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OPTIMAL PACKINGS OF K_4 's INTO A K_n - THE CASE $n \not\equiv 2 \pmod{3}$

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Optimal packings of K_4 's into a K_n - The case $n \not\equiv 2 \pmod{3}$.

by

A.E. Brouwer

ABSTRACT

In this paper we construct a pairwise balanced design $B(\{4, 7^*\}, 1; n)$ (i.e. a design with blocks of size 4 or 7 and exactly one block of size 7, on n points with $\lambda = 1$) for each $n \equiv 7$ or $10 \pmod{12}$ except $n = 10$ or 19 (in which cases such a design cannot exist). From these designs optimal packings of K_4 's into a K_n are derived for $n \not\equiv 2 \pmod{3}$, $n \notin \{9, 10, 18, 19\}$, while the case $n \in \{9, 10, 18\}$ is treated by ad hoc methods. It is not known whether the known packing of 25 K_4 's in K_{19} is optimal.

KEY WORDS & PHRASES: *pairwise balanced design, scarce design, packing, constant weight code*

1. INTRODUCTION

Let I_n be a finite set of n elements. For $n \geq k \geq t$ let $D(n,k,t)$ be the largest integer b such that there exist b subsets B_1, \dots, B_b of I_n , each of k elements, such that every t -element subset of I_n is contained in at most one of them.

In a previous paper ([1]) the present author and A. Schrijver determined $D(n,4,2)$ for $n \equiv 2 \pmod{6}$. Here we treat $n \equiv 0$ or $1 \pmod{3}$ (except $n = 19$), and in a future paper we will discuss the remaining case $n \equiv 5 \pmod{6}$. The overall result is the following:

Define

$$J(n,4,2) = \begin{cases} \left\lfloor \frac{n}{4} \left\lfloor \frac{n-1}{3} \right\rfloor \right\rfloor - 1 & \text{for } n \equiv 7 \text{ or } 10 \pmod{12} \\ \left\lfloor \frac{n}{4} \left\lfloor \frac{n-1}{3} \right\rfloor \right\rfloor & \text{otherwise.} \end{cases}$$

Then

(i) for each n $D(n,4,2) \leq J(n,4,2)$

(this is the so-called Johnson bound, see e.g. JOHNSON [4])

(ii) for almost all n $D(n,4,2) = J(n,4,2)$.

Cases in which $D(n,4,2) \neq J(n,4,2)$ is known:

n	8	9	10	11	17	19
$J(n,4,2)$	4	4	6	8	21	27
$D(n,4,2)$	2	3	5	6	20	25 or 26

I conjecture that for all other n equality holds. (In any case all further exceptions must have $n \equiv 11 \pmod{12}$).

REMARK. The undefined notations, especially for various types of designs, are taken from HANANI ([3]) or WILSON ([10]).

2. OPTIMAL PACKINGS

A. The case $n \equiv 0, 1, 3$ or $4 \pmod{12}$

For $n \equiv 1$ or $4 \pmod{12}$ HANANI ([2]) has constructed a Steiner system $S(2, 4, n)$. Therefore $D(n, 4, 2) = J(n, 4, 2) = \frac{1}{12} n(n-1)$ for these n .

Throwing away a fixed point and all blocks containing it produces a system of $\frac{1}{12} n(n-1) - \frac{1}{3}(n-1) = \frac{1}{12}(n-1)(n-4)$ four-tuples on $n-1$ points, i.e. $D(n, 4, 2) = J(n, 4, 2) = \frac{1}{12} n(n-3)$ for $n \equiv 0$ or $3 \pmod{12}$.

B. The case $n \equiv 6, 7, 9$ or $10 \pmod{12}$

For $n \equiv 7$ or $10 \pmod{12}$, $n \neq 10, 19$, we will construct in the next section a pairwise balanced design on n points with $\lambda = 1$ and blocks of size 4 or 7, using exactly one block of size 7 (notation: $B(\{4, 7^*\}, 1; n)$).

If we replace the block $\{x_0, \dots, x_6\}$ of size 7 of such a design by the two four-tuples $\{x_0, x_1, x_2, x_3\}$ and $\{x_0, x_4, x_5, x_6\}$ we have a collection of $\frac{1}{6}(\binom{n}{2} - \binom{7}{2}) + 2 = \frac{1}{12}(n(n-1) - 18) = J(n, 4, 2)$ four-tuples without a common pair.

Hence $D(n, 4, 2) = J(n, 4, 2) = \frac{1}{12}(n(n-1) - 18)$ for $n \equiv 7$ or $10 \pmod{12}$,
 $n \neq 10, 19$.

Throwing away one point (from the set $\{x_1, \dots, x_6\}$) yields:

$$D(n, 4, 2) = J(n, 4, 2) = \frac{1}{12}(n(n-3) - 6) \text{ for } n \equiv 6 \text{ or } 9 \pmod{12},$$

$$n \neq 9, 18.$$

For the exceptional cases we have

$$D(9, 4, 2) = 3$$

and

$$D(10, 4, 2) = 5$$

as can be immediately verified. Next

$$D(18, 4, 2) = 22$$

as follows from packings constructed by S. Lin and H.R. Phinney.

We give here the packing of H.R. Phinney since it has the largest automorphism group (sc. Z_2 , generated by $\pi := (0\ 10)(1\ 9)(2\ 13)(3\ 4)(5\ 6\ 15)(7\ 14)(8\ 17)(11)(12)(16)$).

0	1	2	3	1	10	14	17	3	9	12	17
0	4	5	6	2	4	15	17	4	9	10	13
0	7	8	9	2	5	11	13	5	7	12	14
0	10	11	12	2	6	7	10	6	9	11	14
0	13	14	15	2	8	14	16	6	12	15	16
1	4	8	12	3	4	11	16	7	13	16	17.
1	5	9	16	3	5	10	15				
1	7	11	15	3	6	8	13				

The value of $D(19,4,2)$ is not yet known; as a lower bound we have $D(19,4,2) \geq 25$ as follows from a packing constructed by H.R. Phinney (which is given below) while on the other hand $D(19,4,2) \leq 26$ as we shall prove below.

First the design:

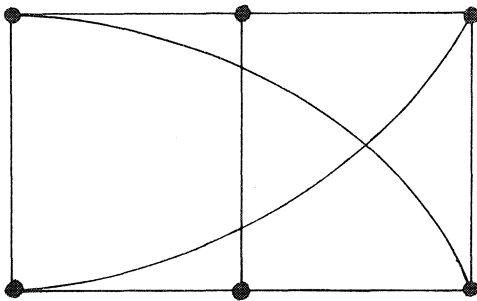
0	1	2	3	1	7	15	17	3	8	11	15
0	4	5	6	1	10	14	18	4	9	10	15
0	7	8	9	2	4	13	17	5	7	11	14
0	10	11	12	2	5	15	18	5	8	10	13
0	13	14	15	2	6	7	10	6	8	14	17
0	16	17	18	2	9	12	14	6	12	15	16
1	4	8	12	3	4	7	16	7	12	13	18.
1	5	9	16	3	5	12	17				
1	6	11	13	3	6	9	18				

PROPOSITION. *If $D(n,4,2) = J(n,4,2)$ for some $n \equiv 7$ or $10 \pmod{12}$ then the edges not covered by a maximal packing of K_4 's into K_n form a regular graph on 6 points with valency 3.*

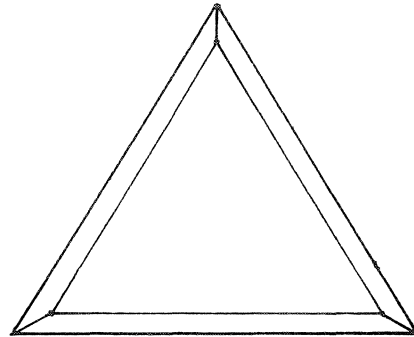
PROOF. $J(n,4,2) = \left\lfloor \frac{n}{4} \left\lfloor \frac{n-1}{3} \right\rfloor \right\rfloor - 1 = \frac{1}{12}(n(n-1)-18)$.

Each quadruple covers six edges, hence $J(n,4,2)$ quadruples cover all edges except nine. Let G be the graph (without isolated vertices) formed by these nine edges. In K_n each point has valency $n-1 \equiv 0 \pmod{3}$, and each quadruple removes 0 or 3 edges incident with a given point, hence in G each point has valency $\equiv 0 \pmod{3}$. Clearly valency ≥ 9 is impossible. If some point p in G has valency 6, then its 6 neighbours need at least 6 other edges in order to reach valency 3 each; but there are only nine edges in all, so valency 6 does not occur and G is regular, and hence has 6 vertices. \square .

LEMMA. *There are only two graphs on 6 points, regular with valency 3: $K_{3,3}$ and the prisma:*



and



\square

PROPOSITION. $D(19,4,2) \neq J(19,4,2)$.

PROOF. The edges of both graphs mentioned in the previous lemma can be covered with 3 K_4 's. Therefore if $D(n,4,2) = J(n,4,2)$ then $C(n,4,2) \leq J(n,4,2) + 3$ (where $C(n,4,2)$ is the number of K_4 's necessary to cover all edges of K_n). But $J(19,4,2) = 27$ and MILLS ([7]) proved that $C(19,4,2) = 31$ (by exhaustive computer search). Hence $D(19,4,2) \leq 26$. \square .

3. THE CLASS $\mathbb{B}\{4,7^*\}$

Let $\mathbb{B}\{4,7^*\}$ be the set of integers n for which there exists a pairwise balanced design on n points with blocks of size 4 or 7 and exactly one block of size 7 (and $\lambda=1$). Then

THEOREM. $\mathbb{B}\{4,7^*\} = \{n \mid n \equiv 7 \text{ or } 10 \pmod{12}\} \setminus \{10,19\}$.

Since by Hanani $\mathbb{B}\{4\} = \{n \mid n \equiv 1 \text{ or } 4 \pmod{12}\}$ we have as an immediate corollary:

COROLLARY. [Wilson] $\mathbb{B}\{4,7,10,19\} = \{n \mid n \equiv 1 \pmod{3}\}$.

PROOF OF THE THEOREM. Suppose $n \in \mathbb{B}\{4,7^*\}$. By considering the valency of a point it follows that $n \equiv 1 \pmod{3}$. Next, since $\binom{4}{2} = 6$ is even and $\binom{7}{2} = 21$ is odd, it follows that $\binom{n}{2}$ must be odd, so that $n \equiv 7 \text{ or } 10 \pmod{12}$. Also we saw in the previous section that $n \notin \{10,19\}$. [Note: this argument used that $D(19,4,2) \neq J(19,4,2)$ which is difficult to verify; on the other hand it is easy to see that if $n \in \mathbb{B}(K)$, where K is minimal (each element of K is used as a block size), then $n \geq (\min K - 1) \cdot \max K + 1$. In our case this means that $n \geq (4 - 1) \cdot 7 + 1 = 22$.] This proves the easy half of the theorem; the remainder of this section is devoted to the other half.

(i) The Truncated Transversal Design.

LEMMA 1. [Truncated Transversal] *If $\{3t+7, 3h+7\} \subset \mathbb{B}\{4,7^*\}$ and $t \geq h$ then $12t + 3h + 7 \in \mathbb{B}\{4,7^*\}$.*

PROOF. As usual: take a transversal design $T(5,1;t)$ (which exists since $t \equiv 0 \text{ or } 1 \pmod{4}$) and throw away $t-h$ points of one group. This leaves a design with blocks of size 4 or 5 and groups of size h or t on a set X with $|X| = 4t+h$. Next split each point into three points, constructing group-divisible designs $GD(4,1,3)$ on the sets of size $3 \times 4 = 12$ and $3 \times 5 = 15$, that is, make a design on the set $X \times I_3$ by taking for each group G of the original design a new group $G \times I_3$, and for each block B the blocks of a $GD(4,1,3;3|B|)$ constructed in such a way that it has groups $\{b\} \times I_3$. We now have a design with blocks of size 4 and groups of size $3h$ or $3t$. Adding a block Z of 7 points and the designs (on the sets $(G \times I_3) \cup Z$) $B(\{4,7^*\}, 1; 3h+7)$ and $B(\{4,7^*\}, 1; 3t+7)$ which exist by hypothesis, we obtain the required design $B(\{4,7^*\}, 1; 12t+3h+7)$. \square .

Let $x \equiv 7 \text{ or } 10 \pmod{12}$. There are 8 cases mod 48:

For $x \equiv 7 \text{ or } 19 \pmod{48}$ write $x = 12t + 7$ ($h=0$, $t \equiv 0 \text{ or } 1 \pmod{4}$).

If we assume that $3t + 7 \in \mathbb{B}\{4,7^*\}$ then $x \in \mathbb{B}\{4,7^*\}$ follows.

We may do this except for $t = 1$ or 4 , hence we get x unless $x = 19$ or 55 .

$19 \notin \mathbb{B}\{4,7^*\}$, and 55 will be done later.

For $x \equiv 22$ or $34 \pmod{48}$ write $x = 12t + 3.5 + 7$, ($h=5$, $t \equiv 0, 1 \pmod{4}$).
 If we assume that $3t + 7 \in \mathbb{B}\{4, 7^*\}$ and $t \geq 5$ then $x \in \mathbb{B}\{4, 7^*\}$ follows.
 We still have to do 22, 34 and 70.

For $x \equiv 31$ or $43 \pmod{48}$ write $x = 12t + 3.8 + 7$, ($h=8$, $t \equiv 0, 1 \pmod{4}$).
 Again for $t \geq 8$ $x \in \mathbb{B}\{4, 7^*\}$ follows provided that we can do 31, 43, 79 and 91.

For $x \equiv 46 \pmod{48}$ write $x = 12t + 3.9 + 7$, ($h=9$, $t \equiv 1 \pmod{4}$), this
 yields $x \geq 142$. We still have to do 46 and 94.

For $x \equiv 10 \pmod{48}$ write $x = 12t + 3.13 + 7$, ($h=13$, $t \equiv 1 \pmod{4}$), this
 yields $x \geq 202$. We still have to do 58, 106 and 154.

Therefore the theorem will be proved if we show that

$$\{22, 31, 34, 43, 46, 55, 58, 70, 79, 91, 94, 106, 154\} \subset \mathbb{B}\{4, 7^*\}.$$

(ii) Kirkman Designs.

LEMMA 2. For each t : $9t + 4 \in \mathbb{B}\{4, (3t+1)^*\}$.

PROOF. For $n \equiv 3 \pmod{6}$ a resolvable $B(\{3\}, 1; n)$ exists; completing such a
 design yields $n + (n-1)/2 \in \mathbb{B}\{4, ((n-1)/2)^*\}$.

Writing $n = 6t + 3$ gives the lemma. \square .

For $t = 2$ we get $22 \in \mathbb{B}\{4, 7^*\}$.

For $t = 10$ we get $94 \in \mathbb{B}\{4, 31^*\}$, and as soon as we know $31 \in \mathbb{B}\{4, 7^*\}$ it
 follows that $94 \in \mathbb{B}\{4, 7^*\}$.

(iii) Two orthogonal Latin Squares with Three Points Outside.

LEMMA 3. If $x \equiv 7$ or $43 \pmod{48}$ then $x \in \mathbb{B}\{4, 7^*\}$.

PROOF. Let $x = 4t + 3$, then $t \equiv 1$ or $10 \pmod{12}$ and hence $t + 3 \in \mathbb{B}\{4\}$.
 Also $t \neq 2, 6$ so $t \in T(4, 1)$. Take a transversal design $T(4, 1; t)$ on a set X
 and choose a fixed block $\{a_1, a_2, a_3, a_4\}$. Adjoin three new points x_0, x_1, x_2
 to X and for each group G make a $B(\{4\}, 1; t+3)$ on each of the sets
 $G \cup \{x_0, x_1, x_2\}$, taking care that the design on the group containing a_i has
 $\{a_i, x_0, x_1, x_2\}$ as a block. Now remove the blocks $\{a_1, a_2, a_3, a_4\}$ and
 $\{a_i, x_0, x_1, x_2\}$ ($1 \leq i \leq 4$) and add the block $\{a_1, a_2, a_3, a_4, x_0, x_1, x_2\}$. This yields
 a $B(\{4, 7^*\}, 1, x)$. \square .

In particular we find 43, 55 and 91.

We still have to do 31,34,46,58,70,79,106 and 154.

(iv) The case $x = 31$ (found by A.E.B. and P.D.P. 11/45 in close cooperation).

A Δ -factor of a graph is a 2-factor consisting of cycles of length 3. Or in design-theoretic terms: a Δ -factor is a parallel class of triples.

Using this definition we clearly have:

LEMMA 4. $n \in \mathbb{B}\{4,7^*\}$ iff there exists a design $B(\{3,4\},1;n-7)$ where the triples form 7 Δ -factors. \square .

In the current case we take for the set of vertices $X = Z_2 \times Z_2 \times Z_6$ (so that $|X|=24=31-7$), and the following blocks:

18 quadruples:

$$\begin{aligned} &\{(0,0,0), (0,1,0), (1,0,0), (1,1,0)\} \quad \text{mod } (-,-,6) \\ &\{(0,0,0), (0,0,3), (1,1,1), (1,1,4)\} \quad \text{mod } (2,2,-) \\ &\{(0,0,0), (0,0,4), (1,1,5), (0,1,2)\} \quad \text{mod } (2,2,-) \\ &\{(0,0,1), (0,0,5), (1,1,2), (0,1,3)\} \quad \text{mod } (2,2,-) \end{aligned}$$

7 Δ -factors:

1. $[\{(0,0,0), (0,0,1), (0,0,2)\}, \{(0,0,3), (0,0,4), (0,0,5)\}] \quad \text{mod } (2,2,-)$.
- 2,3. $[\{(0,0,0), (0,0,5), (0,1,1)\}, \{(0,0,2), (1,1,0), (0,1,3)\},$
 $\{(1,1,1), (1,1,3), (1,0,4)\}, \{(0,0,4), (1,1,2), (1,0,5)\}] \quad \text{mod } (-,2,-)$
 $\text{mod } (2,-,-)$.
- 4,5. $[\{(0,0,2), (0,0,3), (1,0,4)\}, \{(1,1,2), (1,1,5), (0,1,1)\},$
 $\{(0,0,0), (1,0,1), (0,1,4)\}, \{(1,1,0), (1,0,3), (0,1,5)\}] \quad \text{mod } (-,2,-)$
 $\text{mod } (2,-,-)$.
- 6,7. $[\{(0,0,0), (1,1,3), (0,1,5)\}, \{(0,0,2), (0,0,4), (1,0,0)\},$
 $\{(0,0,1), (1,1,5), (1,0,4)\}, \{(0,0,3), (1,1,2), (1,0,1)\}] \quad \text{mod } (-,2,-)$
 $\text{mod } (2,-,-)$.

Clearly it is a finite task to check the correctness of this design.

(v) The Case $x = 34$.

Let $X = (Z_3 \times Z_9) \cup (I_2 \times Z_3) \cup \{x\}$, where the elements of $Z_3 \times Z_9$ are written (i,j) and those of $I_2 \times Z_3$ $[i,j]$.

Take the following blocks:

$$\begin{aligned} \{(i,j), (i+1,j+2), (i+2,j+2), (i+2,j+3)\} & : 27 \text{ blocks} \\ \{(i,j), (i+1,j+3), (i+1,j+5), [0,j-i]\} & : 27 \text{ blocks} \\ \{(i,j), (i+1,j+4), (i+1,j+8), [1,j]\} & : 27 \text{ blocks} \\ \{(i,j), (i,j+3), (i,j+6), \quad \times \} \ (j < 3) & : 9 \text{ blocks.} \end{aligned}$$

(vi) The Case $x = 46$.

Let $X = (Z_3 \times Z_{13}) \cup (I_2 \times Z_3) \cup \{x\}$, and take the following blocks:

$$\begin{aligned} \{(i,j+1), (i,j+3), (i,j+9), (i+1,j)\} & : 39 \text{ blocks} \\ \{(i,j+2), (i,j+6), (i,j+5), (i+1,j)\} & : 39 \text{ blocks} \\ \{(i,j), (i+1,j+1), (i+2,j+4), [0,i]\} & : 39 \text{ blocks} \\ \{(i,j), (i+1,j+2), (i+2,j+7), [1,i]\} & : 39 \text{ blocks} \\ \{(0,j), (1,j), (2,j), \quad \times \} & : 13 \text{ blocks.} \end{aligned}$$

(vii) The Case $x = 58$.

Let $X = (Z_3 \times Z_{17}) \cup (I_2 \times Z_3) \cup \{x\}$, and take the following blocks:

$$\begin{aligned} \{(i,j), (i,j+1), (i,j+4), (i+1,j+5)\} \\ \{(i,j), (i,j+2), (i,j+8), (i+1,j+11)\} \\ \{(i,j), (i,j+5), (i+1,j+2), (i+1,j+12)\} \\ \{(i,j), (i+1,j+8), (i+2,j+7), [0,i]\} \\ \{(i,j), (i+1,j+6), (i+2,j+4), [1,i]\} \\ \{(0,j), (1,j), (2,j), \quad \times \}. \end{aligned}$$

(viii) The Cases 70 and 79.

In [7] Mills showed that $70 \in \mathbb{B}\{4, 22^*\}$ and $79 \in \mathbb{B}\{4, 13^*, 22^*\}$. Since $13 \in \mathbb{B}\{4\}$ and $22 \in \mathbb{B}\{4, 7^*\}$ it immediately follows that $\{70, 79\} \subset \mathbb{B}\{4, 7^*\}$.

(ix) The Cases 106 and 154.

LEMMA 5. *If $t \in \mathbb{B}\{4, 5, 8, 9, 12, k^*\}$ and $3k+1 \in \mathbb{B}\{4, 7^*\}$ then $3t + 1 \in \mathbb{B}\{4, 7^*\}$. In particular this applies for $k = 7$ or 11 .*

PROOF. Let \mathcal{B} be a design on a set I_t with all block sizes congruent 0 or 1 (mod 4) but with one block of size 7 or 11. We can get a $B(\{4, 7^*\}, 1; 3t+1)$ on the set $I_t \times I_3 \cup \{x\}$ by taking for each block $B \in \mathcal{B}$ with $|B| \equiv 0$ or 1

(mod 4) a design $B(\{4\}, 1; 3|B|+1)$ on the set $B \times I_3 \cup \{x\}$, taking care that it contains the blocks $\{b\} \times I_3 \cup \{x\}$ for each $b \in B$; if B_0 is the block with $|B_0| \not\equiv 0$ or $1 \pmod{4}$ then throw away all blocks $\{b\} \times I_3 \cup \{x\}$ for $b \in B_0$, and add the block $B_0 \times I_3 \cup \{x\}$. We now have a $B(\{4, (3|B_0|+1)^*\}, 1; 3t+1)$. Since $\{22, 34\} \subset \mathbb{B}\{4, 7^*\}$ this proves the lemma. \square .

Now for $x = 106 = 3 \cdot 35 + 1$ we take a resolvable $B(\{4\}, 1; 28)$ and partially complete it with 7 points. (This is possible since it has $(28-1)/3 = 9$ parallel classes.) This yields a $B(\{4, 5, 7^*\}, 1; 35)$ and we may apply the lemma. Likewise for $X = 154 = 3 \cdot 51 + 1$ we take a resolvable $B(\{4\}, 1; 40)$ and partially complete it with 11 points which yields a $B(\{4, 5, 11^*\}, 1; 51)$ and we are through.

This completes the proof of our theorem.

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