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A NOTE ON THE SOLVABILITY OF THE DIOPHANTINE EQUATION $1^n + 2^n + \ldots + m^n = G(m+1)^n$

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A note on the solvability of the diophantine equation

$$1^n + 2^n + \dots + m^n = G(m+1)^n$$

by

J. van de Lune & H.J.J. te Riele

ABSTRACT

Concerning the diophantine equation

$$(*) \qquad 1^n + 2^n + \dots + m^n = G(m+1)^n,$$

where G is a fixed positive rational, it is shown that the set of all $n \in \mathbb{N}$ for which (*) has a solution $m \in \mathbb{N}$, has natural density zero, provided that $G > \frac{1}{e^{2\pi}-1}$. As a consequence one has that the diophantine equation

$$1^n + 2^n + \dots + m^n = (m+1)^n$$

is almost never solvable.

KEY WORDS & PHRASES: *Sums of powers of integers, diophantine equation, uniform distribution.*

THEOREM 1. *If* $G \in \mathbb{Q}^+$ *and* $n \in \mathbb{N}$ *are given then there exists at most one* $m \in \mathbb{N}$ *such that*

(1) $$1^n + 2^n + \ldots + m^n = G(m+1)^n.$$

PROOF. Writing

(2) $$\sigma_m(n) = \sum_{k=1}^{m} k^n, \quad (m,n \in \mathbb{N})$$

we have (cf. [3; p.5])

(3) $$\frac{m^{n+1}(m+1)^n}{(m+1)^{n+1}-m^{n+1}} < \sigma_m(n) < \frac{m^n(m+1)^{n+1}}{(m+1)^{n+1}-m^{n+1}} .$$

Hence, defining $\Theta = \Theta(m,n)$ by

(4) $$\sigma_m(n) = \frac{m^n(m+1)^n(m+\Theta)}{(m+1)^{n+1}-m^{n+1}}$$

we have

(5) $$0 < \Theta(m,n) < 1, \quad \forall m,n \in \mathbb{N}.$$

It follows that if (1) is solvable we must have

(6) $$\sigma_m(n) = G(m+1)^n$$

so that (writing Θ instead of $\Theta(m,n)$)

(7) $$\frac{m^n(m+1)^n(m+\Theta)}{(m+1)^{n+1}-m^n} = G(m+1)^n$$

or, equivalently,

(8) $$m^{n+1} + \Theta m^n = G(m+1)^{n+1} - Gm^{n+1}.$$

Since $\Theta > 0$ we obtain

(9) $$(G+1)m^{n+1} < G(m+1)^{n+1}$$

which may also be written as

(10) $$m < -1 + \frac{1}{1-B^{1/(n+1)}}$$

where

(11) $$B \overset{\text{def}}{=} \frac{G}{G+1} .$$

If (1) is solvable we also have

(12) $$\sigma_{m+1}(n) = (G+1)(m+1)^n$$

so that (writing Θ instead of $\Theta(m+1,n)$)

(13) $$\frac{(m+1)^n(m+2)^n(m+1+\Theta)}{(m+2)^{n+1}-(m+1)^{n+1}} = (G+1)(m+1)^n$$

which may also be written as

(14) $$(m+2)^{n+1} + (\Theta-1)(m+2)^n = (G+1)\{(m+2)^{n+1}-(m+1)^{n+1}\}.$$

Since $\Theta < 1$ it follows that

(15) $$(G+1)(m+1)^{n+1} > G(m+2)^{n+1}$$

from which it is easily seen that

(16) $$m > -2 + \frac{1}{1-B^{1/(n+1)}}$$

where B is as in (11).

Combining (10) and (16) we obtain

(17) $$-2 + \frac{1}{1-B^{1/(n+1)}} < m < -1 + \frac{1}{1-B^{1/(n+1)}}$$

completing the proof of theorem 1. □

From the above proof it is clear that if $G \in \mathbb{Q}^+$ and $n \in \mathbb{N}$ are such that (1) is solvable for m, then this solution is given by

(18) $$m = m(n) = [\lambda(n)]$$

where $\lambda(n)$ is defined (for all $n \in \mathbb{N}$) by

(19) $$\lambda(n) = -1 + \frac{1}{1-B^{1/(n+1)}}$$

and [x] denotes the greatest integer not exceeding x.

Now observe that

(20) $$\lambda(n) = -1 + \frac{1}{1-B^{1/(n+1)}} = -1 - \frac{1}{B^{1/(n+1)}-1} = -1 - \frac{1}{\exp(\frac{1}{n+1}\log B)-1} =$$

$$= -1 - \left\{\frac{1}{\frac{1}{n+1}\log B} - \frac{1}{2} + O(\tfrac{1}{n})\right\} = \frac{n+1}{\log\frac{1}{B}} - \frac{1}{2} + O(\tfrac{1}{n}), \quad (n \to \infty).$$

Since e^r is irrational for every positive rational r it follows that $\log\frac{1}{B} = \log(1+\frac{1}{G})$ is irrational. Hence, (see [2; p.92, Satz 9]) the sequence $\left\{(n+1)(\log\frac{1}{B})^{-1}\right\}_{n=1}^{\infty}$ is uniformly distributed modulo 1 (u.d. mod 1).

Now recall that if a real sequence $\{\alpha(n)\}_{n=1}^{\infty}$ is u.d. mod 1 and if $\{\beta(n)\}_{n=1}^{\infty}$ is some *convergent* real sequence than also $\{\alpha(n)+\beta(n)\}_{n=1}^{\infty}$ is u.d. mod 1.

From these observations it follows that $\{\lambda(n)\}_{n=1}^{\infty}$ is u.d. mod 1.

Since m(n), as defined by (18), is an integer and

(21) $$0 < \lambda(n) - m(n) < 1$$

it follows that $\{\lambda(n) - m(n)\}_{n=1}^{\infty}$ is uniformly distributed on the interval (0,1).

Now fix any $G \in \mathbb{Q}$ such that

(22) $$G > \frac{1}{e^{2\pi}-1}$$

and let S be the set of all $n \in \mathbb{N}$ for which (1) is solvable with respect to m. Then we have the following

THEOREM 2. *The set* S *has natural density zero.*

PROOF. If S is finite (possibly empty) then we are done.

If S contains infinitely many elements, it follows from (18), (20) and the definition of B that

$$(23)\qquad \lim_{\substack{n\to\infty \\ n\in S}} \frac{n}{m(n)} = \log\frac{1}{B} = \log(1+\frac{1}{G}).$$

For every $n \in S$ we have

$$(24)\qquad \sigma_m(n) = G(m+1)^n, \quad (m = m(n))$$

so that, in view of (4), we have (writing Θ instead of $\Theta(m,n)$)

$$(25)\qquad \frac{m^n(m+1)^n(m+\Theta)}{(m+1)^{n+1}-m^{n+1}} = G(m+1)^n$$

or, equivalently,

$$(26)\qquad m = \frac{1}{(\frac{G+1}{G}+\frac{\Theta}{mG})^{1/(n+1)}-1}\ .$$

Next we investigate the asymptotic behaviour of $\{\lambda(n) - m(n)\}_{n\in S}$ as $n \to \infty$. Since for $n \in S$, $n \to \infty$, we have

$$(27)\qquad \lambda(n) - m(n) = \frac{n+1}{\log(1+\frac{1}{G})} - \frac{1}{2} + O(\frac{1}{n}) +$$

$$- \frac{1}{\exp\{\frac{1}{n+1}\log(\frac{G+1}{G}+\frac{\Theta}{mG})\}-1} = \frac{n+1}{\log(1+\frac{1}{G})} - \frac{1}{2} + O(\frac{1}{n}) +$$

$$- \left\{\frac{1}{\frac{1}{n+1}\log(\frac{G+1}{G}+\frac{\Theta}{mG})} - \frac{1}{2} + O(\frac{1}{n})\right\} =$$

$$= (n+1)\left\{\frac{1}{\log\frac{G+1}{G}} - \frac{1}{\log(\frac{G+1}{G}+\frac{\Theta}{mG})}\right\} + O(\frac{1}{n}) =$$

$$= (n+1)\frac{\log(1+\frac{\Theta}{m(G+1)})}{\log\frac{G+1}{G}\,\log(\frac{G+1}{G}+\frac{\Theta}{mG})} + O(\frac{1}{n}) =$$

$$= \left\{\frac{1}{(\log\frac{G+1}{G})^2} + o(1)\right\}(n+1)\left\{\frac{\Theta}{m(G+1)} + O(\frac{1}{m^2})\right\} + O(\frac{1}{n}) =$$

$$= \frac{1}{(\log\frac{G+1}{G})^2}\,\frac{(n+1)\Theta(m,n)}{m(G+1)} + o(1),$$

it follows that $\{\lambda(n) - m(n)\}_{n\in S}$ is *convergent* as soon as $\Theta(m(n),n)$ converges for $n \to \infty$.

For the moment let us assume that $\{\Theta(m(n),n)\}_{n=1}^{\infty}$ is convergent. Then $\{\lambda(n) - m(n)\}_{n\in S}$ is a convergent subsequence of $\{\lambda(n) - m(n)\}_{n=1}^{\infty}$, the latter sequence being uniformly distributed on $(0,1)$. From this it follows that S has natural density zero. Hence, in order to complete the proof of theorem 2 it suffices to prove the following

THEOREM 3. *For any* $n \in \mathbb{N}$ *let* $m(n)$ *be defined by* (18). *Then*

(28) $$\lim_{n\to\infty} \Theta(m(n),n)$$

exists and is equal to

(29) $$G+1 - G(G+1)\log(1+\frac{1}{G})$$

provided that $G > \dfrac{1}{e^{2\pi}-1}$.

PROOF. Similarly as in [4] we consider the asymptotic behaviour of the sums $\sigma_m(n)$ as $n \to \infty$ and $\frac{n}{m} \to \alpha = \log(1+\frac{1}{G})$.

By means of the Euler-Maclaurin summation formula one may show that

(30) $$\frac{\sigma_m(n)}{m^n} = \frac{m}{n+1} + \frac{1}{2} + \sum_{r=1}^{[\frac{n}{2}]} \frac{B_{2r}}{2r} \binom{n}{2r-1} m^{-2r+1}$$

where the Bernoulli numbers B_r are defined by

(31) $$\frac{z}{e^z-1} = \sum_{r=0}^{\infty} \frac{B_r}{r!} z^r, \quad (|z| < 2\pi).$$

It is well known that for any real $\alpha \neq 0$ (see [1; p.528])

(32) $$\frac{1}{e^{\alpha}-1} = \frac{1}{\alpha} - \frac{1}{2} + \sum_{r=1}^{k} \frac{B_{2r}}{(2r)!} \alpha^{2r-1} + R_k(\alpha)$$

where

(33) $$R_k(\alpha) = \frac{\alpha^{2k+1}}{e^{\alpha}-1} \int_0^1 P_{2k+1}(x) e^{\alpha x} dx$$

so that

$$(34) \qquad \frac{1}{e^{n/m}-1} - \frac{m}{n} + \frac{1}{2} = \sum_{r=1}^{k} \frac{B_{2r}}{(2r)!} \left(\frac{n}{m}\right)^{2r-1} + R_k\left(\frac{n}{m}\right).$$

Taking $k = [\frac{n}{2}]$ in (34) it follows from (30) and (34) that

$$(35) \qquad \left\{\frac{1}{e^{n/m}-1} - \frac{m}{n} + \frac{1}{2}\right\} - \left\{\frac{\sigma_m(n)}{m^n} - \frac{m}{n+1} - \frac{1}{2}\right\} =$$

$$= \frac{1}{e^{n/m}-1} + 1 - \frac{m}{n(n+1)} - \frac{\sigma_m(n)}{m^n} =$$

$$= \sum_{r=2}^{k} \frac{B_{2r}}{(2r)!} \left(\frac{n}{m}\right)^{2r-1} \left\{1 - \frac{n(n-1)\dots(n-2r+2)}{n^{2r-1}}\right\} + R_k\left(\frac{n}{m}\right) =$$

$$= \sum_{r=2}^{k} \frac{B_{2r}}{(2r)!} \left(\frac{n}{m}\right)^{2r-1} \delta_n(2r-2) + R_k\left(\frac{n}{m}\right)$$

where $\delta_n(\cdot)$ is defined by

$$(36) \qquad \delta_n(a) = 1 - \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\dots\left(1-\frac{a}{n}\right), \quad (a \in \mathbb{N}).$$

From the definition of $\delta_n(a)$ it is easily seen that for any fixed $a \in \mathbb{N}$

$$(37) \qquad \lim_{n\to\infty} n\delta_n(a) = 1 + 2 + \dots + a = \frac{1}{2}a(a+1).$$

By mathematical induction one may show that

$$(38) \qquad (0<)\ \delta_n(a) \leqq \frac{a(a+1)}{2n}, \quad (1 \leqq a < n;\ n \geqq 2).$$

As a consequence we have

$$(39) \qquad \left|\frac{B_{2r}}{(2r)!} \left(\frac{n}{m}\right)^{2r-1} \delta_n(2r-2)\right| \leqq \frac{|B_{2r}|}{(2r)!} \left(\frac{n}{m}\right)^{2r-1} \frac{(2r-2)(2r-1)}{2n}$$

so that in view of the fact that

$$(40) \qquad \lim_{n\to\infty} \frac{n}{m(n)} = \log\left(1+\frac{1}{G}\right)$$

and our assumption that

(41) $$G > \frac{1}{e^{2\pi}-1}, \quad (\text{so that } \log(1+\tfrac{1}{G}) < 2\pi)$$

we have that for some $\varepsilon > 0$

(42) $$\left|\frac{B_{2r}}{(2r)!}\left(\frac{n}{m(n)}\right)^{2r-1} n\delta_n(2r-2)\right| \leqq \frac{1}{2}\frac{|B_{2r}|}{(2r)!}(2\pi-\varepsilon)^{2r-1}(2r-2)(2r-1)$$

if n is sufficiently large (independent of r).

The right hand side of (42) is the general term of a convergent series (compare (31)) so that, by a uniform convergence argument (or by Lebesgue's dominated convergence theorem) we obtain (as before we take $k = [\frac{n}{2}]$)

(43) $$\lim_{n\to\infty}\sum_{r=2}^{k}\frac{B_{2r}}{(2r)!}\left(\frac{n}{m}\right)^{2r-1} n\delta_n(2r-2) =$$

$$= \sum_{r=2}^{\infty}\frac{B_{2r}}{(2r)!}(\log(1+\tfrac{1}{G}))^{2r-1}\,\frac{1}{2}(2r-1)(2r-2) =$$

$$= \frac{1}{2}(\log(1+\tfrac{1}{G}))^2\sum_{r=2}^{\infty}\frac{B_{2r}}{(2r)!}(2r-1)(2r-2)(\log(1+\tfrac{1}{G}))^{2r-3} =$$

$$= \frac{1}{2}(\log(1+\tfrac{1}{G}))^2\,\frac{d^2}{dx^2}\left\{\sum_{r=2}^{\infty}\frac{B_{2r}}{2r!}x^{2r-1}\right\}_{x=\log(1+\frac{1}{G})}.$$

Now observe that (see [1; p.204])

(44) $$x\cot x = 1 - \frac{B_2}{2!}(2x)^2 + \frac{B_4}{4!}(2x)^4 - + \ldots$$

from which it is easily seen that

(45) $$\sum_{r=2}^{\infty}\frac{B_{2r}}{(2r)!}x^{2r-1} = \frac{i}{2}\cot\frac{xi}{2} - \frac{1}{x} - \frac{x}{12}$$

so that

(46) $$\frac{d^2}{dx^2}\left\{\sum_{r=2}^{\infty}\frac{B_{2r}}{(2r)!}x^{2r-1}\right\} = -\frac{2}{x^3} - \frac{e^{-x}-e^{x}}{(e^{-x/2}-e^{x/2})^4}$$

which, for $x = \log(1+\frac{1}{G})$ assumes the value

$$- \frac{2}{(\log(1+\frac{1}{G}))^3} + G(G+1)(2G+1). \tag{47}$$

Hence, defining

$$\rho(n) = n \sum_{r=2}^{k} \frac{B_{2r}}{(2r)!} (\frac{n}{m})^{2r-1} \delta_n(2r-2) \tag{48}$$

it follows from (35) that

$$\frac{\sigma_m(n)}{m^n} = \frac{1}{e^{n/m}-1} + 1 - \frac{m}{n(n+1)} - \frac{\rho(n)}{n} - R_k(\frac{n}{m}) \tag{49}$$

where, in view of (43), (46) and (47)

$$\lim_{n\to\infty} \rho(n) = \frac{1}{2}(\log(1+\frac{1}{G}))^2 \left\{- \frac{2}{(\log(1+\frac{1}{G}))^3} + G(G+1)(2G+1)\right\} = \tag{50}$$

$$= - \frac{1}{\log(1+\frac{1}{G})} + \frac{1}{2} G(G+1)(2G+1)(\log(1+\frac{1}{G}))^2.$$

For $R_k(\frac{n}{m})$ we have the following estimate

$$|R_k(\frac{n}{m})| \leq \frac{(\frac{n}{m})^{2k+1}}{e^{n/m}-1} \int_0^1 |P_{2k+1}(x)| e^{\frac{nx}{m}} dx \leq \frac{(\frac{n}{m})^{2k+1}}{e^{n/m}-1} \cdot \frac{4}{(2\pi)^{2k+1}} e^{n/m} = \tag{51}$$

$$= \frac{4}{\pi} \frac{n/m}{e^{n/m}-1} e^{n/m} (\frac{n}{2\pi m})^{2k} \leq C(\frac{n}{2\pi m})^{2k}.$$

Since $\frac{n}{m} \to \log(1+\frac{1}{G})$ as $n \to \infty$ and $G > \frac{1}{e^{2\pi}-1}$ (so that $\log(1+\frac{1}{G}) < 2\pi$) it follows that $R_k(\frac{n}{m})$ tends exponentially fast to zero as $n \to \infty$.

As a simple consequence of (49), (50) and (51) we have

$$\lim_{n\to\infty} \frac{\sigma_m(n)}{m^n} = \frac{1}{e^{\log(1+\frac{1}{G})}-1} + 1 = G + 1 \tag{52}$$

so that

$$\lim_{n\to\infty} \frac{\sigma_m(n)}{m^n} \left\{1 - (1-\frac{1}{m+1})^{n+1}\right\} = (G+1) \left\{1 - e^{-\log(1+\frac{1}{G})}\right\} = 1. \tag{53}$$

Now observe that

(54) $$\Theta(m,n) = m\left\{\frac{\sigma_m(n)}{m^n}\left(1-\left(1-\frac{1}{m+1}\right)^{n+1}\right) - 1\right\} + \frac{\sigma_m(n)}{m^n}\left(1-\left(1-\frac{1}{m+1}\right)^{n+1}\right).$$

Hence in order to prove that $\lim_{n\to\infty} \Theta(m,n)$ exists we only need to study the asymptotic behaviour of

(55) $$m\left\{\frac{\sigma_m(n)}{m^n}\left(1-\left(1-\frac{1}{m+1}\right)^{n+1}\right) - 1\right\} =$$

$$= m\left\{\left(\frac{1}{e^{n/m}-1} - \frac{m}{n(n+1)} + 1 - \frac{\rho(n)}{n} - R\right)\left(1 - \left(1-\frac{1}{m+1}\right)^{n+1}\right) - 1\right\} =$$

$$= -m\left(\frac{m}{n(n+1)} + \frac{\rho(n)}{n}\right)\left\{1 - \left(1-\frac{1}{m+1}\right)^{n+1}\right\} +$$

$$+ m\left\{\left(\frac{1}{e^{n/m}-1} + 1\right)\left(1-\left(1-\frac{1}{m+1}\right)^{n+1}\right) - 1\right\} - m\, R_k\left(\frac{n}{m}\right)\left\{1-\left(1-\frac{1}{m+1}\right)^{n+1}\right\}.$$

Since $R_k(\frac{n}{m})$ tends exponentially fast to zero and $m = O(n)$ it follows that

(56) $$\lim_{n\to\infty} m\, R_k\left(\frac{n}{m}\right)\left\{1-\left(1-\frac{1}{m+1}\right)^{n+1}\right\} = 0.$$

Next we have

(57) $$\lim_{n\to\infty} m\left\{\frac{m}{n(n+1)} + \frac{\rho(n)}{n}\right\} = \lim_{n\to\infty}\left\{\frac{m^2}{n(n+1)} + \frac{m}{n}\rho(n)\right\} =$$

$$= \frac{1}{(\log(1+\frac{1}{G}))^2} + \frac{1}{\log(1+\frac{1}{G})}\cdot\left\{-\frac{1}{\log(1+\frac{1}{G})} + \frac{1}{2}G(G+1)(2G+1)(\log(1+\frac{1}{G}))^2\right\}$$

$$= \frac{1}{2}G(G+1)(2G+1)\log(1+\frac{1}{G}).$$

Finally we have

(58)
$$m\left\{\left(\frac{1}{e^{n/m}-1}+1\right)\left(1-\left(1-\frac{1}{m+1}\right)^{n+1}\right)-1\right\}=$$

$$=m\left\{\frac{e^{n/m}}{e^{n/m}-1}\left(1-\left(1-\frac{1}{m+1}\right)^{n+1}\right)-1\right\}=m\,\frac{e^{n/m}-e^{n/m}\left(1-\frac{1}{m+1}\right)^{n+1}-e^{n/m}+1}{e^{n/m}-1}=$$

$$=m\,\frac{1-e^{n/m}\left(1-\frac{1}{m+1}\right)^{n+1}}{e^{n/m}-1}=\frac{m}{e^{n/m}-1}\left\{1-\exp\left(\frac{n}{m}+(n+1)\log\left(1-\frac{1}{m+1}\right)\right)\right\}=$$

$$=-\frac{m}{e^{n/m}-1}\,\frac{\exp\left(\frac{n}{m}+(n+1)\log\left(1-\frac{1}{m+1}\right)\right)-1}{(0\neq)\ \frac{n}{m}+(n+1)\log\left(1-\frac{1}{m+1}\right)}\cdot\left\{\frac{n}{m}+(n+1)\log\left(1-\frac{1}{m+1}\right)\right\}.$$

Observing that

(59)
$$\lim_{n\to\infty}\left\{\frac{n}{m}+(n+1)\log\left(1-\frac{1}{m+1}\right)\right\}=\log\left(1+\frac{1}{G}\right)+\log e^{-\log\left(1+\frac{1}{G}\right)}=0$$

it follows that

(60)
$$\lim_{n\to\infty}(58)=$$

$$=-\lim_{n\to\infty}\frac{m}{e^{n/m}-1}\left\{\frac{n}{m}+(n+1)\log\left(1-\frac{1}{m+1}\right)\right\}=$$

$$=-\lim_{n\to\infty}\frac{m}{e^{n/m}-1}\left\{\frac{n}{m}-(n+1)\left(\frac{1}{m+1}+\frac{1}{2(m+1)^2}+O\left(\frac{1}{n^3}\right)\right)\right\}=$$

$$=-\lim_{n\to\infty}\frac{m}{e^{n/m}-1}\left\{\frac{n}{m}-\frac{n+1}{m+1}-\frac{n+1}{2(m+1)^2}+O\left(\frac{1}{n^2}\right)\right\}=$$

$$=-\lim_{n\to\infty}\frac{1}{e^{n/m}-1}\left\{\frac{n-m}{m+1}-\frac{m(n+1)}{2(m+1)^2}\right\}=$$

$$=-\frac{1}{e^{\log\left(1+\frac{1}{G}\right)}-1}\left\{\log\left(1+\frac{1}{G}\right)-1-\frac{1}{2}\log\left(1+\frac{1}{G}\right)\right\}=$$

$$=G\left\{1-\frac{1}{2}\log\left(1+\frac{1}{G}\right)\right\}.$$

Combining (53), (54), (55), (56), (57), (58) and (60) it follows that

$$(61) \qquad \lim_{n\to\infty} \Theta(m(n),n) = 1 + G - G(G+1)\log(1+\frac{1}{G})$$

completing the proof of theorem 3 and hence that of theorem 2. □

REFERENCES

[1] KNOPP, K., *Theory and application of infinite series*, Blackie & Son Ltd. (1928).

[2] KOKSMA, J.F., *Diophantische approximationen*, Springer (1936).

[3] van de LUNE, J., *Monotonic approximation of integrals in relation to some inequalities for sums of powers of integers*, Mathematical Centre, Amsterdam, Report ZW 39/75.

[4] van de LUNE, J. & H.J.J. te RIELE, *On a conjecture of Erdös (II)*, Mathematical Centre, Amsterdam, Report ZW 56/75.