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A GROUP-DIVISIBLE DESIGN GD(4,1,2;n) EXISTS IFF $n \equiv 2 \pmod 6$, $n \neq 8$ (OR: THE PACKING OF COCKTAIL PARTY GRAPHS WITH K_4'S)

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A group-divisible design GD(4,1,2;n) exists iff $n \equiv 2 \pmod 6$, $n \neq 8$ (or: the packing of cocktail party graphs with K_4's)

by

A.E. Brouwer & A. Schrijver

ABSTRACT

In this paper it is shown that for $n \equiv 2 \pmod 6$, $n \neq 8$, the complete graph K_n can be partitioned into $n(n-2)/12$ copies of K_4 and a 1-factor (matching). It follows that the maximum cardinality of a binary constant-weight code with minimum distance $d = 6$ and words of weight 4 is $n(n-2)/12$ for these values of n. The methods used are explicit construction and recursive techniques, as developed by Hanani and Wilson.

KEY WORDS & PHRASES: *group-divisible design, scarce design, packing, constant-weight code.*

0. INTRODUCTION

Let I_n be the set $\{0,\ldots,n-1\}$. For $n \geq k \geq t$ let $C(n,k,t)$ (resp. $D(n,k,t)$) be the smallest (resp. largest) integer b such that there exist b subsets $B_1,\ldots,B_b$ of I_n, each of k elements, such that every t-element subset of I_n is contained in at least (resp. at most) one of them.

Many authors have determined the value of $C(n,k,t)$ or $D(n,k,t)$ for special values of n, k and t. In particular FORT & HEDLUND [1] have shown that

$$C(n,3,2) = \left\lceil \frac{n}{3} \left\lceil \frac{n-1}{2} \right\rceil \right\rceil \qquad \text{for } n \geq 3.$$

In 1966 SCHÖNHEIM [8] showed that

$$D(n,3,2) = \begin{cases} \left\lfloor \frac{n}{3} \left\lfloor \frac{n-1}{2} \right\rfloor \right\rfloor - 1 & \text{for } n \equiv 5 \pmod 6, \\ \left\lfloor \frac{n}{3} \left\lfloor \frac{n-1}{2} \right\rfloor \right\rfloor & \text{otherwise.} \end{cases}$$

(This was done independently, but later, by GUY [2] in 1967 and SPENCER [9] in 1968. The solution of KIRKMAN [4] in 1847 is correct for $n \equiv 0,1,2,3 \pmod 6$, but false for $n \equiv 4$ or $5 \pmod 6$.)

MILLS [5] has shown that $C(n,4,2) = \left\lceil \frac{n}{4} \left\lceil \frac{n-1}{3} \right\rceil \right\rceil$ for $n \geq 4$ except for $n = 7,9,10,19$, while for $n = 7,9,10$ $C(n,4,2)$ is one more, and for $n = 19$ two more than the value given by this formula.

Mills also determined many values of $C(n,4,3)$ (see [6]).

In this paper we do part of the job of computing $D(n,4,2)$ by proving that

$$D(n,4,2) = \left\lfloor \frac{n}{4} \left\lfloor \frac{n-1}{3} \right\rfloor \right\rfloor = \frac{1}{12} n(n-2) \qquad \text{for } n \equiv 2 \pmod 6,\ n \neq 8,$$

while

$$D(8,4,2) = 2.$$

In a subsequent paper we will show that

$$D(n,4,2) = \begin{cases} \lfloor \frac{n}{4} \lfloor \frac{n-1}{3} \rfloor \rfloor - 1 & \text{for } n \equiv 7 \text{ or } 10 \pmod{12}, \\ \lfloor \frac{n}{4} \lfloor \frac{n-1}{3} \rfloor \rfloor & \text{otherwise,} \end{cases}$$

with a few exceptions. (For sufficiently large n this result follows immediately from the theory developed by WILSON (see, e.g., [10]); the difficult part is to fill in the finite gap left.)

This problem can be described in several other ways. Determining $D(n,k,t)$ amounts to the same as finding an optimal binary constant weight code with word length n, constant weight k and minimum distance $2(k-t+1)$.

For $t = 2$, $C(n,k,2)$ (resp. $D(n,k,2)$) is the number of complete graphs K_k required in order to cover (the edges of) K_n (resp. which can be packed (edge-disjoint) in K_n).

Looking in particular at the case $k = 4$, $t = 2$, $n \equiv 2 \pmod 6$ we see that if we pack copies of K_4 into K_n, then each K_4 uses three edges incident with a given point p. Since K_n is regular with valency $n-1 \equiv 1 \pmod 6$, we see that a packing must necessarily leave unused at least one edge for each vertex.

We will show that except for $n = 8$ there exists a packing that uses all edges except for a one-factor, so that

$$D(n,4,2) = \frac{1}{6}\left(\binom{n}{2} - \tfrac{1}{2}n\right) = n(n-2)/12, \quad \text{for } n \equiv 2 \ (6), \quad n \neq 8.$$

(Using Hoffman's terminology this means that we have for these values of n a packing of a cocktail party graph on n points with K_4's.)

In this case (and more generally in case $t = 2$, n in a suitable congruence class mod $k(k-1)$) again different terminology is possible:

A *group-divisible design* $GD(K,\lambda,M;v)$ is a pair $(\mathcal{B},\mathcal{G})$ of collections of subsets of I_v (called *blocks* resp. *groups*) such that

(i) if $B \in \mathcal{B}$, then $|B| \in K$;

(ii) if $G \in \mathcal{G}$, then $|G| \in M$;

(iii) if $\{i,j\}$ is a 2-element subset of I_v, then either there is exactly one group $G \in \mathcal{G}$ such that $\{i,j\} \subset G$ or there are exactly λ blocks

$B \in \mathcal{B}$ such that $\{i,j\} \subset B$;

(iv) if $G_1, G_2 \in \mathcal{G}$, then $G_1 \cap G_2 = \phi$;

(v) $\cup\mathcal{G} = I_v$.

Instead of GD($\{k\},\lambda,\{m\};v$) we write GD($k,\lambda,m;v$).

A *transversal design* TD($k,\lambda;m$) is a group-divisible design GD($k,\lambda,m;km$) (cf. HANANI [3]).

From the description above it is immediately seen that the determination of D(n,4,2) for $n \equiv 2 \pmod 6$ requires the construction of GD(4,1,2;n). This is done in the remaining part of this paper, thus proving the 'if' part of:

THEOREM. *A* GD(4,1,2;n) *exists iff* $n \equiv 2 \pmod 6$, $n \neq 8$.

The 'only if' part is immediate since

(a) A GD(4,1,2;8) would be equivalent to a set of two mutually orthogonal Latin squares of side 2, which does not exist.

(b) If GD(4,1,2;n) exists, then

(i) n is even, since I_n can be partitioned into groups of size 2, and

(ii) $n \equiv 2 \pmod 3$, since the $n-1$ edges incident with a given point are covered by one K_2 (one edge) and a number of K_4's (three edges each).

Therefore, $n \equiv 2 \pmod 6$.

1. THE TRUNCATED TRANSVERSAL DESIGN

THEOREM. *Let* $h \le t$, $2t \in T(5,1)$, $\{6h+2, 6t+2\} \subset GD(4,1,2)$. *Then* $6(4t+h)+2 \in GD(4,1,2)$.

PROOF. Let T be a set with cardinality $|T| = 2t$, and let $\mathcal{T}$ be a transversal design T(5,1;2t) on the set $T \times I_5$ with groups $T \times \{i\}$, $i \in I_5$. Let $H \subset T$ be a subset of cardinality $|H| = 2h$. Then we have the group-divisible design GD($\{4,5\},1,\{2t,2h\}$) on $X = (T \times I_4) \cup (H \times \{4\})$, given by $\mathcal{T}_1 = \{B \in \mathcal{T}, B \subset X\}$, with groups $T \times \{i\}$, $i \in I_4$ and $H \times \{4\}$.

Now let $Y = (X \times I_3) \cup Z$, where $Z = I_2$, and construct a GD(4,1,2) on Y by taking the groups and blocks of the GD(4,1,2) on the sets $(T \times \{i\} \times I_3) \cup Z$, $i \in I_4$, and $(H \times \{4\} \times I_3) \cup Z$, taking care that each of these GD(4,1,2) contains Z as a group, and furthermore the blocks of a GD(4,1,3) on each of the sets $B \times I_3$ where $B \in T_1$, taking care that each of these GD(4,1,3) contains $\{b\} \times I_3$ as a group for each $b \in B$. (Here we use the fact that $\{12,15\} \subset$ GD(4,1,3)). □

This theorem reduces the job of finding all GD(4,1,2) to a finite one. Let t be even, then $2t \in T(5,1)$ (see, e.g., [7]). Now each number r different from 1 can be written as $r = 4t + h$, t even, $h \in \{0,2,3,4,5,6,7,9\}$. If we assume that all designs GD(4,1,2;6s+2) have been constructed for $1 < s < r$, then by the theorem we find a GD(4,1,2;6r+2) provided that $t \geq h$. This means that we have to find GD(4,1,2;6s+2) for $s \in \{0,2,3,4,5,6,7,9,11,12,13,14,15, 17,21,22,23,25,31,33,41\}$.

2. MULTIPLYING BY FOUR

<u>THEOREM</u>. *If* $v \in$ GD(4,1,2) *and* $v \neq 2$, *then* $4v \in$ GD(4,1,2).

<u>PROOF</u>. Let $|X| = v$ and construct a transversal design on $X \times I_4$ with groups $X \times \{i\}$, $i \in I_4$ (such a design exists since $v \notin \{2,6\}$); add to the blocks of this design the blocks and groups of group-divisible designs GD(4,1,2) on each of the sets $X \times \{i\}$ and we get a GD(4,1,2) on $X \times I_4$. □

If we write $v = 6r + 2$, then $4v = 6(4r+1) + 2$. Consequently, from the list given at the end of Section 1, we may drop all numbers congruent to 1 mod 4 (except 5). Therefore, it remains to find GD(4,1,2;6s+2) for $s \in \{0,2,3,4,5,6,7,11,12,14,15,22,23,31\}$.

3. THE CASE v = 2p

<u>THEOREM</u>. *Let* p *be a prime*, $p \equiv 1 \pmod 6$. *Then* $2p \in$ GD(4,1,2).

<u>PROOF</u>. Let $X = Z_p \times Z_2$ and let x be a primitive root mod p. Let $u = (p-1)/6$. Now consider the blocks

$$\{(i,1+k),(x^j+i,k),(x^{j+2u}+i,k),(x^{j+4u}+i,k)\} \quad (i \in Z_p,\ 0 \leq j < u,\ k \in Z_2),$$

and the groups

$$\{(i,0),(i,1)\} \qquad (i \in Z_p).$$

It is not difficult to check that this system gives a GD(4,1,2;2p). □

This disposes of all the even values of s:

s = 0 is the trivial case v = 2 (one group and no blocks),
s = 2 corresponds to v = 14 = 2.7,
s = 4 corresponds to v = 26 = 2.13,
s = 6 corresponds to v = 38 = 2.19,
s = 12 corresponds to v = 74 = 2.37,
s = 14 corresponds to v = 86 = 2.43, and
s = 22 corresponds to v = 134 = 2.67.

We are left with the case $s \in \{3,5,7,11,15,23,31\}$.

4. THE REMAINING SEVEN CASES

(i) The case s = 3, v = 20.
Let $X = I_4 \times Z_5$ and take the groups

$$\{(0,i),(3,i)\} \text{ and } \{(1,i),(2,i)\} \qquad (i \in Z_5),$$

and the blocks

$$\{(0,i),(0,i+1),(1,i+2),(1,i+4)\}$$
$$\{(0,i),(0,i+2),(2,i+3),(2,i+4)\}$$
$$\{(0,i),(1,i),(3,i+1),(3,i+4)\}$$
$$\{(0,i),(2,i),(3,i+2),(3,i+3)\}$$
$$\{(1,i+2),(1,i+3),(2,i),(3,i)\}$$
$$\{(1,i),(2,i+1),(2,i+4),(3,i)\} \qquad (i \in Z_5).$$

(ii) The case s = 5, v = 32.
Let $X = I_2 \times Z_{16}$ and take the groups

$\{(j,i),(j,i+8)\}$ $(i \in Z_{16},\ \ j \in I_2)$,

and the blocks

$\{(0,i),(0,i+6),(1,i),(1,i+2)\}$
$\{(0,i),(0,i+4),(0,i+11),(1,i+15)\}$
$\{(0,i),(0,i+1),(0,i+3),(1,i+6)\}$
$\{(0,i+8),(1,i),(1,i+1),(1,i+5)\}$
$\{(0,i+2),(1,i),(1,i+3),(1,i+9)\}$ $(i \in Z_{16})$.

(iii) The case $s = 7$, $v = 44$.
Let $X = I_2 \times Z_{22}$ and take the groups

$\{(j,i),(j,i+11)\}$ $(i \in Z_{22},\ \ j \in I_2)$,

and the blocks

$\{(0,i),(0,i+5),(1,i)(1,i+15)\}$
$\{(0,i),(0,i+8),(0,i+9),(1,i+6)\}$
$\{(0,i),(0,i+2),(0,i+6),(1,i+9)\}$
$\{(0,i),(0,i+7),(0,i+10),(1,i+11)\}$
$\{(0,i),(1,i+16),(1,i+18),(1,i+21)\}$
$\{(0,i),(1,i+2),(1,i+8),(1,i+12)\}$
$\{(0,i),(1,i+5),(1,i+13),(1,i+14)\}$ $(i \in Z_{22})$.

(iv) The case $s = 11$, $v = 68$.
Let $X = (Z_2 \times Z_3 \times Z_3 \times Z_3) \cup I_{14}$. We will construct a GD(4,1,2) on X by first taking a GD(4,1,2) on I_{14} and then covering $X \backslash I_{14}$ with a 1-factor (matching), 14 Δ-factors (partitions into triples) and 108 4-tuples. The 14 Δ-factors can then be completed to 14.18 4-tuples by adjoining one point of I_{14} to each of the triples in a Δ-factor.

1-factor : $\{(0,0,0,0),(1,0,0,0)\}$ mod $(-,3,3,3)$

Δ-factors: 1. $\{(0,0,0,0),(0,1,0,1),(0,2,0,2)\}$ mod $(2,3,3,3)$

(this gives only 18 triples, since each triple occurs thrice and we retain only one of each three identical triples).

2. $\{(0,0,0,0),(0,1,1,0),(0,2,2,0)\}$ mod $(2,3,3,3)$
3. $\{(0,0,0,0),(0,1,1,1),(0,2,2,2)\}$ mod $(2,3,3,3)$

4. $\{(0,0,0,0),(0,1,2,0),(0,2,1,0)\}$ mod $(2,3,3,3)$
5. $\{(0,0,0,0),(0,1,2,2),(0,2,1,1)\}$ mod $(2,3,3,3)$
6-8. $[\{(0,0,0,0),(1,0,1,0),(1,0,2,2)\}$ mod $(2,3,-,3)]$ mod $(-,-,3,-)$
9-11. $[\{(0,i,0,i),(1,i,0,i+1),(1,i+1,1,i)\}$ $(i=0,1,2)$ mod $(2,-,3,-)]$ mod $(-,3,-,-)$
12-14. $[\{(0,i,i,0),(1,i,i+1,2),(1,i+1,i,0)\}$ $(i=0,1,2)$ mod $(2,-,-,3)]$ mod $(-,3,-,-)$

4-tuples: $\{(0,0,0,0),(0,0,0,1),(0,1,0,0),(1,2,1,1)\}$ mod $(2,3,3,3)$
$\{(0,0,0,0),(0,0,1,1),(1,1,0,2),(1,1,1,2)\}$ mod $(2,3,3,3)$.

(v) The case $s = 15$, $v = 92$.
Let $X = I_4 \times I_{23}$. A GD(4,1,2) on X will be constructed with help of a GD(4,1,{2,5};23) with exactly one block of size 5 and a pair of orthogonal Latin squares of order 23 with orthogonal subsquares of order 5. Hence we first present both these systems.

(v.i) A GD(4,1,{2,5};23) with one block of size 5.
Let $Y = (Z_2 \times Z_3 \times Z_3) \cup I_5$. We will construct the GD(4,1,{2,5}) on Y by taking I_5 as a group and covering $Z_2 \times Z_3 \times Z_3$ with a 1-factor, 5 Δ-factors and 9 4-tuples, as follows:

1-factor: $\{(0,0,0),(1,0,0)\}$ mod $(-,3,3)$

Δ-factors:

1. $\{(0,0,0),(0,1,0),(0,2,0)\}$ mod $(2,-,3)$
2. $\begin{bmatrix}\{(0,0,2),(0,1,1),(0,2,0)\} & \text{mod } (-,3,-) \\ \{(1,0,0),(1,0,1),(1,0,2)\} & \text{mod } (-,3,-)\end{bmatrix}$

3-5. $[\{(0,0,0),(1,0,1),(1,1,2)\}$ mod $(2,3,-)]$ mod $(-,-,3)$

4-tuples: $\{(0,0,0),(0,0,1),(1,1,1),(1,2,0)\}$ mod $(-,3,3)$.

(v.ii) A T(4,1;23) with subdesign T(4,1;5).
This construction is a special case of the construction described by BOSE, PARKER & SHRIKHANDE. Take the affine plane AG(2,5) and delete two points; this gives a pairwise balanced design (cf. HANANI [3]) $\mathcal{B}$ = PBD({3,4,5},1;23) with one block of size 3 on a set I_{23}. Now construct a T(4,1;23) as follows:

for each block $B \in \mathcal{B}$ construct a $T(4,1;4 \cdot |B|)$ on $B \times I_4$ with groups $B \times \{i\}$, $i \in I_4$, taking care that if $|B| > 3$ then the sets $\{b\} \times I_4$, $b \in B$, are blocks of the $T(4,1;4 \cdot |B|)$. Taking all blocks of the thus constructed transversal designs except for the three blocks of the type $\{b\} \times I_4$ with $b \in B_0$, the unique block of size 3, we get a $T(4,1;23)$. It contains subdesigns $T(4,1;5)$ since $\mathcal{B}$ contains blocks of size 5 disjoint from the block of size 3.

(v.iii) A $GD(4,1,2;92)$.
Let $\mathcal{T}$ be a transversal design $T(4,1;23)$ on $X = I_{23} \times I_4$ with a subdesign $\mathcal{T}_0 = T(4,1;5)$ on $Y = I_5 \times I_4$. Let $\mathcal{B}_i$ be a $GD(4,1,\{2,5\};23)$ on $I_{23} \times \{i\}$ with one group of size 5, say $I_5 \times \{i\}$, $(i \in I_4)$. Finally let $\mathcal{D}$ be a $GD(4,1,2;20)$ on Y. Taking the blocks of $\mathcal{T}\backslash\mathcal{T}_0$ and those of $\mathcal{B}_i$ $(i \in I_4)$ and $\mathcal{D}$, and the groups of size 2 of $\mathcal{B}_i$ $(i \in I_4)$ and $\mathcal{D}$, we get a $GD(4,1,2)$ on X.

(vi) The cases $s = 23$ $(v = 140)$ and $s = 31$ $(v = 188)$.
Set for $v = 140, 188$: $v = 48m - 4$ where $m = 3,4$.
Take a resolvable design $RB(4,1;12m+4)$, and form a partial completion by adjoining $4m-6$ points p_i $(i = 1,\dots,4m-6)$, where the point p_i is adjoined to each of the blocks of the i-th parallel class. (This is allowed since there are $4m+1$ parallel classes.) In this way we get a $B(\{4,5,4m-6\},1;16m-2)$ on a set I_n where $n = 16m-2$. Since we did not use all parallel classes of the original design, we can pick the blocks of one parallel class together with the block of size $4m-6$ and call them groups. This gives us a $GD(\{4,5\},1,\{4,4m-6\};n)$. Now let $X = (I_n \times I_3) \cup I_2$ and form a $GD(4,1,3;12)$ resp. $GD(4,1,3;15)$ on $B \times I_3$ for each block B of this design, taking care that each time $\{b\} \times I_3$ becomes a group for each $b \in B$; also form $GD(4,1,2;14)$ resp. $GD(4,1,2;12m-16)$ on $(G \times I_3) \cup I_2$ for each group G of this design, taking care that each time I_2 becomes a group. If we take all the blocks thus obtained, and all groups of size two (where of course I_2 is taken only once) we get a $GD(4,1,2;48m-4)$.

This settles all cases.

REFERENCES

[1] FORT Jr., M.K. & G.A. HEDLUND, *Minimal coverings of pairs by triples*, Pacif. J. Math. 8 (1958) 709-719.

[2] GUY, R.K., *A problem of Zarankiewicz*, Research paper no 12, Univ. of Calgary (1967).

[3] HANANI, H., *Balanced incomplete block designs and related designs*, Discr. Math. 11 (1975) 255-369.

[4] KIRKMAN, T.P., *On a problem in combinations*, Cambridge and Dublin Mathematical Journal 2 (1847) 191-204.

[5] MILLS, W.H., *On the covering of pairs by quadruples I, II*, J. Combinatorial Theory (A) 13 (1972) 55-78 and 15 (1973) 138-166.

[6] MILLS, W.H., *On the covering of triples by quadruples*, Proc. 5th S.E. Conf. on Combinatorics, Graph Theory and Computing (1974) 563-581.

[7] MILLS, W.H., *Three mutually orthogonal Latin squares*, J. Combinatorial Theory (A) 13 (1972) 79-82.

[8] SCHÖNHEIM, J., *On maximal systems of k-tuples*, Studia Scientiarum Mathematicarum Hungarica 1 (1966) 363-368.

[9] SPENCER, J., *Maximal consistent families of triples*, J. Combinatorial Theory 5 (1968) 1-8.

[10] WILSON, R.M., *The construction of group divisible designs and partial planes having the maximum number of lines of a given size*, Proc. Chapel Hill, 1970, pp. 488-497.