AFDELING ZUIVERE WISKUNDE

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TWO OPTIMAL CONSTANT WEIGHT CODES
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Two optimal constant weight codes

by

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ABSTRACT

We write the complete graph $K_{22}$ as a line-disjoint union of 37 copies of $K_4$ and one copy of $K_{3,3}$ and show that it is impossible to pack 38 copies of $K_4$ in $K_{22}$. This implies $A(22,6,4) = 37$ and $A(21,6,4) = 31$ if $A(n,d,w)$ denotes the maximum number of code words of weight $w$ possible in a binary code of word length $n$ and minimum distance $d$.

KEY WORDS & PHRASES: packing, constant weight code, Room square.
0. INTRODUCTION

A (binary) constant weight code with word length $n$ and weight $w$ is a collection of vectors in $\text{GF}(2)^n$ such that each of the vectors has exactly $w$ components equal to 1.

The distance between two vectors is the number of coordinates at which they differ.

Let $A(n,d,w)$ denote the maximum number of vectors possible in a constant weight code with word length $n$ and weight $w$ such that the distance between any two distinct vectors is at least $d$. In this note we determine the values of $A(n,6,4)$ for $n = 21$ and $n = 22$.

Another terminology is possible: if we consider each code word as a characteristic function, then a constant weight code can be identified with a collection of $w$-subsets of a given $n$-set. The requirement on the distance can be translated as follows: Let $\lambda$ be the maximum cardinality of an intersection of two $w$-subsets in the collection; then $2(w-\lambda) \geq d$.

This leads to a problem on hypergraphs. In our case ($d=6$, $w=4$) we find $\lambda \leq 1$, i.e. no two 4-subsets can have a pair in common. So this is the case of a graph and we must try to find as many edge-disjoint $K_4$'s as possible in a $K_n$.

From the work of HANANI it follows that $K_n$ can be partitioned into disjoint $K_4$'s iff $n \equiv 1$ or $4 \pmod{12}$. Shortening these codes we get optimal codes for $n \equiv 0$ or $3 \pmod{12}$. In the remaining cases, only for small $n$ were some values of $A(n,6,4)$ known. Recently we proved the existence of a partition of $K_n$ into a matching and $\frac{1}{12} n(n-2)$ $K_4$'s for each $n \equiv 2 \pmod{12}$, thus proving that $A(n,6,4) = \frac{1}{12} n(n-2)$ for these $n$ (see [2]).

As far as we know the current state of knowledge concerning the value of $A(n,6,4)$ for $n \leq 25$ is reflected by the following table:-
B = transpose of a block design
C = circulant on (11001010...0)
S = Steiner system or shortened Steiner system
T = using a transversal design TD (5,1;20)
X = see [1]
* = this note

1. UPPER BOUNDS

Upper bounds for $A(n,d,w)$ are given by JOHNSON [4] as follows:

$$A(n,d,w) \leq \frac{n}{w} A(n-1,d,w-1)$$

and

$$A(n,d,w) \leq \frac{n}{n-w} A(n-1,d,w).$$

Since obviously $A(n,6,3) = \left\lfloor \frac{n}{3} \right\rfloor$ we get from the first inequality

$$A(n,6,4) \leq \frac{n}{4} \left\lfloor \frac{n-1}{3} \right\rfloor,$$

i.e.,

$$A(n,6,4) \leq \begin{cases} 
\frac{1}{12} n(n-3) & \text{if } n \equiv 0 \pmod{3}, \\
\frac{1}{12} n(n-1) & \text{if } n \equiv 1 \pmod{3}, \\
\frac{1}{12} n(n-2) & \text{if } n \equiv 2 \pmod{3}.
\end{cases}$$
Substituting these bounds in the second inequality,

\[ A(n, 6, 4) \leq \frac{n}{n-4} A(n-1, 6, 4), \]

we obtain better bounds when \( n \equiv 7 \) or \( 10 \pmod{12} \):

\[ A(n, 6, 4) \leq \frac{1}{12} (n(n-1)-18) \quad \text{if } n \equiv 7 \text{ or } 10 \pmod{12} \]

and the same bound as before when \( n \equiv 1 \) or \( 4 \pmod{12} \).

Therefore define the Johnson bound \( J(n, 6, 4) \) to be

\[
J(n, 6, 4) = \begin{cases} \frac{n}{4} \left\lfloor \frac{n-1}{3} \right\rfloor - 1 & \text{if } n \equiv 7 \text{ or } 10 \pmod{12}, \\
\frac{n}{4} \left\lfloor \frac{n-1}{3} \right\rfloor & \text{otherwise.}
\end{cases}
\]

This means that if \( A(3m+1, 6, 4) \) attains the Johnson bound, then so does \( A(3m, 6, 4) \).

In particular, when we have shown that \( A(22, 6, 4) = \frac{1}{12} (22.21-18) = 37 \), then it immediately follows that \( A(21, 6, 4) = \frac{1}{12} \left\lfloor 21.18 \right\rfloor = 31 \).

2. THE CONSTRUCTION

We now construct a collection of 37 4-tuples on 22 points such that each pair occurs together in at most one of the 4-tuples, thus showing that \( A(22, 6, 4) = 37 \).

Let \( V_1 = \{a,b,c,d,e,f,g\} \), \( V_2 = \{A,B,C,D,E,F,G\} \) and \( V_3 = \{0,1,2,3,4,5,6,7\} \).

Consider the following Room square, indexed by \( V_1 \times V_2 \):
\[
\begin{array}{ccccccc}
A & B & C & D & E & F & G \\
a & 01 & - & 45 & 67 & - & - & 23 \\
b & 57 & 02 & - & - & - & 13 & 46 \\
c & - & 56 & 03 & 12 & - & 47 & - \\
d & - & 37 & - & 04 & 26 & - & 15 \\
e & 36 & 14 & 27 & - & 05 & - & - \\
f & 24 & - & - & 35 & 17 & 06 & - \\
g & - & - & 16 & - & 34 & 25 & 07 \\
\end{array}
\]

This Room square has the following properties (verified by inspection):

(i) In each row each digit occurs exactly once.

(ii) In each column each digit occurs exactly once.

(iii) Each of the \(\binom{8}{2}\) unordered pairs from \(V_3\) occurs exactly once in the square.

[These three properties are just the definition of a Room square.]

(iv) \(\binom{-}{-}\) is not a minor of this matrix.

[That is, the positions of the minus signs form the incidence matrix of \(\text{PG}(2, 2)\).]

Let \(V = V_1 \cup V_2 \cup V_3\); then \(|V| = 22\). Choose the following 4-tuples from \(V\):

\(\{xxij\}\), where \(ij\) is the unordered pair at position \(xx\) of the Room square,

\(\{xxYZ\}\), where \(xx\), \(xY\) and \(xZ\) are the unoccupied positions in row \(x\) of the square,

\(\{abcd\}\) and \(\{ae fg\}\).

This gives \(28 + 7 + 2 = 37\) 4-tuples, and it is easily verified that no pair occurs twice. The \(\binom{22}{2} - 37\binom{4}{2} = 9\) pairs which have not been used form a \(K_{3, 3}\) on the set \(\{b, c, d, e, f, g\}\).
REFERENCES


