stichting
mathematisch
centrum

MC

DEPARTMENT OF PURE MATHEMATICS ZW 73/76 APRIL

P. VAN EMDE BOAS, T.M.V. JANSSEN & A. SCHRIJVER

# EACH COMPLETE BIPARTITE GRAPH MINUS A MATCHING IS REPRESENTABLE BY LINE SEGMENTS

Prepublication

2e boerhaavestraat 49 amsterdam

Printed at the Mathematical Centre, 49, 2e Boerhaavestraat, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O), by the Municipality of Amsterdam, by the University of Amsterdam, by the Free University at Amsterdam, and by industries.

AMS(MOS) subject classification scheme (1970): 05C99

# Each complete bipartite graph minus a matching is representable by line segments*)

by

P. van Emde Boas, T.M.V. Janssen, A. Schrijver

ABSTRACT

It is proved that each complete bipartite graph minus a matching is the intersection graph of a collection of line segments.

KEY WORDS & PHRASES: *Intersection graph (representative graph), convex sets, line segments.*

---

*) This report is not for review; it is meant for publication elsewhere.

In [1] M. LAS VERGNAS and L. LOVÁSZ posed the following problem: Is $K_{7,7}$ minus a perfect matching representable by convex sets in the plane $\mathbb{R}^2$? Here, as usual, a graph G is called *representable* by sets of a special kind if G is the representative graph (or intersection graph) of a collection of sets of that kind.

Let $G_n$ be the graph obtained from the complete bipartite graph $K_{n,n}$ by removing a perfect matching. LAS VERGNAS observed that each $G_n$ is representable by arcwise connected subsets of $\mathbb{R}^2$, and that $G_6$ is representable by convex subsets of $\mathbb{R}^2$. Subsequently, the third author showed that $G_7$ is representable by semilines. J BECK (private communication via LOVÁSZ) proved independently that even $G_8$ is representable by semilines, and this inspired LAS VERGNAS (private communication) to arrange a construction of line segments with intersection graph $G_{10}$.

In this note we prove that each $G_n$ is representable by semilines (or by line segments, or by convex subsets) in $\mathbb{R}^2$. In fact, we show that $G_{\aleph_0}$ is representable by line segments or by semilines. A consequence of this is that each graph obtained from some complete bipartite graph by removing a (not necessarily perfect) matching is representable by line segments, since each such graph is an induced subgraph of a $G_n$.

Our construction is as follows. Let C be the curve $\{(x,\tan x) \mid -\frac{1}{2}\pi < x < +\frac{1}{2}\pi\}$. Define, for each natural number n, points $P_n$ and $P'_n$ on the curve C, and line segments $L_n$ and $L'_n$ inductively by:

(i) $P_1 = (\frac{1}{4}\pi, \tan \frac{1}{4}\pi)$, $P'_1 = (-\frac{1}{4}\pi, -\tan \frac{1}{4}\pi)$;

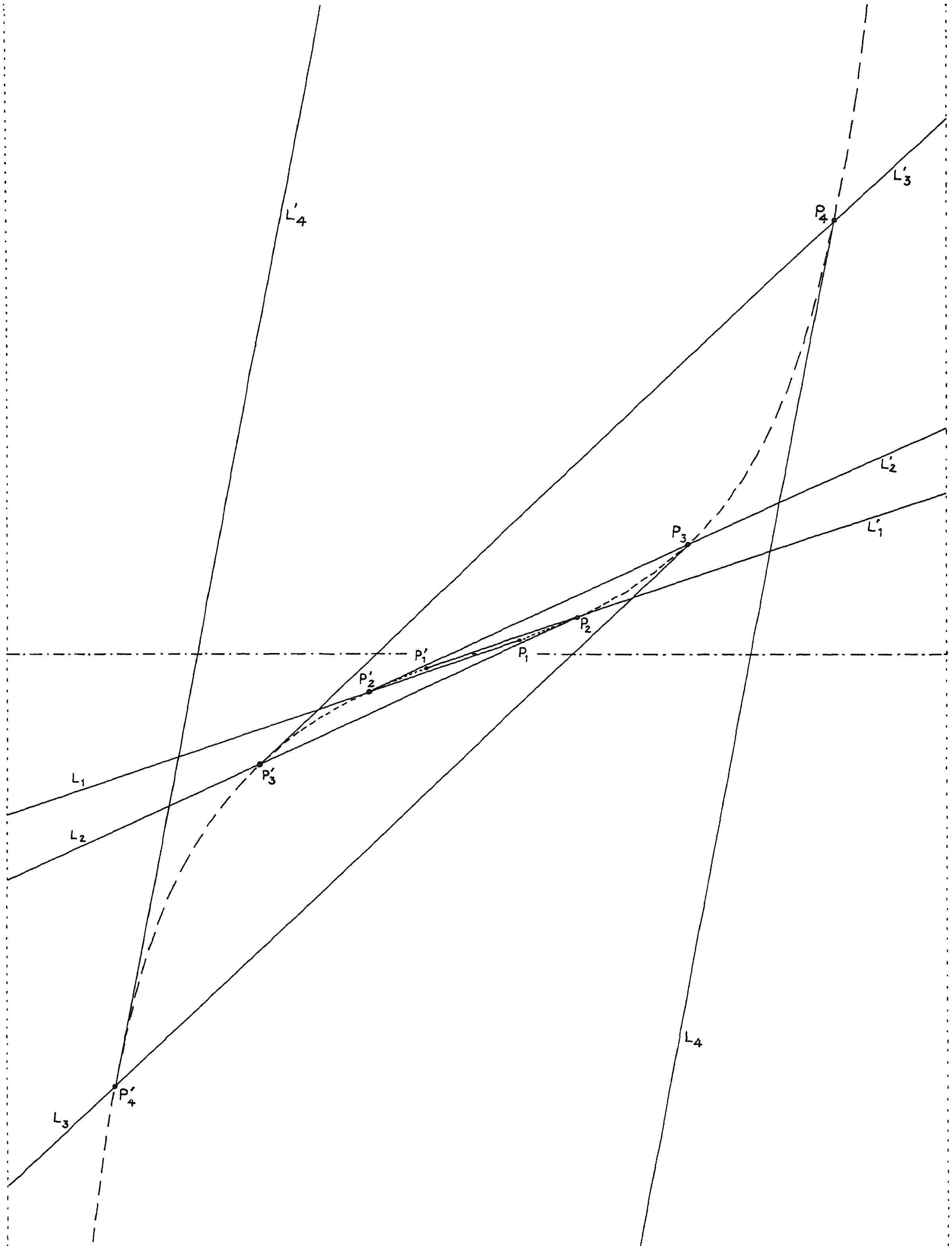
(ii) if $P_1,\ldots,P_n$, $P'_1,\ldots,P'_n$ and $L_1,\ldots,L_{n-1}$, $L'_1,\ldots,L'_{n-1}$ $(n \geq 1)$ are defined, then define line segments $L_n$ and $L'_n$, and points $P_{n+1}$ and $P'_{n+1}$ as follows:

$L_n$ is the segment of the tangent of C in $P_n$ with end points $P_n$ and the intersection point with the vertical line "$x = -\frac{1}{2}\pi$";

$L'_n$ is the segment of the tangent of C in $P'_n$ with end points $P'_n$ and the intersection point with the vertical line "$x = +\frac{1}{2}\pi$";

$P_{n+1}$ is the intersection point of $L'_n$ and the curve C;

$P'_{n+1}$ is the intersection point of $L_n$ and the curve C.



L'4
L'3
P4
L'2
L'1
P3
P2
P'1
P1
P'2
L1
P'3
L2
L4
P'4
L3

It can easily be seen that the line segments $L_1, L_2, \ldots$ and $L'_1, L'_2, \ldots$ satisfy the following conditions: if $n,m \in \mathbb{N}$, $n \neq m$, then $L_n \cap L_m = L'_n \cap L'_m = \emptyset \neq L_n \cap L'_m$, and if $n \in \mathbb{N}$, then $L_n \cap L'_n = \emptyset$. Hence this collection of line segments has as intersection graph the graph $G_{\aleph_0}$. Also it is clear that by extending the line segments $L_n$ beyond the line "$x = -\frac{1}{2}\pi$" and the line segments $L'_n$ beyond the line "$x = +\frac{1}{2}\pi$" we obtain a collection of semilines, again with intersection graph $G_{\aleph_0}$.

The problem of representing $G_7$ by convex sets in $\mathbb{R}^2$ arose from the problem of characterizing all minimal graphs not representable by convex sets in $\mathbb{R}^2$ (similar to the BOLAND & LEKKERKERKER [2] characterization of minimal graphs not representable by convex sets (intervals) in $\mathbb{R}$). This general question is still unanswered.

REFERENCES

[1] BERGE C. & D. RAY-CHAUDHURI (eds), *Hypergraph Seminar* (Proc. First Working Seminar on Hypergraphs, Ohio State University, 1972), Lecture Notes in Math. 411 (Springer, Berlin, 1974).

[2] BOLAND J. & C. LEKKERKERKER, *Representation of a finite graph by a set of intervals on the real line,* Fund. Math. 51 (1962) 45-64.