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J. VAN DE LUNE

A NOTE ON RIEMANN'S ZETA-FUNCTION

2e boerhaavestraat 49 amsterdam

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A note on Riemann's zeta-function

by

J. van de Lune

ABSTRACT

This note deals with the question whether a function of the form $f(t) = \int_X \phi(x) \cos(t\psi(x))\, d\mu(x)$ has infinitely many zeros.

As an application of the technique developped it is shown that the imaginary part of $\zeta(1+it)$ has infinitely many zeros.

KEY WORDS & PHRASES: *Fourier transform, Laplace transform, Riemann zeta-function, zeros.*

0. INTRODUCTION

The subject of this note was inspired by the following observation. Let

(1) $$f(t) = \sum_{n=1}^{N} a_n \cos(t\lambda_n), \qquad (t \in \mathbb{R})$$

where all a's and λ's are *real* and different from zero. Clearly f is continuous and bounded so that we may consider the (one-sided) Laplace transform $\check{f}$ of f for s > 0:

(2) $$\check{f}(s) = \int_0^\infty e^{-st} f(t)dt = \int_0^\infty e^{-st} \sum_{n=1}^{N} a_n \cos(t\lambda_n)dt =$$

$$= \sum_{n=1}^{N} a_n \int_0^\infty e^{-st} \frac{e^{it\lambda_n} + e^{-it\lambda_n}}{2} dt =$$

$$= \tfrac{1}{2} \sum_{n=1}^{N} a_n \left\{\frac{1}{s - i\lambda_n} + \frac{1}{s + i\lambda_n}\right\} .$$

Differentiating k times with respect to s we obtain

(3) $$\int_0^\infty e^{-st} t^k f(t)dt = \frac{k!}{2} \sum_{n=1}^{N} a_n \left\{\frac{1}{(s-i\lambda_n)^{k+1}} + \frac{1}{(s+i\lambda_n)^{k+1}}\right\},$$

from which it is clear that for all $k \in \mathbb{N}$

(4) $$\lim_{s\downarrow 0} \int_0^\infty e^{-st} t^k f(t)dt = \frac{k!}{2i^{k+1}} \sum_{n=1}^{N} a_n \frac{(-1)^{k+1} + 1}{\lambda_n^{k+1}} .$$

Taking k even it follows that

(5) $$\lim_{s\downarrow 0} \int_0^\infty e^{-st} t^{2m} f(t)dt = 0, \qquad (\forall m \in \mathbb{N}) .$$

From (5) it is intuitively clear that f(t) must have infinitely many (real) zeros.

A formal proof may be carried out as follows.

Suppose that (the even function) $f(t)$ has only finitely many zeros (possibly none). Then $f(t)$ is eventually of fixed sign and without loss of generality we may assume that

$$f(t) > 0 \quad \text{for} \quad t \geq t_0 > 0. \tag{6}$$

Define

$$M = \max_{0 \leq t \leq t_0} |f(t)| \tag{7}$$

and

$$\mu = \min_{t_0 \leq t \leq t_0+1} |f(t)| \tag{8}$$

so that $\mu > 0$.

Now choose $m \in \mathbb{N}$ such that

$$\mu\left(1 + \frac{1}{t_0}\right)^{2m+1} - (\mu+M) > 0. \tag{9}$$

Then, for any $s > 0$, we have

$$\int_0^\infty e^{-st} t^{2m} f(t)dt = \left\{ \int_0^{t_0} + \int_{t_0}^{t_0+1} + \int_{t_0+1}^{\infty} \right\} e^{-st} t^{2m} f(t)dt > \tag{10}$$

$$> -M \int_0^{t_0} e^{-st} t^{2m} dt + \mu \int_{t_0}^{t_0+1} e^{-st} t^{2m} dt$$

and letting s tend to zero it follows that

$$0 \geq \mu \int_{t_0}^{t_0+1} t^{2m} dt - M \int_0^{t_0} t^{2m} dt = \mu \frac{(t_0+1)^{2m+1} - t_0^{2m+1}}{2m+1} - M \frac{t_0^{2m+1}}{2m+1} = \tag{11}$$

$$= \frac{t_0^{2m+1}}{2m+1} \left\{\mu\left(1 + \frac{1}{t_0}\right)^{2m+1} - (\mu+M)\right\} > 0$$

which is a contradiction, proving that f(t) must have infinitely many zeros.

In section 1 it will be shown that the technique illustrated above may be generalized considerably and in section 2 we will describe some consequences, the main one being the remarkable fact that the imaginary part of $\zeta(1 + it)$, $(t \in \mathbb{R})$, has infinitely many zeros.

SECTION 1

THEOREM 1. *Let* X *be a locally compact Hausdorff space equipped with a (non-negative) regular measure* μ *such that* X *is sigma- finite with respect to* μ *and* $\mu(K) < \infty$ *for any compact subset* K *of* X.

Let the μ*-measurable functions* $\phi, \psi : X \to \mathbb{R}$ *be such that*

(i) $\phi \in L^1(X,\mu)$

(ii) ψ *is bounded on compact subsets of* X

(iii) *there exists a* μ*-measurable subset* S *of* X *such that*

$$(12) \qquad \int_X |\phi| \, d\mu = \int_S |\phi| \, d\mu$$

$$(13) \qquad \inf_{x \in S} |\psi(x)| > 0$$

Then the function $f: \mathbb{R} \to \mathbb{R}$, *defined by*

$$(14) \qquad f(t) = \int_X \phi(x) \cos(t\psi(x)) \, d\mu(x), \qquad (t \in \mathbb{R})$$

has infinitely many (real) zeros.

PROOF. First of all we note that f is well defined indeed. Next, we observe that f is bounded and continuous on $\mathbb{R}$. It is clear that

$$(15) \qquad |f(t)| \le \int_X |\phi| \, d\mu$$

so that

(16) $$\|f\|_\infty \leq \|\phi\|_1 < \infty$$

In order to see that f is continuous we recall that the set of all continuous complex valued functions on X, having compact support, is dense in $L^1(X,\mu)$. Given $\varepsilon > 0$ we may therefore choose a continuous $\phi^*: X \to \mathbb{C}$ with compact support K such that

(17) $$\int_X |\phi - \phi^*| \, d\mu < \varepsilon.$$

Let

(18) $$M = \sup_{x \in K} |\psi(x)|.$$

In order to prove the continuity of f we may just as well prove that $\tilde{f}: \mathbb{R} \to \mathbb{C}$ defined by

(19) $$\tilde{f}(t) = \int_X \phi(x) \, e^{it\psi(x)} d\mu(x), \qquad (t \in \mathbb{R})$$

is continuous on $\mathbb{R}$. For any $t_1, t_2 \in \mathbb{R}$ we have

(20) $$|\tilde{f}(t_1) - \tilde{f}(t_2)| = \left| \int_X \phi(x) \, \{e^{it_1\psi(x)} - e^{it_2\psi(x)}\} d\mu(x) \right| =$$

$$= \left| \int_X \{\phi(x) - \phi^*(x)\} \, \{e^{it_1\psi(x)} - e^{it_2\psi(x)}\} d\mu(x) \right. +$$

$$+ \left. \int_X \phi^*(x) \, \{e^{it_1\psi(x)} - e^{it_2\psi(x)}\} d\mu(x) \right| \leq$$

$$\leq 2 \int_X |\phi - \phi^*| d\mu + \int_K |\phi^*(x)| \, |e^{i(t_1 - t_2)\psi(x)} - 1| d\mu(x) \leq$$

$$\leq 2 \cdot \varepsilon + |t_1 - t_2| \, M \cdot e^{|t_1 - t_2| M} \int_K |\phi^*| d\mu$$

from which it is clear that $\tilde{f}$, and hence f, is (uniformly) continuous on $\mathbb{R}$. Since f is continuous and bounded we may consider the Laplace transform $\check{f}$ of f for $s > 0$:

$$(21) \qquad \check{f}(s) = \int_0^\infty e^{-st} f(t)dt = \int_0^\infty e^{-st} \int_X \phi(x) \cos(t\psi(x))\, d\mu(x)\, dt.$$

The conditions in our theorem imply (c.f. [1]) that we may apply Fubini's theorem to the repeated integral in (21) so that

$$(22) \qquad \int_0^\infty e^{-st} f(t)dt = \int_X \phi(x) \int_0^\infty e^{-st}\cos(t\psi(x))dt\, d\mu(x) =$$

$$= \int_X \phi(x) \int_0^\infty e^{-st} \frac{e^{it\psi(x)} + e^{-it\psi(x)}}{2}\, dt\, d\mu(x) =$$

$$= \tfrac{1}{2}\int_X \phi(x) \left\{\frac{1}{s - i\psi(x)} + \frac{1}{s + i\psi(x)}\right\} d\mu(x) =$$

$$= \tfrac{1}{2}\int_S \phi(x) \left\{\frac{1}{s - i\psi(x)} + \frac{1}{s + i\psi(x)}\right\} d\mu(x).$$

As before, differentiate k times with respect to s in order to obtain

$$(23) \qquad \int_0^\infty e^{-st} t^k f(t)dt = \frac{k!}{2}\int_S \phi(x) \left\{\frac{1}{(s-i\psi(x))^{k+1}} + \frac{1}{(s+i\psi(x))^{k+1}}\right\} d\mu(x).$$

Taking limits for $s \downarrow 0$ it follows that

$$(24) \qquad \lim_{s\downarrow 0} \int_0^\infty e^{-st} t^k f(t)dt = \frac{k!}{2i^{k+1}} \int_S \phi(x) \frac{(-1)^{k+1} + 1}{(\psi(x))^{k+1}}\, d\mu(x),$$

for all $k \in \mathbb{N}$. Hence, choosing k even we find that

$$(25) \qquad \lim_{s\downarrow 0} \int_0^\infty e^{-st} t^{2m} f(t)dt = 0, \qquad (\forall m \in \mathbb{N}).$$

Similarly as in the introduction, it follows that f(t) has infinitely many (real) zeros. □

REMARK. Theorem 1 also holds true if in the definition of f we replace cos by sin. The proof is virtually the same, the only difference being that we have to choose k odd instead of even.

The theorem also holds if we replace condition (iii) by the following one

$$\text{(26)} \qquad \text{(iii)}^* \qquad \frac{\phi}{\psi^k} \in L^1(X,\mu), \qquad (\forall k \in \mathbb{N}).$$

The details of the proof are left to the reader.

SECTION 2

In this section we will first apply the technique illustrated in sections 0 and 1 to the imaginary part of $\zeta(1+it)$, where ζ denotes Riemann's zeta-function.

THEOREM 2. *Let*

$$\text{(27)} \qquad I(t) = \operatorname{Im}\zeta(1+it) = \frac{\zeta(1+it) - \zeta(1-it)}{2i}, \qquad (t>0).$$

Then $I(t)$ *has infinitely many (real) zeros.*

PROOF. For $\sigma > 1$ we define $I_\sigma : \mathbb{R}^+ \to \mathbb{R}$ by

$$\text{(28)} \qquad I_\sigma(t) = \operatorname{Im}\zeta(\sigma+it) = \frac{\zeta(\sigma+it) - \zeta(\sigma-it)}{2i}, \qquad (t \in \mathbb{R}^+).$$

Then $I_\sigma(t)$, as a function of t, is continuous on $\mathbb{R}^+$ and since $|I_\sigma(t)| \le \zeta(\sigma)$ for all $t \in \mathbb{R}^+$, $I_\sigma(t)$ is also bounded. We consider the Laplace transform $\check{I}_\sigma$ of I_σ for $s > 0$:

$$\text{(29)} \qquad \check{I}_\sigma(s) = \int_0^\infty e^{-st} I_\sigma(t)dt = \int_0^\infty e^{-st}\Big\{\sum_{n=2}^\infty \frac{1}{n^\sigma} \frac{n^{-it} - n^{it}}{2i}\Big\}dt =$$

$$= \frac{1}{2i} \sum_{n=2}^\infty \frac{1}{n^\sigma} \Big\{\frac{1}{s + i\log n} - \frac{1}{s - i\log n}\Big\}, \qquad (s>0).$$

Differentiation with respect to s yields

$$\int_0^\infty e^{-st} t\, I_\sigma(t)dt = \frac{1}{2i} \sum_{n=2}^\infty \frac{1}{n^\sigma} \{\frac{1}{(s+i\log n)^2} - \frac{1}{(s-i\log n)^2}\}. \tag{30}$$

Observing that $(z-1)\zeta(z)$ is an entire function and that (c.f. [2] p.42)

$$\zeta(\sigma+it) = O(\log t), \tag{31}$$

uniformly in the region $\sigma \geq 1$, $t \geq 2$, it is readily seen (using Lebesgue's dominated convergence theorem and the uniform convergence of the series in (30) for $\sigma \geq 1$ and $s \in \mathbb{R}$) that

$$\lim_{\sigma\downarrow 1} \int_0^\infty e^{-st} t\, I_\sigma(t)dt = \int_0^\infty e^{-st} t\, I(t)dt = \tag{32}$$

$$= \frac{1}{2i} \sum_{n=2}^\infty \frac{1}{n} \{\frac{1}{(s+i\log n)^2} - \frac{1}{(s-i\log n)^2}\}.$$

Differentiating (k-1) times with respect to s we obtain for $s > 0$ and $k \in \mathbb{N}$

$$\int_0^\infty e^{-st} t^k I(t)dt = \frac{k!}{2i} \sum_{n=2}^\infty \frac{1}{n} \{\frac{1}{(s+i\log n)^{k+1}} - \frac{1}{(s-i\log n)^{k+1}}\}, \tag{33}$$

Similarly, as before, we find that

$$\lim_{s\downarrow 0} \int_0^\infty e^{-st} t^k I(t)dt = \frac{k!}{2i^k} \sum_{n=2}^\infty \frac{1}{n} \frac{(-1)^{k+1}-1}{(\log n)^{k+1}} \tag{34}$$

and taking k odd it follows that

$$\lim_{s\downarrow 0} \int_0^\infty e^{-st} t^{2m+1} I(t)dt = 0, \qquad (\forall m \in \mathbb{N}). \tag{35}$$

Similarly as before, we arrive at the remarkable result that $I(t) = \mathrm{Im}\, \zeta(1+it)$ has infinitely many (real) zeros. □

REMARKS. In a similar manner it can be shown that Re $\zeta(1+it)$ takes the value 1 infinitely many times.

Also, by the same method, one may show that for any fixed $\sigma > 0$ the function Im $\eta(\sigma+it)$ (where $\eta(s) = \Sigma_{n=1}^{\infty}(-1)^{n+1}n^{-s}$ for Re $s > 0$) has infinitely many real zeros.

As a direct application of theorem 1 we have the following

THEOREM 3. *Let* $\phi\colon \mathbb{R} \to \mathbb{R}$ *be an even (Lebesgue) integrable function which vanishes in a neighborhood of zero. Then the Fourier transform* $\hat{\phi}$ *of* ϕ *has infinitely many (real) zeros.*

PROOF. Observe that

$$\hat{\phi}(t) = \int_{\mathbb{R}} e^{itx}\,\phi(x)dx = \int_{\mathbb{R}} \phi(x)\ (\cos tx + i\sin tx)dx = \tag{36}$$

$$= \int_{\mathbb{R}} \phi(x)\ \cos tx\,dx = 2\int_a^{\infty} \phi(x)\ \cos tx\,dx, \qquad \text{for some } a > 0.$$

It is easily verified that all conditions of theorem 1 are satisfied so that $\hat{\phi}(t)$ has infinitely many zeros. □

REMARK. Note that theorem 3 may by generalized as indicated in the second remark following the proof of theorem 1.

REFERENCES

[1] RUDIN, W., *Real and complex analysis*, McGraw-Hill, 1966.

[2] TITCHMARSH, E.C., *The theory of the Riemann zeta-function*, Oxford, 1951.