A NOTE ON THE PARTIAL SUMS OF $\zeta(s)$
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A note on the partial sums of $\zeta(s)$

by

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ABSTRACT

Defining $\zeta_N(s) = \sum_{n=1}^{N} n^{-s}$, JESSEN proved that if $N \leq 5$ then the real part of $\zeta_N(s+it)$ is positive for $\sigma \geq 1$, $t \in \mathbb{R}$.

Using an electronic computer, SPIRA showed that if $N \leq 9$ then $\zeta_N(s+it) \neq 0$ for $\sigma \geq 1$, $t \in \mathbb{R}$.

In this note the author discusses the relation of these matters to a problem in approximation theory and proves (not making use of a computer) that $\Re \zeta_6(1+it) > 0$ for all $t \in \mathbb{R}$.

Finally he conjectures that if $N$ is any positive integer then $\zeta_N(1+it) \neq 0$ for all $t \in \mathbb{R}$.

KEY WORDS & PHRASES: Partial sums (sections) of the zeta-function, zeros, Tauberian theorems, approximation theory.
We start this note by recalling WIENER's general tauberian theorem for the real line.

For any \( f \in L^1(\mathbb{R}) \) and any \( h \in \mathbb{R} \) let \( f_h \in L^1(\mathbb{R}) \) be the \( h \)-translate of \( f \), i.e.

\[
f_h(x) = f(x+h), \quad (x \in \mathbb{R})
\]

and let \( T_f \) denote the span of the set of all translates of \( f \), i.e.

\[
T_f = \{ \sum_{n=1}^{N} c_n f_{h_n} \mid N \in \mathbb{N}; \; c_n \in \mathbb{C}; \; h_n \in \mathbb{R} \}.
\]

Then WIENER's general tauberian theorem states that \( T_f \) is dense in \( L^1(\mathbb{R}) \) if and only if the Fourier transform \( \hat{f} \) of \( f \) does not vanish on \( \mathbb{R} \), i.e.

\[
\hat{f}(t) = \int_{-\infty}^{\infty} e^{itx} f(x) \, dx \neq 0, \quad \forall t \in \mathbb{R}.
\]

In [4] KOREVAAR showed that WIENER's theorem may be translated into the following theorem for \( L^1(0,1) \): If \( f \in L^1(0,1) \) then the span of the set of functions

\[
\{ x^{\lambda-1} f(x^{\lambda}) \}_{\lambda \in \mathbb{R}^+}, \quad (0<x<1)
\]

is dense in \( L^1(0,1) \) if and only if

\[
(\ast) \quad \int_{0}^{1} f(x) \left( \log \frac{1}{x} \right)^{it} \, dx \neq 0, \quad \forall t \in \mathbb{R}.
\]

In this note we want to discuss the question whether KOREVAAR's theorem applies to the functions \( f_N \in L^1(0,1) \) defined by

\[
f_N(x) = \frac{1 - x^N}{1 - x}, \quad (0<x<1)
\]

where \( N \) is some fixed positive integer.
We first compute the integral
\[
\int_0^1 \frac{1-x^N}{1-x} \left( \log \frac{1}{x} \right) \frac{it}{x} \, dx = \int_0^1 (1+x+...+x^{N-1}) \left( \log \frac{1}{x} \right) \frac{it}{x} \, dx = \\
= \int_0^\infty \left( 1+e^{-u}+...+e^{-(N-1)u} \right) u \frac{it}{e^u} \, du = \\
= \sum_{n=1}^N \int_0^\infty e^{-nu} \frac{it}{e^u} \, du = \\
= \sum_{n=1}^N \int_0^\infty e^{-u} \left( \frac{it}{n} \right) \frac{dv}{n} = \\
= \Gamma(1+it) \cdot \sum_{n=1}^N \frac{1}{n} \frac{1}{1+it}, \quad (t \in \mathbb{R})
\]
from which it is clear that $f_N$ satisfies condition (*) of KOREVAAR's theorem if and only if

\[ (*) \quad \zeta_N(1+it) \overset{\text{def}}{=} \sum_{n=1}^N \frac{1}{n+it} \neq 0, \quad \forall t \in \mathbb{R}. \]

Since for all $t \in \mathbb{R}$

\[ \zeta_1(1+it) = 1 \]
\[ |\zeta_2(1+it)| \geq 1 - \frac{1}{2} = \frac{1}{2} \]
\[ |\zeta_3(1+it)| \geq 1 - \frac{1}{2} - \frac{1}{3} = \frac{1}{6} \]
it is clear that (**) holds for $N = 1,2,3$. Note that the case $N = 1$ also follows from the wellknown Stone-Weierstrass theorem.

Following JESSEN, we have for $N = 4$ (compare [3])
\[
\text{Re } \zeta_4(1 + it) = \sum_{n=1}^{\infty} \frac{1}{n} \cos(t \log n) \\
\text{ (we write } x = t \log 2)\\n\geq 1 + \frac{1}{2} \cos x - \frac{1}{3} + \frac{1}{4} \cos 2x =
\]
\[
= 1 - \frac{1}{3} + \frac{1}{2} \cos x + \frac{1}{4} \{2 \cos^2 x - 1\} =
\]
\[
\text{ (we write } u = \cos x)
\]
\[
= \frac{5}{12} + \frac{1}{2}(u+u^2) \\
\geq \frac{5}{12} + \frac{1}{2} \min_{|u| \leq 1} (u+u^2) = \frac{5}{12} + \frac{1}{2} \cdot \left(-\frac{1}{4}\right) = \frac{7}{24}
\]
so that

\[
\text{Re } \zeta_4(1 + it) \geq \frac{7}{24}, \quad \forall t \in \mathbb{R}.
\]

Consequently we have

\[
\zeta_4(1 + it) \neq 0, \quad \forall t \in \mathbb{R}.
\]

From the above calculation it is also clear that

\[
\text{Re } \zeta_5(1 + it) \geq \text{Re } \zeta_4(1 + it) - \frac{1}{5} \geq \frac{7}{24} - \frac{1}{5} = \frac{11}{120}
\]
so that also

\[
\zeta_5(1 + it) \neq 0, \quad \forall t \in \mathbb{R}
\]
a result which is also due to JESSEN (cf. [3]).

The case \( N = 6 \) is somewhat less transparent. In [1] SPIRA claims to have shown that

\[
(***) \quad \text{Re } \zeta_N(s) > 0, \quad (\text{Re } s \geq 1)
\]
for \( N = 6 \) and \( N = 8 \) and promises to return to these matters in [2].

However, in [2] SPIRA presents machine proofs.

In this note we will present a theoretical proof of the following

**Proposition.** \( \text{Re} \ z_6(1+it) > 0, \quad \forall t \in \mathbb{R}. \)

**Proof.** In this proof we will write

\[
x = t \log 2; \quad y = t \log 3
\]
\[
u = \cos x; \quad v = \cos y.
\]

We will prove our proposition by showing that \( \text{Re} \ z_6(1+it) > \frac{1}{100}, \quad \forall t \in \mathbb{R}. \)

Observe that

\[
\text{Re} \ z_6(1+it) = \sum_{n=1}^{6} \frac{1}{n} \cos(t \log n) = \\
= 1 + \frac{1}{2} \cos x + \frac{1}{3} \cos y + \frac{1}{4} \cos 2x + \frac{1}{5} \cos(t \log 5) + \frac{1}{6} \cos(x+y) \geq \\
\geq 1 + \frac{1}{2} \cos x + \frac{1}{3} \cos y + \frac{1}{4}(2 \cos^2 x - 1) - \frac{1}{5} + \\
+ \frac{1}{6} (\cos x \cos y - \sin x \sin y) \geq \\
\geq \frac{11}{20} + \frac{1}{2}(u+u^2) + \frac{1}{3} v + \frac{1}{6} uv - \frac{1}{6} \sqrt{1-u^2} \sqrt{1-v^2} = \\
\text{def} \quad \phi(u,v), \quad (-1 \leq u, v \leq 1).
\]

It is clear that \( \phi \) is continuous on the compact square \(-1 \leq u, v \leq 1\) so that \( \phi \) assumes an absolute minimum.

The partial derivatives of \( \phi \) on the open square \(-1 < u, v < 1\) may be written as

\[
\frac{\partial \phi}{\partial u} = \frac{1}{2} + u + \frac{1}{6} v + \frac{u}{6} \sqrt{1-v^2} \\
\frac{\partial \phi}{\partial v} = \frac{1}{2} + v + \frac{1}{6} u + \frac{v}{6} \sqrt{1-u^2}
\]

and
\[ \frac{\partial \phi}{\partial v} = \frac{1}{3} + \frac{1}{6} u + \frac{\sqrt{1-u^2}}{6} \frac{\sqrt{1-u^2}}{\sqrt{1-v^2}}. \]

We first prove that \( \phi \) assumes its minimal value in the interior of the square \(-1 \leq u, v \leq 1\).

a. On the segment \( v = 1, -1 \leq u \leq 1 \), we have

\[ \phi(u,v) = \phi(u,1) = \frac{53}{60} + \frac{2}{3} u + \frac{1}{2} u^2 \]

which is minimal for \( u = -\frac{2}{3} \) with minimal value \( \frac{119}{180} \). At \((u,v) = (-\frac{2}{3},1)\) we have \( \frac{\partial \phi}{\partial v} = +\infty \) so that \( \phi \) does not assume its minimal value on the segment \( v = 1, -1 \leq u \leq 1 \).

b. On the segment \( u = 1, -1 \leq v \leq 1 \) we have

\[ \phi(u,v) = \phi(1,v) = \frac{31}{20} + \frac{1}{2} v \]

which is minimal for \( v = -1 \) with minimal value \( \frac{21}{20} > \frac{119}{180} \) so that \( \phi \) does not assume its minimal value on the segment \( u = 1, -1 \leq v \leq 1 \).

c. For \( v = -1, -1 \leq u \leq 1 \), we have

\[ \phi(u,v) = \phi(u,-1) = \frac{13}{60} + \frac{1}{3} u + \frac{1}{2} u^2 \]

which is minimal for \( u = -\frac{1}{3} \) with minimal value \( \frac{29}{180} \). Since \( \frac{\partial \phi}{\partial v} (-\frac{1}{3}, -1) = -\infty \) it follows that \( \phi \) does not assume its minimal value on the segment \( v = -1, -1 \leq u \leq 1 \).

d. Finally, for \( u = -1, -1 \leq v \leq 1 \), we have

\[ \phi(u,v) = \phi(-1,v) = \frac{11}{20} + \frac{1}{6} v \]

which is minimal for \( v = -1 \) with minimal value \( \frac{23}{60} > \frac{29}{180} \) so that \( \phi \) does not assume its minimal value on the segment \( u = -1, -1 \leq v \leq 1 \).

It follows that we may restrict ourselves to the open square \(-1 < u, v < 1\).
In order to show that the minimal value of $\phi$ is positive we consider the equations

$$\frac{\partial \phi}{\partial u} = \frac{\partial \phi}{\partial v} = 0, \quad (-1 < u, v < 1).$$

Observe that if $u \geq 0$ then

$$\frac{\partial \phi}{\partial u} = \frac{1}{2} + \frac{1}{6} v > \frac{1}{2} - \frac{1}{6} > 0$$

and that if $v \geq 0$ then

$$\frac{\partial \phi}{\partial v} = \frac{1}{3} + \frac{1}{6} u > \frac{1}{3} - \frac{1}{6} > 0$$

so that we only need to consider $\phi(u,v)$ on the open square $-1 < u, v < 0$. The equations $\frac{\partial \phi}{\partial u} = 0$ and $\frac{\partial \phi}{\partial v} = 0$ are equivalent to

$$3 + 6u + v + u \frac{\sqrt{1-v^2}}{\sqrt{1-u^2}} = 0$$

respectively

$$2 + u + v \frac{\sqrt{1-u^2}}{\sqrt{1-v^2}} = 0,$$

from which we obtain

$$3 + 6u + v = \frac{v}{2 + u}$$

or, equivalently, that

$$v = -\frac{1}{2} (6 + 15u + 6u^2).$$

Since $6 + 15u + 6u^2 = 0$ for $u = -\frac{1}{2}$ and $u = -2$ and $-\frac{1}{2} (6 + 15u + 6u^2) = -1$ for

$$u = \frac{-15 + \sqrt{129}}{12} \quad \text{and} \quad u = \frac{-15 - \sqrt{129}}{12} \quad (< -1)$$
we find that \( \phi \) is minimal on the curve

\[
v = -\frac{1}{2}(6+15u+6u^2)
\]

where

\[-\frac{1}{2} < u < \frac{-15 + \sqrt{129}}{12} (< -\frac{1}{4}).\]

From \( \frac{\partial \phi}{\partial v} = 0 \) it follows that

\[
\frac{1}{3} + \frac{1}{6} u = -v \frac{\sqrt{1-u^2}}{\sqrt{1-v^2}}
\]

from which we obtain subsequently

\[
\frac{1}{3} + \frac{1}{6} (-\frac{1}{2}) < \frac{-v}{6\sqrt{1-v^2}},
\]

\[
\frac{3}{2} < \frac{-v}{\sqrt{1-v^2}},
\]

\[
\frac{9}{4} < \frac{v^2}{1-v^2} = -1 + \frac{1}{1-v^2},
\]

\[
v^2 > \frac{9}{13},
\]

\[-v = |v| > \frac{3}{\sqrt{13}} (> \frac{3}{4}).\]

Hence, if \( u \) and \( v \) satisfy the restrictions described above we have

\[
\phi(u,v) > \frac{11}{20} + \frac{1}{2} \inf(u+u^2) - \frac{1}{3} + \frac{1}{6} (\inf |u|)(\inf |v|) - \frac{1}{6} \sqrt{1-\frac{1}{16}} \sqrt{1-\frac{9}{13}} >
\]

\[
> \frac{11}{20} - \frac{1}{8} - \frac{1}{3} + \frac{1}{6} \cdot \frac{1}{4} \cdot \frac{3}{4} - \frac{1}{12} \sqrt{15} >
\]

\[
> \frac{11}{20} - \frac{1}{8} - \frac{1}{3} + \frac{1}{12} - \frac{1}{4} \cdot \frac{4}{3} =
\]

\[
= \frac{11}{20} + \frac{1}{32} - \frac{1}{8} - \frac{4}{9} >
\]

\[
> 0.55 + 0.03 - 0.125 - 0.445 = 0.01,
\]
so that $\Re \zeta_6(1+it) > 0.01$ for all $t \in \mathbb{R}$, proving the proposition.

**REMARK.** Numerical computations show that $\phi$ assumes its minimal value $0.1197...$

at the point $u = -0.3266..., v = -0.8705...$.

Computing $\Re \zeta_7(1+it)$ for $t = n.10^{-1}$, $(n=1,2,3,...)$, we found that

$$\Re \zeta_7(1+it) = -0.0136... < 0 \quad \text{for } t = 1,009$$

so that the above proposition does not hold when $\zeta_6$ is replaced $\zeta_7$.

We conclude this note by stating the following

**CONJECTURE.** For every positive integer $N$ one has that

$$\zeta_N(1+it) \neq 0, \quad \forall t \in \mathbb{R}.$$

REFERENCES


