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AFDELING ZUIVERE WISKUNDE ZW 81/76 AUGUSTUS
(DEPARTMENT OF PURE MATHEMATICS)

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A GENERALIZATION OF BARANYAI'S THEOREM

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Printed at the Mathematical Centre, 49, 2e Boerhaavestraat, Amsterdam.

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AMS(MOS) subject classification scheme (1970): 05A17, 05B30

A generalization of Baranyai's theorem

by

A.E. Brouwer

ABSTRACT

The existence of resolvable parallelisms on a complete multipartite hypergraph is shown. As an application a question of P.J. Cameron is answered.

KEY WORDS & PHRASES: *parallelism*

1. INTRODUCTION

Let X be a finite set which is the disjoint union of r subsets X_i:

$$X = \bigcup_{j=1}^{r} X_j .$$

Let $n = |X|$ and $n_j = |X_j|$ $(1 \le j \le r)$.
Let $N = \prod_{j=1}^{r} \{0,\dots,n_j\}$ and define for $\underline{i} \in N$:

$$\binom{\underline{n}}{\underline{i}} = \prod_{j=1}^{r} \binom{n_j}{i_j}$$

For each subset $A \subset X$ we define its *characteristic* as the rowvector $\underline{i} = \underline{i}_A = (|A \cap X_1|,\dots,|A \cap X_r|) \in N$. Observe that $\binom{\underline{n}}{\underline{i}}$ is just the number of subsets of X with characteristic $\underline{i}$. Let a map $\underline{a}: N \to \mathbb{N}$ be given. (We shall often write $a_{\underline{i}}$ instead of $a(\underline{i})$.) A collection $\mathcal{C}$ of subsets of X is called an (a)-*spread* if

(i) for each $\underline{i} \in N$ it contains exactly $a_{\underline{i}}$ sets of characteristic $\underline{i}$ and

(ii) each point of X_j is contained in the same number λ_j $(1 \le j \le r)$ of elements of $\mathcal{C}$.

If $\underline{\lambda} = \underline{1}$ it is called an (a)-*partition*.
Observe that $\underline{\lambda}$ is uniquely determined by the function a:

$$\sum_{\underline{i} \in N} a_{\underline{i}}\, \underline{i} = \underline{\lambda} \cdot \underline{n} = (\lambda_1 n_1,\dots,\lambda_r n_r).$$

We now have the following theorem:

<u>THEOREM 1</u>. *A collection of* ℓ (a)-*spreads on* X *such that each subset of* X *with characteristic* $\underline{i}$ *occurs exactly* $\alpha_{\underline{i}}$ *times among the members of the spreads exists if and only if*

(i) *for each* $\underline{i} \in N$: $\ell\, a_{\underline{i}} = \binom{\underline{n}}{\underline{i}} \alpha_{\underline{i}}$,

(ii) $\sum_{\underline{i} \in N} a_{\underline{i}}\, \underline{i} = \underline{n} \cdot \underline{\lambda}$

where (if $\ell \ne 0$*)* ℓ *and the* $\alpha_{\underline{i}}$ ($\underline{i} \in N$) *and* λ_j $(1 \le j \le r)$ *must be integers.*

The stated conditions are obviously necessary: (i) counts the number of sets with characteristic $\underline{i}$ in two ways, while (ii) counts in two ways the number of times a point is covered. The sufficiency will be proved in the next section.

Now consider some special cases:

First, if we set $r = 1$ and $\alpha_{\underline{i}} = \delta_{\underline{i}h}$ (then $a_{\underline{i}} = \delta_{\underline{i}h} \cdot \frac{n\lambda}{h}$ and $\ell = \frac{1}{\lambda}\binom{n-1}{h-1}$) we get the theorem of BARANYAI [1]:

COROLLARY 1.1. *If* $h|n\lambda$ *and* $\lambda|\binom{n-1}{h-1}$ *then the complete* h*-uniform hypergraph on* n *vertices is* λ*-factorizable; in particular this is true for* $\lambda = \frac{h}{(n,h)}$.

Here a λ-factorization of a hypergraph (X,E) is a partition of its edge-set $E = \bigcup_j E_j$ such that for each j and each $x \in X$ $|\{E \in E_j \mid x \in E\}| = \lambda$ holds.

A 1-factorization is also called a *parallelism*.

The next special case, $r = 2$, will provide an answer to the question of P.J. CAMERON [2]: For which h and n does there exist a parallelism on the collection of all h-subsets of a given n-set X such that it induces a parallelism on some $\frac{1}{2}$n-subset X_1 of X?

That is, we would like to have a parallelism on X such that each parallel class either contains only h-sets intersecting both X_1 and $X_2 := X \setminus X_1$ or contains only h-sets entirely contained within X_1 or X_2. Clearly $2h|n$ is necessary. Cameron knew of solutions for $h = 2$ or $h = 3$ and $n = 12$ or $n = 2h$, while J.-C. Bermond, J.I. Hall and the author constructed solutions for $h = 3$ and $6|n$ using resolvable triple systems.

But from the theorem above, taking $r = 2$, $n_1 = n_2 = \frac{1}{2}n$, $\lambda_1 = \lambda_2 = 1$ and some fixed g : $\alpha_{g,h-g} = \alpha_{h-g,g} = 1$ and all other α's zero (so that $a_{g,h-g} - a_{h-g,g} = \frac{n}{2h}$ if $2g \neq h$ and $a_{g,g} = \frac{n}{h}$ if $2g = h$, while it is also easy to check that ℓ is integral), it follows that there exists a parallelism on all h-subsets intersecting X_1 in g or h-g points; now take the union of these parallelisms for $g = 0,1,\ldots, \lfloor \frac{1}{2}h \rfloor$ to get the required system:

COROLLARY 1.2. *If* $2h|n$ *then there exists a parallelism on the collection of all* h*-subsets of a given* n*-set which induces a parallelism on a* $\frac{1}{2}$n*-subset.*

Finally we mention a result anounced in BARANYAI [1]:
Let $K^h_{r\times m}$ be the collection of all h-subsets $A \subset X$ such that

$$|A \cap X_j| \le 1 \ (1 \le j \le r),$$

where $|X_1| = \ldots = |X_r| = m$ (so that $n = rm$). Then

COROLLARY 1.3. *Let* $1 \le h \le r$ *and* $h|n\lambda$ *and* $\lambda|\binom{r-1}{h-1}m^{h-1}$. *Then* $K^h_{r\times m}$ *is* λ*-factorizable.*

PROOF. If $\binom{r-1}{h-1}|\lambda m$ we can directly apply Theorem 1 to get a λ-factorization in which every λ-factor is an (a)-spread for the same function a. In the general case however, just as in the proof of the corollary 1.2, we need λ-factors of several types. The choice of the types can be done by an application of corollary 1.1 as follows: Let

$$\mu = \frac{h}{(h,r)}, \quad \text{and let } K^h_r = \bigcup_j E_j \quad (j=1,\ldots,\binom{r-1}{h-1}/\mu)$$

be a μ-factorization of the complete h-uniform hypergraph on r vertices. Identifying sets $E \in E_j$ with 0-1 vectors of length r, we can consider each E_j as a subset of N. Now apply Theorem 1 for each j with $\alpha_{\underline{i}} = 1$ if $\underline{i} \in E_j$ and $\alpha_{\underline{i}} = 0$ otherwise. (Then $\ell = \frac{\mu}{\lambda} m^{h-1}$ and $a_{\underline{i}} = \frac{\lambda}{\mu} m$ (if $\underline{i} \in E_j$) are integers.) This yields that for each j the collection of subsets of X with characteristic in E_j is λ-factorizable, and hence $K^h_{r\times m}$ is λ-factorizable. □

PROOF OF THE THEOREM. Let

$$X = \{x_1,\ldots,x_n\}, \text{ and } X_j = \{x_{m_{j-1}+1},\ldots,x_{m_j}\}$$

where

$$m_s = \sum_{j\le s} n_j.$$

We prove the theorem using induction with respect to k and s, where k ranges from 0 to n and either $x_k \in X_s$ or $k = m_{s-1}$. The inductive assertion is:

Let $X^{(k)} = \{x_1,\ldots,x_k\}$. *There exists a collection of* ℓ $\underline{\lambda}$*-factors* $F_g^{(k)}$ $(1 \le g \le \ell)$ *on the set* $X^{(k)}$, *where each* $F_g^{(k)}$ *is the disjoint union of sets* $F_{g,\underline{i}}^{(k)}$ $(\underline{i} \in N)$ *such that*

1. $|F_{g,\underline{i}}^{(k)}| = a_{\underline{i}}$ *for* $\underline{i} \in N$ *and* $1 \le g \le \ell$.
2. *If* $Y \in F_{g,\underline{i}}^{(k)}$ *then for* $j < s$: $|Y \cap X_j| = i_j$.
3. *If* $Y \subset X^{(k)}$ *then for each* $\underline{i}$ *such that* $Y \cap X_j = i_j$ *for* $j < s$ Y *occurs* $\alpha_{\underline{i}} M_{\underline{i}} \binom{m_s - k}{i_s - |Y \cap X_s|}$ *times in some* $F_{g,\underline{i}}^{(k)}$, *where*

$$M_{\underline{i}} = \prod_{j=s+1}^{r} \binom{n_j}{i_j}.$$

The idea is that the $F_g^{(n)}$ are the required $\underline{\lambda}$-factors, and the $F_{g,\underline{i}}^{(n)}$ are the subsets of $F_g^{(n)}$ consisting precisely of the sets with characteristic $\underline{i}$. The $F_g^{(k)}$ and $F_{g,\underline{i}}^{(k)}$ will be their restrictions to $X^{(k)}$, i.e. $F_g^{(k)} =$
$= \{A \cap X^{(k)} \mid A \in F_g^{(n)}\}$ and for $F_{g,\underline{i}}^{(k)}$ likewise.

Note that $F_g^{(k)}$ may contain the same set more than once, i.e. it is a selection rather than a set.
Given this interpretation, the conditions of the inductive hypothesis are clearly necessary, and it will appear below that they suffice.

Starting the induction with $k = 0$, $s = 1$, we are to construct collections $F_{g,\underline{i}}^{(0)}$ containing empty sets only, where the empty set occurs for each $\underline{i} \in N$ $\alpha_{\underline{i}} \binom{\underline{n}}{\underline{i}}$ times in some $F_{g,\underline{i}}^{(0)}$, and $|F_{g,\underline{i}}^{(0)}| = a_{\underline{i}}$. This is possible since by assumption $\alpha_{\underline{i}}$ and $a_{\underline{i}}$ are integers and $\alpha_{\underline{i}} \binom{\underline{n}}{\underline{i}} = \ell a_{\underline{i}}$.

There are two kinds of induction step: steps that increment k and steps that increment s if $k = m_s$.
The latter are only a formality: suppose the induction hypothesis has been verified for $s = t$ and $k = m_t$, and let now $s = t + 1$.
2. requires that for $Y \in F_{g,\underline{i}}^{(k)}$ $|Y \cap X_t| = i_t$ but this follows from 3. since $\binom{m_t - k}{i_t - |Y \cap X_t|}$ is nonzero only if $|Y \cap X_t| = i_t$.

3. requires that Y occurs $\alpha_{\underline{i}} \prod_{j=t+1}^{r} \binom{n_j}{i_j}$ times for such Y, and this equals the hypothesis.

The former are implemented using a flow-through-network argument: Suppose the collections $F_{g,\underline{i}}^{(k)}$ constructed for some $k < m_s$. Then in order to get them for k+1 we have to choose λ_s sets from each collection $F_g^{(k)}$ and adjoin the point x_{k+1} to them so that

$$F_g^{(k+1)} = \{Y \in F_g^{(k)} \mid Y \text{ not chosen}\} \cup \{Y \cup \{x_{k+1}\} \mid Y \text{ chosen}\}.$$

Consider a directed network with vertices: source, sink, $F^{(k)}$ $(1 \leq g \leq \ell)$, $F_{g,\underline{i}}^{(k)}$ $(1 \leq g \leq \ell,\ \underline{i} \in N)$, Y $(Y \subset X^{(k)})$, $Y_{\underline{i}}$ $(Y \subset X^{(k)},\ |Y \cap X_j| = i_j\ (j<s))$ and edges from the source to each $F_g^{(k)}$, from $F_g^{(k)}$ to each $F_{g,\underline{i}}^{(k)}$, from $F_{g,\underline{i}}^{(k)}$ to $Y_{\underline{i}}$ iff $Y \in F_{g,\underline{i}}^{(k)}$, from $Y_{\underline{i}}$ to Y and from each Y to the sink.

A flow through this network is completely defined by its value on each of the edges $(F_{g,\underline{i}}^{(k)}, Y_{\underline{i}})$. Consider the flow with value $\frac{i_s - |Y \cap X_s|}{m_s - k}$ on each such edge. Through the vertex $F_g^{(k)}$ the flow is

$$\frac{1}{m_s - k} \sum_{\underline{i} \in N} \sum_{Y \in F_{g,\underline{i}}^{(k)}} (i_s - |Y \cap X_s|) = \frac{\lambda_s}{m_s - k}(n_s - (k - m_{s-1})) = \lambda_s$$

since $\sum_{\underline{i} \in N} a_{\underline{i}}\, i_s = \lambda_s n_s$ and $F_g^{(k)}$ restricted to $X_s \cap X^{(k)}$ is a λ_s-factor.

Through the vertex $Y_{\underline{i}}$ the flow is

$$\frac{i_s - |Y \cap X_s|}{m_s - k} \cdot \alpha_{\underline{i}} M_{\underline{i}} \binom{m_s - k}{i_s - |Y \cap X_s|} = \alpha_{\underline{i}} M_{\underline{i}} \binom{m_s - k - 1}{i_s - |Y \cap X_s| - 1}$$

which is an integer.

Now use the integrety theorem on flows in networks in the following form:

> If there is a flow in a network with value ϕ_i on edge e_i, then there is a flow with value ψ_i on edge e_i, where $\phi_i - 1 < \psi_i < \phi_i + 1$ and ψ_i is integral for each i. [I.e. all flow values may be rounded either up or down in such a way that again a flow results. In particular if some flow value was integral it is not changed.]

(cf. Ford & Fulkerson [3], p. 19).

In this particular case the integrity theorem yields an integer flow through the network with flow λ_s through each vertex $F_g^{(k)}$, i.e. the flow defines for each collection $F_g^{(k)}$ λ_s elements Y, each belonging to some known $F_{g,\underline{i}}^{(k)}$. Now if we adjoin the point x_{k+1} to these sets Y then, using that

$$\binom{m_s-k}{i_s-|Y\cap X_s|} = \binom{m_s-k-1}{i_s-|Y\cap X_s|} + \binom{m_s-k-1}{i_s-|Y\cap X_s|-1},$$

it is readily verified that the new collections $F_g^{(k+1)}$ and $F_{g,\underline{i}}^{(k+1)}$ satisfy the conditions 1,2 and 3.

This shows that the inductive hypothesis is true for k = n and s = r, and therefore the theorem holds.

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