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A CONNECTED METRIC SPACE WITHOUT NONTRIVIAL OPEN  
CONNECTED SUBSPACES

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A connected metric space without nontrivial open connected subspaces

by

A.E. Brouwer

ABSTRACT

An example is given of an arcwise connected metric space (which is not separable) without proper non-empty open connected subspaces.

KEY WORDS & PHRASES: *connected metric space*

## 0. INTRODUCTION

While trying to generalize known results for separable metric spaces to general topological spaces it was found that outside of the class of separable metric spaces requirements on all open connected subsets of a connected set are much weaker than within that class. This note shows the reason for this: while each nondegenerated connected separable metric space contains proper nonempty open connected subspaces, this is not true in general.

PROPOSITION *Let  $X$  be a connected topological space,  $C$  a closed connected subspace,  $\emptyset \neq C \neq X$ . If  $X$  does not contain nontrivial open connected subspaces then  $X \setminus C$  decomposes into infinitely many components, and if  $S$  is such a component then  $C \cup S$  is connected.*

COROLLARY *If, in addition,  $X$  is  $T_1$  then each point of  $X$  is a ramification point. In particular  $X$  is not separable metric:*

PROPOSITION *Let  $X$  be a connected  $T_1$  space in which each point is a ramification point.*

*Equivalent are:*

- (i) (assuming SH)  $X$  has the Suslin property
- (ii)  $X$  is separable
- (iii)  $X$  is countable

While a connected  $T_1$  space without open connected subsets cannot be separable metric, it can be separable (hence countable) and it can be metric as we will show by examples.

## 1. PROOF OF THE PROPOSITIONS

- (i) Let  $X$  be a connected space without nontrivial open subspaces. Let  $C$  be a closed connected subspace,  $\emptyset \neq C \neq X$ . Let  $S$  be a component of  $X \setminus C$ . Since  $S$  is not open while  $X \setminus C$  is,  $X \setminus C$  decomposes into infinitely many components. Also  $X \setminus S$  is connected and not open, so  $S$  is not closed. But  $S$  is closed in  $X \setminus C$  so that  $C \cap \bar{S} \neq \emptyset$ . Therefore  $C \cup S$  is connected.  $\square$

(iia) Let  $X$  be a connected  $T_1$  space, and  $Y$  its collection of ramification points (that is,  $Y$  is the set of all  $y \in X$  such that  $X \setminus y$  has at least three components).

Now consider the map  $\kappa: (x_1, x_2, x_3) \mapsto y$  if  $X \setminus y = A + B + C$  .  
 $x_1 \quad x_2 \quad x_3$

If  $D$  is a dense subset of  $X$ , and  $M \subset X^3$  is the domain of  $\kappa$ , then it follows that  $|Y| \leq |D^3 \cap M| \leq |D|^3 = |D|$  if  $D$  is infinite.

In particular if  $X$  is separable and  $Y = X$  then  $Y$  is countable.

(iib) Now let  $X$  have the Suslin property (i.e. any collection of disjoint open nonempty subsets of  $X$  is countable). We shall show that if  $Y$  is not countable then it is a Suslin tree (i.e. an uncountable partial order which does not contain the Boolean algebra  $2$ , and in which all chains and anti-chains are countable.) It is well-known that the non-existence of a Suslin tree is equivalent to the Suslin Hypothesis (SH) [E.W. Miller, A note on Souslin's problem, Amer. J. Math. 65 (1943) 673-678]. To this end define a partial order on  $X$  as follows. Choose  $x_0 \in X$  and set  $x > x_0$  for each  $x \in X \setminus x_0$ . Set  $x_1 < x_2$  for  $x_1, x_2 \in X \setminus x_0$  iff  $x_1$  separates  $x_2$  from  $x_0$ . Let  $S$  be an anti-chain in  $Y$ . If we choose for each  $s \in S \setminus x_0$  a clopen subset  $U_s$  of  $X \setminus s$  not containing  $x_0$  then  $\{U_s\}$  is a disjoint collection of open sets, hence countable. Next, let  $C$  be a chain in  $Y$ . Let for  $c \in C$   $X \setminus c = P(c) + Q(c) + R(c)$  be a decomposition of  $X \setminus c$  such that  $x_0 \in P(c)$  and  $C \subset P(c) \cup Q(c) \cup \{c\}$  and  $R(c) \neq \emptyset$ . Then  $\{R(c) \mid c \in C\}$  is a disjoint collection of open sets, hence countable. Therefore  $(Y, <)$  is a Suslin tree.  $\square$

## 2. THE EXAMPLE

Let  $Y = \{(x, y) \in I^2 \mid x = qy, 1 < q \in \mathbb{Q}\}$  with Euclidean topology.  $Y$  is separable metric arcwise connected space. Moreover any nonempty open connected subset of  $Y$  must contain the point  $0 = (0, 0) \in Y$ . Also  $Y$  has no cut points different from  $0$ . Starting from this space we construct our example  $X$  as a direct limit:

Let  $Y^* = Y \setminus \{(0,0)\}$ . Let  $X_0 = (Y^*)^0 = \{\lambda\}$  where  $\lambda$  is the empty sequence over  $Y^*$ . Let for  $n > 0$   $X_n = (X_{n-1} \times \{0\}) \cup (Y^*)^n$ , the set of all sequences of length  $n$  over  $Y$  such that the nonzero entries form an initial subsequence. We will identify  $X_{n-1}$  with  $X_{n-1} \times \{0\} \subset X_n$ . Now we may write  $X := \bigcup_{n=0}^{\infty} X_n$ .

If  $X = (x_1, \dots, x_n) \in X_n$  and  $n \geq 1$  then define  $x' := (x_1, \dots, x_{n-1})$ .

Define a metric  $d$  on  $X$  as follows: Let  $d_0$  be the Euclidean metric of  $I^2$  (and  $Y$ ). If  $x \in X_n \setminus X_{n-1} = (Y^*)^n$  and  $y \in (Y^*)^m$  then define

$$d_1(x, y) = d_1(y, x) := \begin{cases} d_0(x_n, y_n) & \text{if } m = n \text{ and } x' = y' \\ d_0(x_n, (0,0)) & \text{if } (m=n-1 \text{ and}) y = x' \\ d_0(x_n, (1,1)) & \text{if } (m=n-2 \text{ and}) y = x'' \\ \infty & \text{otherwise} \end{cases}$$

Let  $d(x, y) = \min\left\{ \sum_{i=0}^{\ell} d_1(x_i, x_{i+1}) \mid x_0 = x, x_{\ell+1} = y \right\}$ , then  $d$  is a metric on  $X$ :

$d$  is finite because for each  $x$  the chain  $x_0 = x, x_1 = x', \dots, x_n = \lambda$  shows that  $d(x, \lambda) < \infty$ . If  $x$  and  $y$  are points in  $X_n$  then it is not necessary to consider chains with points in  $X \setminus X_n$  since for each  $z \in X$  we have  $d_1(z', z'') \leq \sqrt{2} \leq d(z', z) + d(z, z'')$ , as soon as  $z''$  is defined. This shows that 'min' instead of 'inf' is justified and that  $d(x, y)$  is nonzero if  $x \neq y$ .

Now we have:

(i) Each  $X_n$  is a closed subspace of  $X$ : If  $x \in X_m \setminus X_{m-1}$  and  $m > n$  and  $\epsilon_x = \min(d(x, x'), d(x, x''))$  then  $d(x, X_n) \geq \epsilon_x$ .

(ii) The map  $j_u : Y \rightarrow X_{n+1}$  defined by

$$y \mapsto (u, y)$$

where  $u \in (Y^*)^n$ , is a homeomorphic embedding;

for  $n = 0$   $j_\lambda : Y \xrightarrow{\sim} X_1$  is a homeomorphism onto  $X_1$ ,

for  $n \geq 1$   $\text{cl } j_u[Y] = j_u[Y] \cup \{u'\}$ .

For: Obviously  $j_u$  is 1-1 and open since  $d(j_u y_1, j_u y_2) \leq d_0(y_1, y_2)$ .  $j_u$  is also continuous since if  $x = j_u a$  then  $d(x, j_u b) \leq \epsilon_x \Rightarrow d(x, j_u b) = d_0(a, b)$  ( $a \neq 0$ ) and  $d(u, j_u b) < \frac{1}{2}\sqrt{2} \Rightarrow d(u, j_u b) = d_0(0, b)$ .

Since  $X_{n+1}$  is closed and  $j_u[Y] \subset X_{n+1}$ , and from the definition of  $d_0$  it is easily seen that  $\text{cl } j_u[Y] = j_u[Y] \cup \{u'\}$ .

(iii) Each  $X_n$  is (arcwise) connected, as is  $X$  itself. For:  $x$  is joined by an arc with  $x'$ , hence by induction each point is in the arc component of  $\lambda$ .

(iv) Let  $L_q = \{(x,y) \in Y \mid x = qy\}$  and  $L_q^* = L_q \setminus \{0\}$ . Then  $Y \setminus \{0\}$  decomposes into the components  $L_q^*$  ( $1 < q \in \mathbb{Q}$ ).

(v) Define the projection  $\pi_m : X \rightarrow X_m$  by

$$x = (x_1, \dots, x_m, \dots) \mapsto (x_1, \dots, x_m).$$

$\pi_m$  is not continuous (the image of the connected  $j_u[Y] \cup \{u'\}$  is the doubleton  $\{u, u'\}$  if  $u \in X_m \setminus X_{m-1}$ ), but it is continuous on  $X \setminus X_{m-1}$ .

(vi) Let  $u \in (Y^*)^{n-1}$ . Then  $X \setminus u$  decomposes into the components

$$X \setminus \pi_{n-1}^{-1}\{u\} = X \setminus \pi_n^{-1} j_u[Y] \text{ and } \pi_n^{-1} j_u L_q^* \quad (1 < q \in \mathbb{Q}).$$

For: for each point  $a \in (Y^*)^n$   $\pi_n^{-1}\{a\}$  is homeomorphic to  $X$  and hence connected. Therefore since  $j_u L_q^*$  is connected each  $\pi_n^{-1} j_u L_q^*$  is connected. Also  $X \setminus \pi_{n-1}^{-1}\{u\}$  is connected (and has at most 2 arc components). If  $M_r = \cup\{L_q^* \mid r < q \in \mathbb{Q}\}$  then  $M_r$  is clopen in  $Y^*$  and  $\pi_n^{-1} j_u M_r$  is clopen in  $X \setminus u$  ( $r > 1$ ), so the candidate components are indeed maximal connected.

(vii)  $X$  is a non-separable metric arcwise connected space without proper nonempty open subspaces.

For: Let  $U \subset X$  be open and connected and let  $u \in U$ ,  $u = (u_1, \dots, u_n) \in (Y^*)^n$ . If  $n > 0$  and  $v = u'$  then  $u$  is not an interior point of its component in  $X \setminus v$ . Therefore  $U$  is not contained in  $X \setminus v$ , that is,  $v \in U$ . By induction it follows that  $\lambda \in U$ . Next if  $u \in U$  and  $u = v'$  then again  $u$  is accumulation point of components of  $X \setminus v$  so  $v \in U$ . This proves  $U = X$ .

This finishes the description of a metric example.

The space  $Y$  was a fan of copies of the unit interval  $I$ . When  $I$  is replaced by a countable connected  $T_2$  space and the construction is carried out as before (now defining an appropriate topology instead of a metric) then we obtain a countable connected  $T_2$  space not containing nontrivial open connected subspace.

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