A.E. Brouwer

Optimal packings of $K_4$'s into $K_n$ - the case $n \not\equiv 2 \pmod{3}$
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Optimal packings of $K_4$'s into a $K_n$ - The case $n \not\equiv 2 \pmod{3}$.

by

A.E. Brouwer

ABSTRACT

In this paper we construct a pairwise balanced design $B((4,7^*),1;n)$ (i.e., a design with blocks of size 4 or 7 and exactly one block of size 7, on $n$ points with $\lambda = 1$) for each $n \equiv 7$ or 10 (mod 12) except $n = 10$ or 19 (in which cases such a design cannot exist). From these designs optimal packings of $K_4$'s into a $K_n$ are derived for $n \not\equiv 2 \pmod{3}$, $n \not\in \{9,13,18,19\}$, while the case $n \in \{9,10,18\}$ is treated by ad hoc methods. It is not known whether the known packing of 25 $K_4$'s in $K_{19}$ is optimal.

KEY WORDS & PHRASES: pairwise balanced design, scarce design, packing, constant weight code
1. INTRODUCTION

Let \( I_n \) be a finite set of \( n \) elements. For \( n \geq k \geq t \) let \( D(n,k,t) \) be the largest integer \( b \) such that there exist \( b \) subsets \( B_1, \ldots, B_b \) of \( I_n \) each of \( k \) elements, such that every \( t \)-element subset of \( I_n \) is contained in at most one of them.

In a previous paper ([1]) the present author and A. Schrijver determined \( D(n,4,2) \) for \( n \equiv 2 \pmod{6} \). Here we treat \( n \equiv 0 \) or \( 1 \pmod{3} \) (except \( n = 19 \)), and in a future paper we will discuss the remaining case \( n \equiv 5 \pmod{6} \). The overall result is the following:

Define

\[
J(n,4,2) = \begin{cases} 
\left\lfloor \frac{n}{4} \right\rfloor \left\lfloor \frac{n-1}{3} \right\rfloor - 1 & \text{for } n \equiv 7 \text{ or } 10 \pmod{12} \\
\left\lfloor \frac{n}{4} \right\rfloor \left\lfloor \frac{n-1}{3} \right\rfloor & \text{otherwise.}
\end{cases}
\]

Then

(i) for each \( n \) \( D(n,4,2) \leq J(n,4,2) \)

(this is the so-called Johnson bound, see e.g. JOHNSON [4])

(ii) for almost all \( n \) \( D(n,4,2) = J(n,4,2) \).

Cases in which \( D(n,4,2) \neq J(n,4,2) \) is known:

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<tr>
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<tr>
<td>( J(n,4,2) )</td>
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<td>25 or 26</td>
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I conjecture that for all other \( n \) equality holds. (In any case all further exceptions must have \( n \equiv 11 \pmod{12} \)).

REMARK. The undefined notations, especially for various types of designs, are taken from HANANI ([3]) or WILSON ([10]).
2. OPTIMAL PACKINGS

A. The case \( n \equiv 0,1,3 \) or 4 (mod 12)

For \( n \equiv 1 \) or 4 (mod 12) HANANI [12] has constructed a Steiner system S(2,4,n). Therefore \( D(n,4,2) = J(n,4,2) = \frac{1}{12} n(n-1) \) for these \( n \).

Throwing away a fixed point and all blocks containing it produces a system of \( \frac{1}{12} n(n-1) - \frac{1}{3}(n-1) = \frac{1}{12}(n-1)(n-4) \) four-tuples on \( n-1 \) points, i.e. \( D(n,4,2) = J(n,4,2) = \frac{1}{12} n(n-3) \) for \( n \equiv 0 \) or 3 (mod 12).

B. The case \( n \equiv 6,7,9 \) or 10 (mod 12)

For \( n \equiv 7 \) or 10 (mod 12), \( n \neq 10,19 \), we will construct in the next section a pairwise balanced design on \( n \) points with \( \lambda = 1 \) and blocks of size 4 or 7, using exactly one block of size 7 (notation: \( B([4,7^8],1;n) \)).

If we replace the block \( \{x_0, \ldots, x_6\} \) of size 7 of such a design by the two four-tuples \( \{x_0, x_1, x_2, x_3\} \) and \( \{x_0, x_4, x_5, x_6\} \) we have a collection of \( \frac{1}{6}(\binom{n}{2} - \binom{7}{2}) + 2 = \frac{1}{12}(n(n-1)-18) = J(n,4,2) \) four-tuples without a common pair.

Hence \( D(n,4,2) = J(n,4,2) = \frac{1}{12}(n(n-1)-18) \) for \( n \equiv 7 \) or 10 (mod 12), \( n \neq 10,19 \).

Throwing away one point (from the set \( \{x_1, \ldots, x_6\} \)) yields:

\[ D(n,4,2) = J(n,4,2) = \frac{1}{12}(n(n-3)-6) \] for \( n \equiv 6 \) or 9 (mod 12), \( n \neq 9,18 \).

For the exceptional cases we have

\[ D(9,4,2) = 3 \]

and

\[ D(10,4,2) = 5 \]

as can be immediately verified. Next

\[ D(18,4,2) = 22 \]

as follows from packings constructed by S. Lin and H.R. Phinney.
We give here the packing of H.R. Phinney since it has the largest automorphism group (sc. $\mathbb{Z}_2$, generated by $\pi := (0\ 10)(1\ 9)(2\ 13)(3\ 4)(5\ 6\ 15)$ $(7\ 14)(8\ 17)(11)(12)(16))$.

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The value of $D(19,4,2)$ is not yet known; as a lower bound we have $D(19,4,2) \geq 25$ as follows from a packing constructed by H.R. Phinney (which is given below) while on the other hand $D(19,4,2) \leq 26$ as we shall prove below.

First the design:

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**PROPOSITION.** If $D(n,4,2) = J(n,4,2)$ for some $n \equiv 7$ or $10 \pmod{12}$ then the edges not covered by a maximal packing of $K_4$'s into $K_n$ form a regular graph on 6 points with valency 3.

**PROOF.** $J(n,4,2) = \left\lceil \frac{n}{4} \right\rceil \left\lfloor \frac{n-1}{3} \right\rfloor - 1 = \frac{1}{12}(n(n-1)-18)$.
Each quadruple covers six edges, hence $J(n,4,2)$ quadruples cover all edges except nine. Let $G$ be the graph (without isolated vertices) formed by these nine edges. In $K_n^4$, each point has valency $n-1 \equiv 0 \pmod{3}$, and each quadruple removes 0 or 3 edges incident with a given point, hence in $G$ each point has valency $\equiv 0 \pmod{3}$. Clearly valency $\geq 9$ is impossible. If some point $p$ in $G$ has valency 6, then its 6 neighbours need at least 6 other edges in order to reach valency 3 each; but there are only nine edges in all, so valency 6 does not occur and $G$ is regular, and hence has 6 vertices. $\Box$.

**Lemma.** There are only two graphs on 6 points, regular with valency 3: $K_3,3$ and the prism:

PROPOSITION. $D(19,4,2) \neq J(19,4,2)$.

**Proof.** The edges of both graphs mentioned in the previous lemma can be covered with 3 $K_4$'s. Therefore if $D(n,4,2) = J(n,4,2)$ then $C(n,4,2) \leq J(n,4,2) + 3$ (where $C(n,4,2)$ is the number of $K_4$'s necessary to cover all edges of $K_n^4$). But $J(19,4,2) = 27$ and MILLS ([7]) proved that $C(19,4,2) = 31$ (by exhaustive computer search). Hence $D(19,4,2) \leq 26$. $\Box$.

3. THE CLASS $\mathcal{B}(4,7^\ast)$

Let $\mathcal{B}(4,7^\ast)$ be the set of integers $n$ for which there exists a pairwise balanced design on $n$ points with blocks of size 4 or 7 and exactly one block of size 7 (and $\lambda=1$). Then
THEOREM. $\mathcal{B}\{4,7^*\} = \{n \mid n \equiv 7 \text{ or } 10 \pmod{12}\}\setminus\{10,19\}$.

Since by Hanani $\mathcal{B}\{4\} = \{n \mid n \equiv 1 \text{ or } 4 \pmod{12}\}$ we have as an immediate corollary:

COROLLARY. [Wilson] $\mathcal{B}\{4,7,10,19\} = \{n \mid n \equiv 1 \pmod{3}\}$.

PROOF OF THE THEOREM. Suppose $n \in \mathcal{B}\{4,7^*\}$. By considering the valency of a point it follows that $n \equiv 1 \pmod{3}$. Next, since $\binom{4}{2} = 6$ is even and $\binom{7}{2} = 21$ is odd, it follows that $\binom{n}{2}$ must be odd, so that $n \equiv 7$ or $10 \pmod{12}$. Also we saw in the previous section that $n \notin \{10,19\}$. [Note: this argument used that $D(19,4,2) \neq J(19,4,2)$ which is difficult to verify; on the other hand it is easy to see that if $n \in \mathcal{B}(K)$, where $K$ is minimal (each element of $K$ is used as a block size), then $n \geq (\min K-1)\max K + 1$. In our case this means that $n \geq (4-1)7+1 = 22$. This proves the easy half of the theorem; the remainder of this section is devoted to the other half.

(i) The Truncated Transversal Design.

LEMMA 1. [Truncated Transversal] If $\{3t+7,3h+7\} \in \mathcal{B}\{4,7^*\}$ and $t \geq h$ then $12t + 3h + 7 \in \mathcal{B}\{4,7^*\}$.

PROOF. As usual: take a transversal design $T(5,1; t)$ (which exists since $t \equiv 0$ or $1 \pmod{4}$) and throw away $t-h$ points of one group. This leaves a design with blocks of size 4 or 5 and groups of size $h$ or $t$ on a set $X$ with $|X| = 4t+h$. Next split each point into three points, constructing group-divisible designs $GD(4,1,3)$ on the sets of size $3 \times 4 = 12$ and $3 \times 5 = 15$, that is, make a design on the set $XI^3_I$ by taking for each group $G$ of the original design a new group $G \times I^3_I$, and for each block $B$ the blocks of a $GD(4,1,3; 3|B|)$ constructed in such a way that it has groups $\{b\} \times I^3_I$. We now have a design with blocks of size 4 and groups of size $3h$ or $3t$. Adding a block $Z$ of 7 points and the designs (on the sets $(G \times I^3_I)\cup Z$) $B(\{4,7^*\}, 1; 3h+7)$ and $B(\{4,7^*\}, 1; 3t+7)$ which exist by hypothesis, we obtain the required design $B(\{4,7^*\}, 1; 12t+3h+7)$.

Let $x \equiv 7$ or $10 \pmod{12}$. There are 8 cases mod 48:

For $x \equiv 7$ or 19 (mod 48) write $x = 12t + 7$ ($h=0$, $x=0$ or $1 \pmod{4}$).

If we assume that $3t + 7 \in \mathcal{B}\{4,7^*\}$ then $x \in \mathcal{B}\{4,7^*\}$ follows.

We may do this except for $t = 1$ or 4, hence we get $x$ unless $x = 19$ or 55.

$19 \notin \mathcal{B}\{4,7^*\}$, and 55 will be done later.
For \( x \equiv 22 \) or \( 34 \) (mod 48) write \( x = 12t + 3.5 + 7 \), (h=5, t=0,1 (mod 4)).
If we assume that \( 3t + 7 \in B\{4,7^*\} \) and \( t \geq 5 \) then \( x \in B\{4,7^*\} \) follows.
We still have to do 22, 34 and 70.

For \( x \equiv 31 \) or \( 43 \) (mod 48) write \( x = 12t + 3.8 + 7 \), (h=8, t=0,1 (mod 4)).
Again for \( t \geq 8 \) \( x \in B\{4,7^*\} \) follows provided that we can do 31, 43, 79 and 91.

For \( x \equiv 46 \) (mod 48) write \( x = 12t + 3.9 + 7 \), (h=9, t = 1 (mod 4)), this
yields \( x \geq 142 \). We still have to do 46 and 94.

For \( x \equiv 10 \) (mod 48) write \( x = 12t + 3.13 + 7 \), (h=13, t = 1 (mod 4)), this
yields \( x \geq 202 \). We still have to do 58, 106 and 154.

Therefore the theorem will be proved if we show that

\[
\{22,31,34,43,46,55,58,70,79,91,94,106,154\} \in B\{4,7^*\}.
\]

(ii) Kirkman Designs.

**Lemma 2.** For each \( t \): \( 9t + 4 \in B\{4,(3t+1)^*\} \).

**Proof.** For \( n \equiv 3 \) (mod 6) a resolvable \( B\{(3),1;n\} \) exists; completing such a
design yields \( n + (n-1)/2 \in B\{(n-1)/2\} \).
Writing \( n = 6t + 3 \) gives the lemma. \( \Box \).

For \( t = 2 \) we get 22 \( \in B\{4,7^*\} \).

For \( t = 10 \) we get 94 \( \in B\{4,31^*\} \), and as soon as we know 31 \( \in B\{4,7^*\} \) it
follows that 94 \( \in B\{4,7^*\} \).

(iii) Two orthogonal Latin Squares with Three Points Outside.

**Lemma 3.** If \( x \equiv 7 \) or \( 43 \) (mod 48) then \( x \in B\{4,7^*\} \).

**Proof.** Let \( x = 4t + 3 \), then \( t \equiv 1 \) or \( 10 \) (mod 12) and hence \( t + 3 \in B\{4\} \).
Also \( t \not\equiv 2,6 \) so \( t \in T\{4,1\} \). Take a transversal design \( T\{4,1; t\} \) on a set \( X \)
and choose a fixed block \( \{a_1,a_2,a_3,a_4\} \). Adjoin three new points \( x_0,x_1,x_2 \)
to \( X \) and for each group \( G \) make a \( B\{(4,1);t+3\} \) on each of the sets
\( G \cup \{x_0,x_1,x_2\} \), taking care that the design on the group containing \( a_1 \) has
\( \{a_1,x_0,x_1,x_2\} \) as a block. Now remove the blocks \( \{a_1,a_2,a_3,a_4\} \) and
\( \{a_1,x_0,x_1,x_2\} \) (1\leq i\leq 4) and add the block \( \{a_1,a_2,a_3,a_4,x_0,x_1,x_2\} \). This yields
a \( B\{(4,7^*),1,x\} \). \( \Box \).
In particular we find 43, 55 and 91.
We still have to do 31, 34, 46, 58, 70, 79, 106 and 154.

(iv) The case \( x = 31 \) (found by A.E.B. and P.D.P. 11/45 in close cooperation).
A \( \Delta \)-factor of a graph is a 2-factor consisting of cycles of length 3. Or in design-theoretic terms: a \( \Delta \)-factor is a parallel class of triples.
Using this definition we clearly have:

**LEMMA 4.** \( n \in \mathbb{B} \{4,7^*\} \iff \) there exists a design \( B(\{3,4\},1; n-7) \) where the triples form 7 \( \Delta \)-factors. \( \Box \).

In the current case we take for the set of vertices \( X = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_6 \) (so that \( |X|=24=31-7 \)), and the following blocks:

18 quadruples:

\[
\begin{align*}
(0,0,0), (0,1,0), & (1,0,0), (1,1,0) \mod (-,-,-,6) \\
(0,0,0), (0,0,3), & (1,1,1), (1,1,4) \mod (2,2,-) \\
(0,0,0), (0,0,4), & (1,1,5), (0,1,2) \mod (2,2,-) \\
(0,0,1), (0,0,5), & (1,1,2), (0,1,3) \mod (2,2,-)
\end{align*}
\]

7 \( \Delta \)-factors:

1. \[
\begin{align*}
\{(0,0,0), (0,0,1), (0,0,2)\}, \{(0,0,3), (0,0,4), (0,0,5)\} \mod (2,2,-).
\end{align*}
\]

2, 3. \[
\begin{align*}
\{(0,0,0), (0,0,5), (0,1,1)\}, \{(0,0,2), (1,1,0), (0,1,3)\}, \\
\{(1,1,1), (1,1,3), (1,0,4)\}, \{(0,0,4), (1,1,2), (1,0,5)\} \mod (2,-,-).
\end{align*}
\]

4, 5. \[
\begin{align*}
\{(0,0,2), (0,0,3), (1,0,4)\}, \{(1,1,2), (1,1,5), (0,1,1)\}, \\
\{(0,0,0), (1,0,1), (0,1,4)\}, \{(1,1,0), (1,0,3), (0,1,5)\} \mod (2,-,-).
\end{align*}
\]

6, 7. \[
\begin{align*}
\{(0,0,0), (1,1,3), (0,1,5)\}, \{(0,0,2), (0,0,4), (1,0,0)\}, \\
\{(0,0,1), (1,1,5), (1,0,4)\}, \{(0,0,3), (1,1,2), (1,0,1)\} \mod (2,-,-).
\end{align*}
\]

Clearly it is a finite task to check the correctness of this design.

(v) The case \( x = 34 \).

Let \( X = (\mathbb{Z}_3 \times \mathbb{Z}_9) \cup (\mathbb{I}_2 \times \mathbb{Z}_3) \cup \{x\} \), where the elements of \( \mathbb{Z}_3 \times \mathbb{Z}_9 \) are written \((i,j)\) and those of \( \mathbb{I}_2 \times \mathbb{Z}_3 \) \([i,j]\).
Take the following blocks:

\{
(i,j), (i+1,j+2), (i+2,j+2), (i+2,j+3) \} : 27 blocks
\{(i,j), (i+1,j+3), (i+1,j+5), [0,j-i] \} : 27 blocks
\{(i,j), (i+1,j+4), (i+1,j+8), [1,j] \} : 27 blocks
\{(i,j), (i,j+3), (i,j+6), \times \} (j<3) : 9 blocks.

(vi) The Case $x = 46$.

Let $X = (Z_3 \times Z_{13}) \cup (I_2 \times Z_3) \cup \{x\}$, and take the following blocks:

\{
(i,j+1), (i,j+3), (i,j+9), (i+1,j) \} : 39 blocks
\{(i,j+2), (i,j+6), (i,j+5), (i+1,j) \} : 39 blocks
\{(i,j), (i+1,j+4), (i+2,j+4), [0,i] \} : 39 blocks
\{(i,j), (i+1,j+2), (i+2,j+7), [1,i] \} : 39 blocks
\{(0,j), (1,j), (2,j), \times \} : 13 blocks.

(vii) The Case $x = 58$.

Let $X = (Z_3 \times Z_{17}) \cup (I_2 \times Z_3) \cup \{x\}$, and take the following blocks:

\{
(i,j), (i,j+1), (i,j+4), (i+1,j+5) \}
\{(i,j), (i,j+2), (i,j+8), (i+1,j+11) \}
\{(i,j), (i,j+5), (i+1,j+2), (i+1,j+12) \}
\{(i,j), (i+1,j+8), (i+2,j+7), [0,i] \}
\{(i,j), (i+1,j+6), (i+2,j+4), [1,i] \}
\{(0,j), (1,j), (2,j), \times \}.

(viii) The Cases 70 and 79.

In [7] Mills showed that $70 \in \mathbb{B}\{4, 22^*\}$ and $79 \in \mathbb{B}\{4, 13^*, 22^*\}$. Since $13 \in \mathbb{B}\{4\}$ and $22 \in \mathbb{B}\{47^*\}$ it immediately follows that $\{70, 79\} \subset \mathbb{B}\{47^*\}$.

(ix) The Cases 106 and 154.

\textbf{Lemma 5.} If $t \in \mathbb{B}\{4, 5, 8, 9, 12, k^*\}$ and $3k+1 \in \mathbb{B}\{4, 7^*\}$ then $3t + 1 \in \mathbb{B}\{4, 7^*\}$. In particular this applies for $k = 7$ or 11.

\textbf{Proof.} Let $\mathcal{B}$ be a design on a set $I_t$ with all block sizes congruent 0 or 1 \textup{(mod 4)} but with one block of size 7 or 11. We can get a $\mathcal{B}\{(4, 7^*), 1; 3t+1\}$ on the set $I_t \times I_3 \cup \{x\}$ by taking for each block $B \in \mathcal{B}$ with $|B| \equiv 0$ or 1
(mod 4) a design \( B(\{4\}, 1; 3|B|+1) \) on the set \( B \times I_3 \cup \{x\} \), taking care that it contains the blocks \( \{b\} \times I_3 \cup \{x\} \) for each \( b \in B \); if \( B_0 \) is the block with \( |B_0| \neq 0 \) or 1 (mod 4) then throw away all blocks \( \{b\} \times I_3 \cup \{x\} \) for \( b \in B_0 \), and add the block \( B_0 \times I_3 \cup \{x\} \). We now have a \( B(\{4, (3|B_0|+1)*\}, 1; 3t+1) \).
Since \( \{22,34\} \subseteq B(\{4,7\}^*) \) this proves the lemma. \( \square \).

Now for \( x = 106 = 3.35 + 1 \) we take a resolvable \( B(\{4\}, 1; 28) \) and partially complete it with 7 points. (This is possible since it has \( (28-1)/3 = 9 \) parallel classes.) This yields a \( B(\{4,5,7\}^*, 1; 35) \) and we may apply the lemma. Likewise for \( X = 154 = 3.51 + 1 \) we take a resolvable \( B(\{4\}, 1; 40) \) and partially complete it with 11 points which yields a \( B(\{4,5,11\}^*, 1; 51) \) and we are through.

This completes the proof of our theorem.

REFERENCES

[1] Brouwer, A.E. & A. Schrijver, A group-divisible design G\(D(4,1,2;n) \) exists iff \( n \equiv 2 \) (mod 6), \( n \neq 8 \) (or: the packing of cocktail party graphs with \( K_4 \)')s, Math. Centr. report ZW 64 (1976).


