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A.E. BROUWER & A. SCHRIJVER

TWO OPTIMAL CONSTANT WEIGHT CODES

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Two optimal constant weight codes

by

A.E. Brouwer & A. Schrijver

ABSTRACT

We write the complete graph K_{22} as a line-disjoint union of 37 copies of K_4 and one copy of $K_{3,3}$ and show that it is impossible to pack 38 copies of K_4 in K_{22}. This implies $A(22,6,4) = 37$ and $A(21,6,4) = 31$ if $A(n,d,w)$ denotes the maximum number of code words of weight w possible in a binary code of word length n and minimum distance d.

KEY WORDS & PHRASES: *packing, constant weight code, Room square.*

0. INTRODUCTION

A (binary) *constant weight code* with *word length* n and *weight* w is a collection of vectors in $GF(2)^n$ such that each of the vectors has exactly w components equal to 1.

The *distance* between two vectors is the number of coordinates at which they differ.

Let $A(n,d,w)$ denote the maximum number of vectors possible in a constant weight code with word length n and weight w such that the distance between any two distinct vectors is at least d. In this note we determine the values of $A(n,6,4)$ for $n = 21$ and $n = 22$.

Another terminology is possible: if we consider each code word as a characteristic function, then a constant weight code can be identified with a collection of w-subsets of a given n-set. The requirement on the distance can be translated as follows: Let λ be the maximum cardinality of an intersection of two w-subsets in the collection; then $2(w-\lambda) \geq d$.

This leads to a problem on hypergraphs. In our case ($d=6, w=4$) we find $\lambda \leq 1$, i.e. no two 4-subsets can have a pair in common. So this is the case of a graph and we must try to find as many edge-disjoint K_4's as possible in a K_n.

From the work of HANANI it follows that K_n can be partitioned into disjoint K_4's iff $n \equiv 1$ or $4 \pmod{12}$. Shortening these codes we get optimal codes for $n \equiv 0$ or $3 \pmod{12}$. In the remaining cases, only for small n were some values of $A(n,6,4)$ known. Recently we proved the existence of a partition of K_n into a matching and $\frac{1}{12}n(n-2)$ K_4's for each $n \equiv 2 \pmod{12}$, thus proving that $A(n,6,4) = \frac{1}{12}n(n-2)$ for these n (see [2]).

As far as we know the current state of knowledge concerning the value of $A(n,6,4)$ for $n \leq 25$ is reflected by the following table:-

n	0-3	4-6	7-8	9	10	11	12	13	14	15	16
A(n,6,4)	0	1	2	3	5^B	6	9^{BS}	13^{BCS}	14^C	15^{CS}	20^S

n	17	18	19	20	21	22	23	24	25
A(n,6,4)	20^X	20-22	24^T-27	29^T-30	31^*	37^*	37^*-40	42^S	50^S

B = transpose of a block design

C = circulant on (11001010...0)

S = Steiner system or shortened Steiner system

T = using a transversal design TD (5,1;20)

X = see [1]

* = this note

1. UPPER BOUNDS

Upper bounds for A(n,d,w) are given by JOHNSON [4] as follows:

$$A(n,d,w) \le \frac{n}{w} A(n-1,d,w-1)$$

and

$$A(n,d,w) \le \frac{n}{n-w} A(n-1,d,w).$$

Since obviously $A(n,6,3) = \lfloor \frac{n}{3} \rfloor$ we get from the first inequality

$$A(n,6,4) \le \frac{n}{4} \lfloor \frac{n-1}{3} \rfloor,$$

i.e.,

$$A(n,6,4) \le \begin{cases} \frac{1}{12} n(n-3) & \text{if } n \equiv 0 \pmod 3, \\ \frac{1}{12} n(n-1) & \text{if } n \equiv 1 \pmod 3, \\ \frac{1}{12} n(n-2) & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

Substituting these bounds in the second inequality,

$$A(n,6,4) \le \frac{n}{n-4}\, A(n-1,6,4),$$

we obtain better bounds when $n \equiv 7$ or $10 \pmod{12}$:

$$A(n,6,4) \le \frac{1}{12}\,(n(n-1)-18) \qquad \text{if } n \equiv 7 \text{ or } 10 \pmod{12}$$

and the same bound as before when $n \equiv 1$ or $4 \pmod{12}$.

Therefore define the Johnson bound $J(n,6,4)$ to be

$$J(n,6,4) = \begin{cases} \left\lfloor \frac{n}{4} \left\lfloor \frac{n-1}{3} \right\rfloor \right\rfloor - 1 & \text{if } n \equiv 7 \text{ or } 10 \pmod{12}, \\ \left\lfloor \frac{n}{4} \left\lfloor \frac{n-1}{3} \right\rfloor \right\rfloor & \text{otherwise.} \end{cases}$$

This means that if $A(3m+1,6,4)$ attains the Johnson bound, then so does $A(3m,6,4)$.

In particular, when we have shown that $A(22,6,4) = \frac{1}{12}(22.21-18) = 37$, then it immediately follows that $A(21,6,4) = \left\lfloor \frac{1}{12} .21.18 \right\rfloor = 31$.

2. THE CONSTRUCTION

We now construct a collection of 37 4-tuples on 22 points such that each pair occurs together in at most one of the 4-tuples, thus showing that $A(22,6,4) = 37$.

Let $V_1 = \{a,b,c,d,e,f,g\}$, $V_2 = \{A,B,C,D,E,F,G\}$ and $V_3 = \{0,1,2,3,4,5,6,7\}$.

Consider the following Room square, indexed by $V_1 \times V_2$:

	A	B	C	D	E	F	G
a	01	-	45	67	-	-	23
b	57	02	-	-	-	13	46
c	-	56	03	12	-	47	-
d	-	37	-	04	26	-	15
e	36	14	27	-	05	-	-
f	24	-	-	35	17	06	-
g	-	-	16	-	34	25	07

This Room square has the following properties (verified by inspection):

(i) In each row each digit occurs exactly once.

(ii) In each column each digit occurs exactly once.

(iii) Each of the $\binom{8}{2}$ unordered pairs from V_3 occurs exactly once in the square.

[These three properties are just the definition of a Room square.]

(iv) $\begin{pmatrix} - & - \\ - & - \end{pmatrix}$ is not a minor of this matrix.

[That is, the positions of the minus signs form the incidence matrix of PG(2,2).]

Let $V = V_1 \cup V_2 \cup V_3$; then $|V| = 22$. Choose the following 4-tuples from V:

> {xXij}, where {ij} is the unordered pair at position xX of the Room square,
>
> {xXYZ}, where xX, xY and xZ are the unoccupied positions in row x of the square,
>
> {abcd} and {aefg}.

This gives 28 + 7 + 2 = 37 4-tuples, and it is easily verified that no pair occurs twice. The $\binom{22}{2} - 37\binom{4}{2} = 9$ pairs which have not been used form a $K_{3,3}$ on the set {b,c,d,e,f,g}.

REFERENCES

[1] BROUWER, A.E., A(17,6,4) = 20, Report ZW 62 (Mathematisch Centrum, Amsterdam, 1975).

[2] BROUWER, A.E. & A. SCHRIJVER, *An infinite sequence of optimal constant weight codes, or the packing of cocktail party graphs with* K_4's, Report ZW 64 (Mathematisch Centrum, Amsterdam, 1975).

[3] HANANI, H., *Balanced incomplete block designs and related designs,* Discrete Math. 11 (1975) 255-370.

[4] JOHNSON, S.M., *A new upper bound for error-correcting codes,* IRE Trans. Inf. Theory IT-8 (1962) 203-207.