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A NOTE ON THE PARTIAL SUMS OF $\zeta(s)$ (II)

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A note on the partial sums of $\zeta(s)$ (II)

by

J. van de Lune & H.J.J. te Riele

ABSTRACT

This note is a continuation of the first named author's Mathematical Centre Report ZW 53/75. It is shown that for N = 8,9 and 10 one has that $\zeta_N(1+it) \neq 0$, $\forall t \in \mathbb{R}$, where $\zeta_N(s) = \sum_{n=1}^{N} n^{-s}$, $s \in \mathbb{C}$. The case N = 10 is handled by use of a partially numerical argument.

KEY WORDS & PHRASES: *Partial sums (sections) of the zeta-function, zeros.*

0. INTRODUCTION

This note is a continuation of the first named author's Mathematical Centre Report ZW 53/75.

In this note it will be shown that for $N = 8$ and $N = 9$ one has that

$$(0.1) \qquad \zeta_N(1+it) \neq 0, \quad \forall t \in \mathbb{R}$$

where

$$(0.2) \qquad \zeta_N(s) = \sum_{n=1}^{N} n^{-s}, \quad (s \in \mathbb{C}).$$

Actually we will prove the above assertion by showing a little more, namely that

$$(0.3) \qquad R_N(t) \overset{\text{def}}{=} \operatorname{Re} \zeta_N(1+it) > 0, \quad \forall t \in \mathbb{R}$$

for $N = 8$ and $N = 9$. Next, it will be shown (by a partially numerical argument) that also $R_{10}(t) > 0$ for all $t \in \mathbb{R}$. Finally, the smallest positive zeros of $R_N(t)$ are listed for $N = 7, 11(1)100$.

1. N = 8

<u>THEOREM 1.1</u>. $R_8(t) > 0, \quad t \in \mathbb{R}$.

<u>PROOF</u>. We will use the following notation

$$(1.1) \qquad \begin{cases} u = t \log 2; & v = t \log 3; \\ x = \cos u; & y = \cos v. \end{cases}$$

It is clear that

$$(1.2) \qquad R_8(t) = \sum_{n=1}^{8} \frac{1}{n} \cos(t \log n) \geqq 1 + \frac{1}{2} \cos u + \frac{1}{3} \cos v + \frac{1}{4} \cos(2u) +$$
$$- \frac{1}{5} + \frac{1}{6} \cos(u+v) - \frac{1}{7} + \frac{1}{8} \cos(3u) \geqq 1 + \frac{x}{2} + \frac{y}{3} + \frac{1}{4}(2x^2-1) - \frac{1}{5} +$$
$$+ \frac{1}{6}\left(xy - \sqrt{(1-x^2)(1-y^2)}\right) - \frac{1}{7} + \frac{1}{8}(4x^3-3x) = 1 - \frac{1}{4} - \frac{1}{5} - \frac{1}{7} +$$
$$+ \frac{x}{8} + \frac{x^2}{2} + \frac{x^3}{2} + \frac{y}{3} + \frac{xy}{6} - \frac{1}{6}\sqrt{(1-x^2)(1-y^2)} \overset{\text{def}}{=} \phi(x,y), \quad (-1 \leqq x,y \leqq 1).$$

Note that

$$(1.3) \qquad 1 - \frac{1}{4} - \frac{1}{5} - \frac{1}{7} > 0.407$$

and that for $-1 < x,y < 1$ we have

$$(1.4) \qquad \frac{\partial\phi}{\partial x} = \frac{1}{8} + x + \frac{3}{2}x^2 + \frac{y}{6} + \frac{x}{6}\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$$

and

$$(1.5) \qquad \frac{\partial\phi}{\partial y} = \frac{1}{3} + \frac{x}{6} + \frac{y}{6}\frac{\sqrt{1-x^2}}{\sqrt{1-y^2}} .$$

We will show first that $\phi(x,y)$ assumes its minimal value in the *interior* of the square $[-1,1] \times [-1,1]$. First, observe that the minimum of $\phi(x,y)$ over the vertices of the square $[-1,1] \times [-1,1]$ lies in the vertex $[-1,-1]$. Furthermore, it is easily verified that the minimum of $\phi(x,y)$ on the *edge* $y = -1$ is *not* assumed at the point $x = -1$, so that $\phi(x,y)$ does not assume its minimum in one of the vertices of the square $[-1,1] \times [-1,1]$.

Next, observe that for $-1 < x < 1$ we have

$$(1.6) \qquad \lim_{y\uparrow 1} \frac{\partial\phi}{\partial y} = +\infty \quad \text{and} \quad \lim_{y\downarrow -1} \frac{\partial\phi}{\partial y} = -\infty,$$

so that $\phi(x,y)$ does *not* assume its minimum on the edges $y = \pm 1$. Similarly, it follows that $\phi(x,y)$ does not assume its minimum on the edges $x = \pm 1$, so that indeed its minimal value is assumed in the *interior* of the square $[-1,1] \times [-1,1]$.

It follows that $\phi(x,y)$ is minimal at a point (x,y) satisfying $-1 < x,y < 1$ and

$$(1.7) \qquad \frac{\partial\phi}{\partial x} = \frac{\partial\phi}{\partial y} = 0$$

or, more explicitly,

$$(1.8) \qquad \frac{1}{8} + x + \frac{3}{2}x^2 + \frac{y}{6} + \frac{x}{6}\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}} = 0$$

and

(1.9) $$\frac{1}{3} + \frac{x}{6} + \frac{y}{6}\frac{\sqrt{1-x^2}}{\sqrt{1-y^2}} = 0.$$

Since $x > -1$ we have

(1.10) $$\frac{1}{3} + \frac{x}{6} > \frac{1}{3} - \frac{1}{6} > 0$$

so that in view of (1.9) we must have

(1.11) $$y < 0.$$

It is easily verified that (1.8) and (1.9) do not admit $x = 0$.

From (1.8), (1.9) and the fact that $x \neq 0$ it follows that

(1.12) $$\frac{6}{x}\left\{\frac{1}{8} + x + \frac{3}{2}x^2 + \frac{y}{6}\right\} = -\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$$

and (recall that $y \neq 0$)

(1.13) $$\frac{6}{y}\left\{\frac{1}{3} + \frac{x}{6}\right\} = -\frac{\sqrt{1-x^2}}{\sqrt{1-y^2}}$$

so that

(1.14) $$\frac{6}{x}\left\{\frac{1}{8} + x + \frac{3}{2}x^2 + \frac{y}{6}\right\} = \frac{y}{2+x}$$

or, equivalently,

(1.15) $$-y = \frac{3}{2}(2+x)(3x^2 + 2x + \frac{1}{4}).$$

Now observe that

(1.16) $$-y < 1,$$

so that we must have

(1.17) $$\frac{3}{2}(2+x)(3x^2 + 2x + \frac{1}{4}) < 1.$$

Since $2 + x > 0$ this may also be written as

(1.18) $$\frac{3}{2}(3x^2 + 2x + \frac{1}{4}) - \frac{1}{2+x} < 0.$$

Since the left hand side of (1.18) is increasing for $x > 0$ and takes a positive value at $x = 0.04$, we may conclude that

(1.19) $$x < 0.04.$$

From (1.9) it follows that

(1.20) $$1 < 2 + x = -y\,\frac{\sqrt{1-x^2}}{\sqrt{1-y^2}} \leq \frac{-y}{\sqrt{1-y^2}}$$

so that

(1.21) $$1 < \frac{y^2}{1-y^2}$$

from which we obtain (recall that $y < 0$)

(1.22) $$y < -\frac{1}{2}\sqrt{2}.$$

From (1.8) it is clear that

(1.23) $$\frac{1}{8} + x + \frac{3}{2}x^2 + \frac{y}{6} = \frac{-x}{6}\,\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$$

so that, in case $x < 0$, we must have

(1.24) $$\frac{1}{8} + x + \frac{3}{2}x^2 + \frac{y}{6} > 0$$

so that in view of (1.22)

(1.25) $$\frac{3}{2}x^2 + x + (\frac{1}{8} - \frac{1}{12}\sqrt{2}) > 0.$$

From this it is easily seen that

(1.26) $$x < -0.659 \quad \text{or} \quad x > -0.008.$$

Case I. $-0.008 < x < 0.04$.

In this case we have

$$(1.27) \qquad \phi(x,y) > 0.407 + \frac{1}{8}x - \frac{1}{6}|x||y| + \frac{1}{3}y - \frac{1}{6}\sqrt{1-y^2} >$$

$$> 0.407 - \frac{1}{8} * 0.008 - \frac{1}{6} * 0.04 + \frac{1}{6}\{2y - \sqrt{1-y^2}\}.$$

Defining

$$(1.28) \qquad f(y) = 2y - \sqrt{1-y^2}, \qquad (-1 < y < -\tfrac{1}{2}\sqrt{2})$$

we find that

$$(1.29) \qquad \min_{-1<y<-\frac{1}{2}\sqrt{2}} f(y) = f\left(-\frac{2}{\sqrt{5}}\right) = -\sqrt{5},$$

so that

$$(1.30) \qquad \phi(x,y) > 0.407 - 0.001 - 0.007 - \frac{1}{6}\sqrt{5} > 0.026.$$

Case II. $-1 < x < -0.659$

In this case we have

$$(1.31) \qquad \phi(x,y) > 0.407 + \frac{1}{8}x + \frac{1}{6}xy + \frac{1}{3}y - \frac{1}{6}\sqrt{(1-x^2)(1-y^2)} >$$

$$> 0.407 + \left(\frac{x}{8} + \frac{xy}{6} + \frac{y}{3}\right) - \frac{1}{6}\sqrt{(1-(0.659)^2)(1-(\tfrac{1}{2}\sqrt{2})^2)}.$$

Defining

$$(1.32) \qquad \psi(x,y) = \frac{x}{8} + \frac{xy}{6} + \frac{y}{3}$$

for $-1 \leq x \leq -0.659$ and $-1 \leq y \leq -\frac{1}{2}\sqrt{2}$ we have

$$(1.33) \qquad \frac{\partial\psi}{\partial y} = \frac{x}{6} + \frac{1}{3} > 0$$

so that $\psi(x,y)$ is minimal on the edge with $y = -1$.

Since

(1.34) $\psi(x,-1) = -\frac{1}{3} - \frac{x}{24}$

we obtain that

(1.35) $\psi(x,y) \geqq -\frac{1}{3} + \frac{1}{24} * 0.659 > -0.306$

so that, in view of (1.31), it follows that

(1.36) $\phi(x,y) > 0.407 - 0.306 - 0.089 = 0.012.$

This completes the proof of theorem 1.1. □

REMARK. Substituting (1.15) in $\phi(x,y)$ we found numerically that $\phi(x,y)$ assumes its minimal value 0.03419... at the point (x_0,y_0) where $x_0 = 0.02204...$ and $y_0 = -0.89641...$.

2. N = 9.

THEOREM 2.1. $R_9(t) > 0, \quad \forall t \in \mathbb{R}$.

PROOF. Using the same notation as in the proof of theorem 1.1 we have

(2.1)
$$R_9(t) = \sum_{n=1}^{9} \frac{1}{n}\cos(t \log n) \geqq 1 - \frac{1}{4} - \frac{1}{5} - \frac{1}{7} - \frac{1}{9} +$$
$$+ \frac{1}{8}x + \frac{1}{2}x^2 + \frac{1}{2}x^3 + \frac{1}{3}y + \frac{1}{6}xy + \frac{2}{9}y^2 - \frac{1}{6}\sqrt{(1-x^2)(1-y^2)} =$$
$$\stackrel{\text{def}}{=} \phi(x,y), \quad (-1 \leqq x,y \leqq 1).$$

Similarly as in the proof of theorem 1.1 it is easily seen that $\phi(x,y)$ assumes its minimal value in the interior of the square $[-1,1] \times [-1,1]$.

It follows that $\phi(x,y)$ is minimal at a point (x,y) satisfying $-1 < x,y < 1$ and

(2.2)
$$\frac{\partial\phi}{\partial x} = \frac{1}{8} + x + \frac{3}{2}x^2 + \frac{1}{6}y + \frac{x}{6}\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}} = 0$$

and

(2.3) $$\frac{\partial\phi}{\partial y} = \frac{1}{3} + \frac{4}{9}y + \frac{1}{6}x + \frac{y}{6}\frac{\sqrt{1-x^2}}{\sqrt{1-y^2}} = 0\,.$$

Since

(2.4) $$\frac{1}{3} + \frac{1}{6}x > \frac{1}{3} - \frac{1}{6} > 0$$

it follows from (2.3) that

(2.5) $$y < 0.$$

Next we show that $x < 0$. From (2.2) and (2.3) it is easily seen that

(2.6) $$x \neq 0.$$

Therefore we assume that $x > 0$ and derive a contradiction. If $x > 0$ it follows from (2.2) that

(2.7) $$\frac{1}{8} + \frac{1}{6}y < 0$$

so that

(2.8) $$y < -\frac{3}{4}, \quad -y > \frac{3}{4}.$$

Consequently, in view of (2.3), we have

(2.9) $$\frac{1}{3} + \frac{4}{9}y + \frac{1}{6}x = -\frac{y}{6}\frac{\sqrt{1-x^2}}{\sqrt{1-y^2}} > \frac{3}{4}\cdot\frac{1}{6}\frac{\sqrt{1-x^2}}{\sqrt{1-(\frac{3}{4})^2}} = \frac{\sqrt{1-x^2}}{2\sqrt{7}}$$

whereas

(2.10) $$\frac{1}{3} + \frac{4}{9}y + \frac{1}{6}x < \frac{1}{3} + \frac{1}{6}x - \frac{4}{9}\cdot\frac{3}{4} = \frac{1}{6}x.$$

Hence

(2.11) $$\frac{1}{6}x > \frac{\sqrt{1-x^2}}{2\sqrt{7}}$$

from which it is easily seen that

(2.12) $\quad x > \frac{3}{4}.$

Combining this with (2.2) we arrive at

(2.13) $\quad 0 > \frac{1}{8} + x + \frac{3}{2}x^2 + \frac{1}{6}y > \frac{1}{8} + \frac{3}{4} + \frac{3}{2}\ \frac{9}{16} - \frac{1}{6} > 0$

which is a palpable contradiction.

Conclusion:

(2.14) $\quad x < 0.$

In combination with (2.2) it then follows that

(2.15) $\quad \frac{1}{8} + x + \frac{3}{2}x^2 > 0$

so that

(2.16) $\quad -1 < x < -\frac{1}{2} \quad \text{or} \quad -\frac{1}{6} < x < 0.$

Since $y < 0$ it follows from (2.3) that

(2.17) $\quad \frac{1}{3} + \frac{4}{9}y + \frac{1}{6}x > 0$

so that

(2.18) $\quad y > -\frac{3}{4} - \frac{3}{8}x.$

Before proceeding we insert two lemmas, the proofs of which are easily supplied.

LEMMA 2.1. *The function* g *defined by*

(2.19A) $\quad g(x) = \frac{1}{8}x - \frac{1}{6}\sqrt{1-x^2}, \qquad (-1 < x < 0)$

is convex and assumes its minimal value at $x = -\frac{3}{5}$.

LEMMA 2.2. *The function* h *defined by*

(2.19B) $\quad h(y) = \frac{1}{3}y + \frac{2}{9}y^2, \qquad y \in \mathbb{R}$

is increasing for $y > -\frac{3}{4}$.

In the sequel these lemmas will be used a number of times without further notice.

In order to complete the proof we consider a number of cases.

Case I. $-1 < x < -\frac{1}{2}$.

Case Ia. $-1 < x \leq -0.9$.

From (2.18) it follows that

$$y > -\frac{33}{80} \; (> -\frac{3}{4}) \tag{2.20}$$

so that (note that $1 - \frac{1}{4} - \frac{1}{5} - \frac{1}{7} - \frac{1}{9} > 0.296$)

$$\phi(x,y) > 0.296 + \frac{1}{2}x^2(1+x) + \left\{\frac{x}{8} - \frac{1}{6}\sqrt{1-x^2}\right\} + \frac{1}{3}y + \frac{2}{9}y^2 > \tag{2.21}$$

$$> 0.296 + \left\{\frac{-0.9}{8} - \frac{1}{6}\sqrt{1-(0.9)^2}\right\} - \frac{1}{3}\cdot\frac{33}{80} + \frac{2}{9}(\frac{33}{80})^2 > 0.011.$$

Case Ib. $-0.9 < x \leq -0.7$.

From (2.18) it follows that

$$y > -\frac{39}{80} \; (> -\frac{3}{4}) \tag{2.22}$$

so that

$$\phi(x,y) > 0.296 + \frac{1}{2}(0.7)^2(1-0.9) - \frac{1}{8}*0.7 - \frac{1}{6}\sqrt{1-(0.7)^2} + \tag{2.23}$$

$$- \frac{1}{3}\cdot\frac{39}{80} + \frac{2}{9}(\frac{39}{80})^2 > 0.004.$$

Case Ic. $-0.7 < x < -0.5$.

From (2.18) it follows that

$$y > -\frac{9}{16} \; (> -\frac{3}{4}) \tag{2.24}$$

so that

$$\phi(x,y) > 0.296 + \frac{1}{2}(0.5)^2(1-0.7) - \frac{1}{8}*0.6 - \frac{1}{6}\sqrt{1-(0.6)^2} + \tag{2.25}$$

$$- \frac{1}{3}\cdot\frac{9}{16} + \frac{2}{9}(\frac{9}{16})^2 > 0.007.$$

Case II. $-\frac{1}{6} < x < 0$.
From (2.18) if follows that

(2.26) $$y > -\frac{3}{4}.$$

Defining

(2.27) $$f(x,y) = \frac{1}{8}x + \frac{1}{2}x^2 + \frac{1}{2}x^3 + \frac{1}{3}y + \frac{1}{6}xy + \frac{2}{9}y^2$$

on the rectangle $-\frac{1}{6} \leqq x \leqq 0$; $-\frac{3}{4} \leqq y \leqq 0$, we have that

(2.28) $$\phi(x,y) > 0.296 - \frac{1}{6} + f(x,y).$$

If f(x,y) is minimal in the interior of its domain then

(2.29) $$\frac{\partial f}{\partial x} = \frac{1}{8} + x + \frac{3}{2}x^2 + \frac{1}{6}y = 0$$

and

(2.30) $$\frac{\partial f}{\partial y} = \frac{1}{3} + \frac{1}{6}x + \frac{4}{9}y = 0$$

so that, in view of (2.30)

(2.31) $$y = -\frac{3}{4} - \frac{3}{8}x.$$

In combination with (2.29) this yields

(2.32) $$\frac{1}{8} + x + \frac{3}{2}x^2 + \frac{1}{6}\left(-\frac{3}{4} - \frac{3}{8}x\right) = x\left(\frac{15}{16} + \frac{3}{2}x\right) = 0,$$

the solutions of which are

(2.33) $$x_1 = 0 \quad \text{and} \quad x_2 = -\frac{5}{8}.$$

It follows that f(x,y) is minimal on the boundary of its domain.

Case IIa. $y = 0$.
In this case we have

(2.34) $$f(x,0) = \frac{1}{8} x + \frac{1}{2} x^2 + \frac{1}{2} x^3$$

so that

(2.35) $$\frac{d}{dx} f(x,0) = \frac{1}{8} + x + \frac{3}{2} x^2 > 0, \qquad (x > -\frac{1}{6}).$$

Hence f(x,0) is minimal at $x = -\frac{1}{6}$ with minimal value $f(-\frac{1}{6}, 0) = -\frac{1}{108}$.

Case IIb. $x = 0$.
In this case we have

(2.36) $$f(0,y) = \frac{1}{3} y + \frac{2}{9} y^2$$

so that, in view of lemma 2.1, f(0,y) is minimal at $y = -\frac{3}{4}$ with minimal value $f(0, -\frac{3}{4}) = -\frac{1}{8}$.

Case IIc. $y = -\frac{3}{4}$.
In this case we have

(2.37) $$f(x, -\frac{3}{4}) = -\frac{1}{8} + \frac{1}{2} x^2 + \frac{1}{2} x^3 = -\frac{1}{8} + \frac{1}{2} x^2 (1+x) \geqq -\frac{1}{8}.$$

Case IId. $x = -\frac{1}{6}$.
In this case we have

(2.38) $$f(-\frac{1}{6}, y) = \frac{2}{9} y^2 + \frac{11}{36} y - \frac{1}{108}$$

so that

(2.39) $$\frac{d}{dy} f(-\frac{1}{6}, y) = \frac{4}{9} y + \frac{11}{36}$$

which equals 0 if $y = -\frac{11}{16}$ $(> -\frac{3}{4})$. Hence $f(-\frac{1}{6}, y)$ is minimal at $y = -\frac{11}{16}$ with minimal value $f(-\frac{1}{6}, -\frac{11}{16}) = -0.11429\ldots > -\frac{1}{8}$.

From cases IIa through IId it follows that

(2.40) $$f(x,y) > -\frac{1}{8}, \quad (-\frac{1}{6} < x < 0;\ -\frac{3}{4} < y < 0)$$

so that in view of (2.28)

(2.41) $\phi(x,y) > 0.296 - \frac{1}{6} - \frac{1}{8} > 0.004.$

Combining case I and case II the proof of theorem 2.1 is complete. □

REMARK. From (2.2) and (2.3) one may derive that

(2.42) $$\frac{8}{3}y^2 + 4(1+4x+6x^2)y + 3(2+x)(\frac{1}{4} + 2x+3x^2) = 0$$

so that

(2.43) $$y = \frac{3}{4}\{-B \pm \sqrt{B^2 - (2+x)(B-\frac{1}{2})}\}$$

where

(2.44) $$B = 1 + 4x + 6x^2.$$

Substituting (2.43) in the right hand side of (2.1) we found numerically that $\phi(x,y)$ assumes its minimal value 0.03974... at the point (x_0,y_0) where

(2.45) $$x_0 = -0.03633... \quad \text{and} \quad y_0 = -0.51262... \,.$$

3. N = 10.

THEOREM 3.1. $R_{10}(t) > 0$, $\forall t \in \mathbb{R}$.

PROOF. In addition to the notational conventions used in sections 1 and 2 we will write

(3.1) $$w = t \log 5 \quad \text{and} \quad \cos w = z.$$

Then we have that

(3.2) $$\begin{aligned} R_{10}(t) \geqq 1 - \frac{1}{4} - \frac{1}{7} - \frac{1}{9} + \frac{x}{8} + \frac{x^2}{2} + \frac{x^3}{2} + \\ + \frac{y}{3} + \frac{2y^2}{9} + \frac{xy}{6} - \frac{1}{6}\sqrt{(1-x^2)(1-y^2)} + \\ + \frac{z}{5} + \frac{xz}{10} - \frac{1}{10}\sqrt{(1-x^2)(1-z^2)} \overset{\text{def}}{=} \phi(x,y,z) \end{aligned}$$

for

(3.3) $$(x,y,z) \in K \stackrel{\text{def}}{=} [-1,1]^3.$$

We want to prove that the continuous function ϕ is positive on the cube K. Some tedious but easy calculations reveal that ϕ is positive on the skeleton of all edges of K, its minimal value on this skeleton being 0.17598... .

In the interior of K we have

(3.4) $$\frac{\partial\phi}{\partial x} = \frac{1}{8} + x + \frac{3}{2}x^2 + \frac{1}{6}y + \frac{x}{6}\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}} + \frac{1}{10}z + \frac{x}{10}\frac{\sqrt{1-z^2}}{\sqrt{1-x^2}},$$

(3.5) $$\frac{\partial\phi}{\partial y} = \frac{1}{3} + \frac{4}{9}y + \frac{1}{6}x + \frac{y}{6}\frac{\sqrt{1-x^2}}{\sqrt{1-y^2}}$$

and

(3.6) $$\frac{\partial\phi}{\partial z} = \frac{1}{5} + \frac{1}{10}x + \frac{z}{10}\frac{\sqrt{1-x^2}}{\sqrt{1-z^2}},$$

from which it is easily seen (compare (1.6)) that ϕ cannot assume its minimal value in the interior of any one of the faces of the cube K. Hence, if ϕ is minimal on the boundary of K we are done.

Therefore we assume that ϕ is minimal in the interior of K. Then we must have

(3.7) $$\frac{\partial\phi}{\partial x} = \frac{\partial\phi}{\partial y} = \frac{\partial\phi}{\partial z} = 0.$$

Similarly as before it follows from (3.5) and (3.6) that

(3.8) $$y < 0 \quad \text{and} \quad z < 0.$$

If $x > 0$ then it follows from (3.4) that

(3.9) $$\frac{1}{8} + x + \frac{3}{2}x^2 + \frac{1}{6}y + \frac{1}{10}z < 0$$

so that certainly (put $y = z = -1$)

(3.10) $$-\frac{17}{120} + x + \frac{3}{2}x^2 < 0$$

from which it is easily seen that

(3.11) $$x < 0.121.$$

From (3.6) we obtain

(3.12) $$\frac{2+x}{\sqrt{1-x^2}} = \frac{-z}{\sqrt{1-z^2}}$$

so that in view of (3.8) we must have

(3.13) $$z = -\frac{2+x}{\sqrt{5+4x}}.$$

Substitution of (3.13) in $\phi(x,y,z)$ yields

(3.14) $$\phi(x,y,-\frac{2+x}{\sqrt{5+4x}}) = c_0 + \frac{x}{8} + \frac{x^2}{2} + \frac{x^3}{2} + \frac{y}{3} + \frac{2y^2}{9} + \frac{xy}{6} + - \frac{1}{6}\sqrt{(1-x^2)(1-y^2)} - \frac{1}{10}\sqrt{5+4x},$$

where

(3.15) $$c_0 = 1 - \frac{1}{4} - \frac{1}{7} - \frac{1}{9} > 0.496.$$

Using the numerical result mentioned at the end of section 2 we thus find that

(3.16) $$\phi(x,y,z) > 0.039 + \frac{1}{5} - \frac{1}{10}\sqrt{5+4x}$$

so that in view of (3.11) we have

(3.17) $$\phi(x,y,z) > 0.239 - \frac{1}{10}\sqrt{5 + 4 * 0.121} > 0.004$$

proving (by a partially numerical argument) that $R_{10}(t) > 0$ for all $t \in \mathbb{R}$.

REMARK. From (3.4), (3.5), (3.7) and (3.13) one may derive that

(3.18) $$\frac{8}{3}y^2 + 4By + \frac{3}{2}(2+x)(B-\frac{1}{2}) = 0$$

where

(3.19) $$B = 1 + 4x + 6x^2 - \frac{4}{5}\frac{1}{\sqrt{5+4x}}.$$

Solving (3.18) for y we obtain

(3.20) $$y = \frac{3}{4}\{-B \pm \sqrt{B^2 - (2+x)(B-\frac{1}{2})}\}.$$

Substituting (3.13) and (3.20) in $\phi(x,y,z)$ we found numerically that ϕ assumes its minimal value 0.01570... at the point (x_0,y_0) where

(3.21) $$x_0 = 0.04270\ldots \quad \text{and} \quad y_0 = -0.53115\ldots .$$

FINAL REMARKS. In ZW 53/75 it was already mentioned that $R_7(t)$ has at least one real zero. Furthermore, numerical computations indicate that $R_N(t)$ has at least one real zero for every $N \geq 11$. Hence, in order to prove our *conjecture* that

$$\zeta_N(1+it) \neq 0, \ \forall t \in \mathbb{R},$$

for *all* $N \in \mathbb{N}$ one will have to search for a method of proof also involving the imaginary part of $\zeta_N(1+it)$. To the best of our knowledge this is still an unsolved problem.

For the sake of completeness we list, in the table below, the smallest positive zero $t_1(N)$ of (the even function) $R_N(t)$ for $N = 7$ and $N = 11(1)100$, the machine proof of the table being based on the following almost trivial

PROPOSITION. *If the differentiable function* $f\colon \mathbb{R} \to \mathbb{R}$ *is such that*

$$f(t_0) > 0 \quad \textit{for some } t_0 \in \mathbb{R}$$

and

$$|f'(t)| < h \quad \textit{for all } t \in \mathbb{R}$$

where h *is a constant, then*

$$f(t) > 0 \quad \textit{for } t_0 \leq t \leq t_0 + \delta,$$

where

$$\delta = \frac{1}{h} f(t_0).$$

TABLE

N	$t_1(N)$	N	$t_1(N)$	N	$t_1(N)$
7	1008.9095				
11	1180.3887	41	1.0124	71	0.8580
12	3098.0590	42	1.0044	72	0.8547
13	1919.3622	43	0.9968	73	0.8514
14	1379.8280	44	0.9894	74	0.8483
15	1.5897	45	0.9823	75	0.8452
16	1.5120	46	0.9754	76	0.8421
17	1.4566	47	0.9688	77	0.8392
18	1.4114	48	0.9625	78	0.8362
19	1.3727	49	0.9563	79	0.8334
20	1.3388	50	0.9504	80	0.8306
21	1.3086	51	0.9446	81	0.8278
22	1.2814	52	0.9390	82	0.8251
23	1.2567	53	0.9336	83	0.8225
24	1.2342	54	0.9284	84	0.8199
25	1.2135	55	0.9233	85	0.8173
26	1.1943	56	0.9183	86	0.8148
27	1.1765	57	0.9135	87	0.8124
28	1.1599	58	0.9089	88	0.8100
29	1.1444	59	0.9043	89	0.8076
30	1.1298	60	0.8999	90	0.8052
31	1.1161	61	0.8956	91	0.8029
32	1.1032	62	0.8914	92	0.8007
33	1.0910	63	0.8873	93	0.7985
34	1.0794	64	0.8833	94	0.7963
35	1.0684	65	0.8794	95	0.7941
36	1.0580	66	0.8757	96	0.7920
37	1.0480	67	0.8720	97	0.7899
38	1.0385	68	0.8683	98	0.7879
39	1.0294	69	0.8648	99	0.7859
40	1.0208	70	0.8613	100	0.7839