification provided the topological group G is locally compact. As an application we improve an embedding theorem of CARLSON.

KEY WORDS & PHRASES: G-*space, equivariant embedding, compactification.*

---

*) This paper is not for review; it is meant for publication elsewhere.

1. INTRODUCTION

In this paper we contribute to the solution of the following problem: *Can every topological transformation group* $\langle G,X,\pi\rangle$ *with* X *a Tychonoff space equivariantly be embedded in a topological transformation group* $\langle G,Y,\sigma\rangle$ *with* Y *a compact Hausdorff space? If so, how "small" can* Y *be chosen?* Several authors have worked earlier on this problem. For the case of discrete groups (where only the second part of the problem is non-trivial) we refer to [6], [2] (section 3.4), and [1]. For the case $G = \mathbb{R}$ we mention [4], and for more general groups,[3], [11] and [13]. For a categorical motivation of this question, see [11] or [12], the final remarks in 4.3.13. Our result improves all earlier results by showing that the answer is affirmative for every locally compact group G, no matter what the Tychonoff space X and the action π look like, and that one can choose the compact Hausdorff space Y such that $w(Y) \leq \max\{w(G), w(X)\}$. Here $w(Z)$ denotes the weight of the topological space Z.

As a typical application of our result we mention the following. In [5], Theorem 1, CARLSON describes a dynamical system (τ, C_V^{∞}) which is universal for the class of all dynamical systems (π, X) on separable metrizable spaces X such that the action π is, what he calls, *bounded* (in the sequel, we shall call his notion of boundedness: *metrical* boundedness). A consequence of the main result of our paper is that the condition of being bounded is superfluous in CARLSON's theorem: any dynamical system with a separable metrizable phase space is metrically bounded (cf. 4.1 below). No reparametrization is needed, as was suggested at the end of [5].

2. PRELIMINARIES

Notation will be as in [12], but for convenience we recall here some definitions and terminology. A *topological transformation group* (ttg) or a G-space is a triple $\langle G,X,\pi\rangle$ where G is a topological group, X is a topological space, and π is a (left) *action* of G on X, that is, $\pi : G \times X \to X$ is a continuous function, $\pi(e,x) = x$ and $\pi(t,\pi(s,x)) = \pi(ts,x)$ for every $x \in X$ and $s,t \in G$ (e denotes the identity of G). The *transitions* $\pi^t : X \to X$

and the *motions* $\pi_x : G \to X$ are defined by $\pi^t x := \pi(t,x) =: \pi_x t$ for $(t,x) \in G \times X$. If $<G,X,\pi>$ and $<G,Y,\sigma>$ are G-spaces, then a function $f : X \to Y$ is called *equivariant* whenever $f \circ \pi^t = \sigma^t \circ f$ for every $t \in G$. If f is an equivariant dense topological embedding of X into Y, and Y is a compact Hausdorff space, then we call $<G,Y,\sigma>$ a *G-compactification of* $<G,X,\pi>$. A necessary condition for $<G,X,\pi>$ to have a G-compactification is that X is a Tychonoff space and from now on we assume that X is such a space. Recall that X admits a uniformity which is compatible with the topology of X (shortly: a uniformity *for* X). We shall call a ttg $<G,X,\pi>$ *$\mathcal{U}$-bounded* or *bounded w.r.t.* $\mathcal{U}$ whenever $\mathcal{U}$ is a uniformity for X and

$$\forall \alpha \in \mathcal{U},\ \exists U \in \mathcal{V}_e \,\vdots\, (\pi^t x, x) \in \alpha \quad \text{for all } t \in U,\ x \in X$$

(here $\mathcal{V}_e$ denotes the neighbourhood filter of e in G). In this case we shall also call the action π of G on X bounded w.r.t. $\mathcal{U}$ or $\mathcal{U}$-bounded. The relevance of the notion of boundedness for the problem of the existence of G-compactifications is immediate from the following result, which generalizes Theorem 3.1(1) of [3]:

2.1. THEOREM. *Let* $<G,X,\pi>$ *be a ttg with* X *a Tychonoff space. The following conditions are equivalent:*

(i) *There exists a* G-*compactification* $<G,Y,\sigma>$ *of* $<G,X,\pi>$.

(ii) *The action* π *of* G *on* X *is bounded w.r.t. some uniformity* $\mathcal{U}$ *for* X.

If these conditions are fulfilled, then $<G,X,\pi>$ *has a* G-*compactification* $<G,Y_0,\sigma_0>$ *such that*

$$(*) \qquad w(Y_0) \le \max\{w(G), w(X)\}.$$

PROOF. Cf. [12], Proposition 7.3.12. □

2.2. REMARK. The proof of (i) ⇒ (ii) is easy. A quick proof of the converse imlication (ii) ⇒ (i), using Theorem 3.1 of [3], is included in [13]; the difference between the result in [3] and ours is, that in [3] condition (ii) is replaced by the stronger condition

(ii)' The action π is bounded w.r.t. some uniformity $\mathcal{U}$ for X and each π^t ($t \in G$) is a unimorphism w.r.t. $\mathcal{U}$.

In addition, [3] and [13] contain no proof of the inequality (*).

3. MAIN RESULT

Throughout this section $\langle G,X,\pi\rangle$ shall denote a ttg with G an arbitrary locally compact Hausdorff group and X a Tychonoff space. The basic result is:

3.1. PROPOSITION. *There exists a uniformity $\mathcal{U}$ for* X *such that the action* π *of* G *on* X *is $\mathcal{U}$-bounded.*

<u>PROOF</u>. It is sufficient to prove that there exists a set $\{g_i : i \in I\}$ of continuous, [0,1]-valued functions on X satisfying the conditions

(1) $\{g_i : i \in I\}$ separates points and closed subsets of X, i.e. for any closed subset A of X and $x \in X \sim A$ there exists $i \in I$ with $g_i(x) \notin \mathrm{cl}\, g_i[A]$.

(2) $\forall i \in I : \{g_i \circ \pi_x : x \in X\}$ is equicontinuous at e.

Indeed, if we have such a family, let $\mathcal{U}$ denote the weakest uniformity on X making every $g_i : X \to [0,1]$ uniformly continuous. Then the topology generated by $\mathcal{U}$ coincides with the weakest topology on X, making every g_i continuous. By (1), this topology is just the original topology of X, hence $\mathcal{U}$ is a uniformity for X. In addition, π is $\mathcal{U}$-bounded. For if $\alpha \in \mathcal{U}$, then there are a finite subset I_α of I and a real number $\varepsilon > 0$ such that for all $(x,y) \in X \times X$:

$$|g_i(x) - g_i(y)| < \varepsilon \quad \text{for all } i \in I_\alpha \quad \Rightarrow \quad (x,y) \in \alpha.$$

In view of (2), there is for every $i \in I$ a neighbourhood U_i of e in G such that $|g_i(\pi(t,x)) - g_i(x)| < \varepsilon$ for every $t \in U_i$ and $x \in X$. Hence $(\pi(t,x),x) \in \alpha$ for all $t \in \bigcap\{U_i : i \in I_\alpha\}$ and $x \in X$, as desired.

We shall demonstrate now that a family $\{g_i : i \in I\}$ with properties (1) and (2) exists. The proof will be interrupted by several lemmas.

<u>3.2. LEMMA</u>. *There exists a set* Φ *of left uniformly continuous, real valued functions on* G *such that*

(3) $\forall\varphi \in \Phi : \varphi(e) = 0 \ \& \ \varphi(t) \geq 0 \ (t\in G)$;

(4) $\forall t \in G \ [t \neq e \ \Rightarrow \ \exists\varphi \in \Phi \ \vdots \ \varphi(t) > 0)]$;

(5) $\forall\varphi \in \Phi$: the set $A_\varphi := \{t \in G \ \vdots \ \varphi(t) \leq 2\}$ *is a compact subset of* G.

<u>PROOF</u>. Since G is a locally compact Hausdorff group, there exists a local base $\mathcal{B}$ at e such that cl U is compact for every $U \in \mathcal{B}$. Choose for every $U \in \mathcal{B}$ a continuous function $\varphi_U : G \to [0,3]$ such that $\varphi_U(e) = 0$ and $\varphi_U(t) = 3$ if $t \in G \sim U$. Now take $\Phi := \{\varphi_U \ \vdots \ U \in \mathcal{B}\}$. Since for each $U \in \mathcal{B}$, φ_U is constant outside a compact set (viz.cl U) it is clear that φ_U is left uniformly continuous. □

3.3. If Φ is as above and if for $(n,\varphi) \in \mathbb{N} \times \Phi$ we define

$$U_{n,\varphi} := \{t \in G \ \vdots \ \varphi(t) \leq 1/n\},$$

then $U_{n,\varphi}$ is a compact neighbourhood of e in G, and

$$\cap\{U_{n,\varphi} \ \vdots \ (n,\varphi) \in \mathbb{N} \times \Phi\} = \{e\}.$$

It follows easily that $\{U_{n,\varphi} \ \vdots \ (n,\varphi) \in \mathbb{N} \times \Phi\}$ *is a local base at* e (see e.g.[8], the proof of 8.5, which can easily be adepted to the present situation). In particular, if Φ is countable, then G is metrizable (cf.[8],8.3). Conversely, if G is metrizable, one can choose Φ such that it contains only one element: set $\varphi(t) := d(e,t)$ $(t\in G)$ where d is a left invariant metric for G such that $\{t \in G \ \vdots \ d(e,t) \leq 2\}$ is compact.

3.4. Fix a set Φ as indicated in 3.2. For every $f \in C(X,[0,1])$ and $\varphi \in \Phi$, a real-valued function $\tilde{f}_\varphi$ on X can be defined by

(6) $$\tilde{f}_\varphi(x) := \inf_{t \in G} \{\varphi(t) + f(\pi^t x)\}$$

for $x \in X$. Incidentally, this definition and the lemmas 3.5 and 3.7 below are motivated by Lemma 7 in [7].

3.5. LEMMA. *The functions* $\tilde{f}_\varphi$ *(*$f \in C(X,[0,1])$ *and* $\varphi \in \Phi$*) map* X *continuously into the interval* $[0,1]$.

PROOF. Clearly, $0 \leq \tilde{f}_\varphi(x) \leq \varphi(e) + f(x) = f(x) \leq 1$ for every $x \in X$. So we need only to prove continuity of $\tilde{f}_\varphi$. In order to do so, first observe that for every $t \in G$ with $\varphi(t) \geq 2$ we have $\varphi(t) + f(\pi^t x) \geq 2 > 1 \geq \tilde{f}_\varphi(x)$. Consequently, with A_φ as defined in (5), we have

$$\tilde{f}_\varphi(x) = \inf_{t \in A_\varphi} \{\varphi(t) + f(\pi^t x)\}. \tag{7}$$

However, the function $t \mapsto \varphi(t) + f(\pi^t x) : A_\varphi \to \mathbb{R}$ is continuous and A_φ is compact. Hence the infinum in (7) is not only actually attained at some point $t_x \in A_\varphi$ but it follows also that $\tilde{f}_\varphi$ is continuous, as is well-known and easy to prove. □

3.6. LEMMA. *If* $f(x) = 0$ *then* $\tilde{f}_\varphi(x) = 0$ *for every* $\varphi \in \Phi$. *If* $f(x) > 0$ *then there exists* $\varphi \in \Phi$ *such that* $\tilde{f}_\varphi(x) > 0$.

PROOF. If $f(x) = 0$, then the inequalities $0 \leq \tilde{f}_\varphi(x) \leq f(x)$ (cf. the proof of 3.5) imply that $\tilde{f}_\varphi(x) = 0$. If $f(x) > 0$, then there is $U \in \mathcal{V}_e$ such that $f(\pi^t x) > \frac{1}{2}f(x)$ for all $t \in U$. By 3.3, there exists $\varphi \in \Phi$ and $n \in \mathbb{N}$ such that $U_{n,\varphi} \subseteq U$. We may and shall assume that $1/n \leq \frac{1}{2}f(x)$. Then we have for every $t \in G$, $\varphi(t) + f(\pi^t x) > 1/n$, whence $\tilde{f}_\varphi(x) \geq 1/n > 0$. □

3.7. LEMMA. *For every* $f \in C(X,[0,1])$ *and* $\varphi \in \Phi$, *the family* $\{\tilde{f}_\varphi \circ \pi_x : x \in X\}$ *is equicontinuous at* e.

PROOF. Fix f and φ as indicated. For every $(t,x) \in G \times X$ we have

$$\begin{aligned} \tilde{f}_\varphi(\pi(t,x)) &= \inf_{s \in G} \{\varphi(s) + f(\pi(st,x))\} \\ &= \inf_{u \in G} \{\varphi(ut^{-1}) - \varphi(u) + \varphi(u) + f(\pi^u x)\} \\ &\geq \inf_{u \in G} \{\varphi(ut^{-1}) - \varphi(u)\} + \tilde{f}_\varphi(x) \end{aligned}$$

Since φ is left uniformly continuous on G, there is for every $\varepsilon > 0$ a neighbourhood U_ε of e in G such that $|\varphi(ut^{-1}) - \varphi(u)| < \varepsilon$ for all $t \in U_\varepsilon$ and $u \in G$.

Consequently, $\tilde{f}_\varphi(\pi(t,x)) \geq \tilde{f}_\varphi(x) - \varepsilon$ for all $t \in U_\varepsilon$ and all $x \in X$. Similarly, there is $V_\varepsilon \in \mathcal{V}_e$ such that $\tilde{f}_\varphi(x) \geq \tilde{f}_\varphi(\pi(t,x)) - \varepsilon$ for all $t \in V_\varepsilon$ and all $x \in X$. Hence

$$(8) \qquad |\tilde{f}_\varphi(\pi(t,x)) - \tilde{f}_\varphi(x)| < \varepsilon$$

for every $t \in U_\varepsilon \cap V_\varepsilon$ and every $x \in X$. □

3.8. In the preceding proof we have shown a little bit more than was actually needed; namely, if $t \in U_\varepsilon \cap V_\varepsilon$ then (8) holds not only uniformly in $x \in X$, but also uniformly in $f \in C(X,[0,1])$. Hence $\{\tilde{f}_\varphi \circ \pi_x : x \in X \;\&\; f \in C(X,[0,1])\}$ is equicontinuous at e. However, the statement of lemma 3.7 is sufficient for our purposes.

3.9. PROOF OF 3.1 (continued). Consider the family $\{\tilde{f}_\varphi : (\varphi,f) \in \Phi \times C(X,[0,1])\}$. By 3.5, this is a set of continuous, $[0,1]$-valued functions, and it is easy to see that it satisfies condition (1) (use lemma 3.6 and the fact that for any closed set $A \subseteq X$ and any point $x \in X \sim A$ there is $f \in C(X,[0,1])$ with $f(x) = 1$ and $f[A] = \{0\}$). In addition, our family fulfills condition (2): this is exactly lemma 3.7. □

3.10 THEOREM. *Any ttg* $\langle G,X,\pi\rangle$ *with* G *a locally compact Hausdorff group and* X *a Tychonoff space has a* G*-compactification* $\langle G,Y,\sigma\rangle$. *Moreover, one may assume that*

$$w(Y) \leq \max\{w(G), w(X)\}.$$

PROOF. Combine 2.1 and 3.1. □

3.11. The restriction that G is Hausdorff can be omitted from 3.1 and 3.10 This can be seen as follows. Suppose we are given a ttg $\langle H,X,\pi'\rangle$ with H locally compact but not Hausdorff, and X a Tychonoff space. Then the stability subgroup $H_0 := \{t \in H : \pi'(t,x) = x$ for every $x \in X\}$ is a closed normal subgroup of H, hence $G := H/H_0$ is a locally compact *Hausdorff* topological group. Let π denote the naturally induced action of G on X. Then theorem 3.10 can be applied to $\langle G,X,\pi\rangle$ so as to produce a G-compactification

$\langle G,Y,\sigma\rangle$ of $\langle G,X,\pi\rangle$. If $\psi\colon H \to G$ is the quotient mapping, then an action σ^{ψ} of H on Y can be defined by $\sigma^{\psi}(t,y) := \sigma(\psi(t),y)$ for $(t,y) \in H \times Y$. It is plain that now $\langle H,Y,\sigma^{\psi}\rangle$ is the desired H-compactification of $\langle H,X,\pi'\rangle$..

4. AN APPLICATION

In [5], a dynamical system (that is, an $\mathbb{R}$-space in our terminology) is described, which is defined by a Cauchy problem for an autonomous partial differential equation and which has the following property: every "bounded" dynamical system on a separable metrizable space can equivariantly be embedded in this "universal" system. However, the notion of boundedness which occurs in [5] differs slightly from ours, and we shall call it therefore *metrical* boundedness. Here is the definition: a ttg $\langle G,X,\pi\rangle$ with X a metrizable space is called *metrically bounded w.r.t. a metric* d provided it is bounded w.r.t. the uniformity $\mathcal{U}_d$ which corresponds with d. Here the situation is somewhat subtle: a bounded action on a metrizable space X (w.r.t. some uniformity $\mathcal{U}$ for X) may be *not* metrically bounded w.r.t. any metric d for X, even if the acting group is a separable locally compact group (cf.[10], p.110, where an example is given with a σ-compact locally compact Hausdorff group G; if we take in that example for the index set A a set with the cardinality of the continuum, we obtain a separable group: a product of continuously many separable spaces is still separable). However, if G is a σ-compact locally compact Hausdorff group (in particular, if $G = \mathbb{R}$) and X is a *separable* metrizable space, then boundedness of $\langle G,X,\pi\rangle$ w.r.t. some uniformity $\mathcal{U}$ implies metric boundedness of $\langle G,X,\pi\rangle$ w.r.t. some metric d.

For a proof of this fact in its full generality, we refer to [10], Corollary 4.11, or to [12], 7.3.14. For the special case of $G = \mathbb{R}$ we present here a quick proof:

<u>4.1. PROPOSITION</u>. *Every ttg* $\langle \mathbb{R},X,\pi\rangle$ *with* X *a separable metrizable space is metrically bounded w.r.t. some metric* d *for* X.

<u>PROOF</u>. According to 3.10, the ttg $\langle \mathbb{R},X,\pi\rangle$ has an $\mathbb{R}$-compactification $\langle \mathbb{R},Y,\sigma\rangle$ with $w(Y) \le \max\{w(\mathbb{R}), w(X)\} = \aleph_0$. Hence Y is metrizable. Clearly, the action of σ of $\mathbb{R}$ on Y is bounded w.r.t. any metric d for Y, hence the

action π of $\mathbb{R}$ on X is bounded w.r.t. the restriction of d to X. □

4.2. COROLLARY. *Every ttg* $<\mathbb{R},X,\pi>$ *with* X *a separable metrizable space can equivariantly be embedded in* CARLSON'*s universal system* $<\mathbb{R},C_V^\infty,\tau>$.

PROOF. Use proposition 4.1 above and [5], Theorem 1. □

4.3. REMARK. In [5], boundedness is used only in order to prove that the equivariant embedding mapping F constructed there is actually a relatively open mapping: for injectivity and continuity of F no boundedness condition is needed. Hence a different proof of 4.2 can be given as follows: if $<\mathbb{R},X,\pi>$ is a ttg with X a separable metrizable space, then there is an $\mathbb{R}$-compactification $<\mathbb{R},Y,\sigma>$ with Y compact metrizable and also separable (cf. the proof of 4.1). Apply CARLSON's proof to $<\mathbb{R},Y,\sigma>$; note that a continuous injection F of Y into C_V^∞ is automatically a topological embedding (Y is compact and C_V^∞ is Hausdorff). Hence the restriction of F to X is a topological embedding of X in C_V^∞.

A similar application of theorem 3.10 to another embedding problem is the following one: Let G be an infinite σ-compact, locally compact Hausdorff group. In [9] we constructed a linear action π of G on the Hilbert space $L^2(G\times G)$ such that every bounded ttg $<G,X,\sigma>$ with X a separable metrizable space can equivariantly be embedded in $<G,L^2(G\times G),\pi>$. By 3.10, we can remove here the boundedness condition as well, provided G is second countable, i.e. separable and metrizable. For such groups G we infer that every separable metrizable G-space can equivariantly be embedded in the Hilbert G-space $<G,L^2(G\times G),\pi>$.

Further applications of 3.10 will be published in the future.

REFERENCES

[1] ANDERSON, R.D., Universal and quasi-universal flows, *Topological Dynamics* (Intern. Symp., Colorado State Univ., Ft Collins, 1967), Benjamin, New York, 1968, pp. 1-16.

[2] BAAYEN, P.C., *Universal morphisms*, Mathematical Centre Tracts no. 9, Mathematisch Centrum, Amsterdam, 1964.

[3] BROOK, R.B., A construction of the greatest ambit, *Math. Systems Theory* 4 (1970), 243-248.

[4] CARLSON, D.H., Extensions of dynamical systems via prolongations, *Funkcial. Ekvac.* 14 (1971), 35-46.

[5] ———, Universal dynamical systems, *Math. Systems Theory* 6 (1972), 90-95.

[6] GROOT, J. DE & R.H. McDOWELL, Extension of mappings on metric spaces, *Fund. Math.* 68 (1960), 251-263.

[7] HAJEK, O., Representation of dynamical systems, *Funkcial. Ekvac.* 14 (1971), 25-34.

[8] HEWITT, E. & K.A. ROSS, *Abstract Harmonic Analysis* I, Springer- Verlag, Berlin, Heidelberg, New York, 1963.

[9] VRIES, J. DE, A universal topological transformation group in $L^2(G\times G)$. *Math. Systems Theory* 9 (1975), 46-50.

[10] ———, Universal topological transformation groups, *General Topology and Appl.* 5 (1975), 107-122.

[11] ———, *Can every Tychonoff* G-*space equivariantly be embedded in a compact Hausdorff* G-*space?* Math. Centrum Amsterdam, Afd. Zuivere Wisk., ZW 36, 1975.

[12] ———, *Topological Transformation Groups I (A categorical approach)*, Mathematical Centre Tracts no 65, Mathematisch Centrum, Amsterdam, 1975.

[13] ———, *A note on compactifications of* G-*spaces*, Math. Centrum, Amsterdam, Afd. Zuivere Wisk., ZW 61, 1976 (preprint).