

**stichting
mathematisch
centrum**



AFDELING ZUIVERE WISKUNDE
(DEPARTMENT OF PURE MATHEMATICS)

ZW 84/76

DECEMBER

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A NOTE ON THE PARTIAL SUMS OF $\zeta(s)$, III

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Printed at the Mathematical Centre, 49, 2e Boerhaavestraat, Amsterdam.

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A note on the partial sums of $\zeta(s)$, III

by

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ABSTRACT

This note is a continuation of the Mathematical Centre Reports ZW 53/75 and ZW 58/75.

For $N \geq 2$ let $\zeta_N(s) = \sum_{n=1}^N n^{-s}$, ($s \in \mathbb{C}$). It is shown that for $N = 2(1)6$ and $N = 8(1)10$ the functions ζ_N have no zeros in the half-plane $\sigma = \operatorname{Re}(s) > 1$.

In addition it is shown that all zeros of ζ_N lie in the strip $1 - N \log 2 < \sigma < 1.72865$.

With respect to the orders of the zeros of ζ_N it is shown that for every N there exists a number $\omega(N)$ such that the orders of the zeros of ζ_N do not exceed $\omega(N)$. For $N = 2(1)4$ all zeros of ζ_N are shown to be simple.

Finally an improvement is given of a theorem of SPIRA on the location of the lowest zero of ζ_N for large N .

KEY WORDS & PHRASES: *Partial sums (sections) of the zeta-function, zeros.*

0. INTRODUCTION.

This note is a continuation of the Mathematical Centre Reports ZW 53/75 and ZW 58/75.

In section 1 we present a device by means of which one may easily show that for $N = 2(1)6$ and $N = 8(1)10$ the functions ζ_N defined by

$$(0.1) \quad \zeta_N(s) = \sum_{n=1}^N n^{-s}, \quad (s \in \mathbb{C}; N=2,3,4,\dots)$$

have no zeros in the half-plane $\sigma = \operatorname{Re}(s) > 1$. (Compare the methods of JESSEN and SPIRA described in TURAN [8], SPIRA [6] and SPIRA [7]).

Section 2 is concerned with the width of the vertical strip in \mathbb{C} containing all zeros of $\zeta_N(s)$.

In SPIRA [6] it was shown that for $N \geq 2$ all zeros of $\zeta_N(s)$ satisfy

$$(0.2) \quad \sigma > 1 - N.$$

It will be shown here that this result may be improved by utilizing some new inequalities for sums of powers of positive integers (also see [4]). As a result we have that (0.2) may be replaced by

$$(0.3.1) \quad \sigma > 1 - \frac{\log 2}{\log(1 + \frac{1}{N-\frac{1}{2}})}.$$

Also it will be shown that (roughly speaking) the south-west corner of the halfstrip

$$(0.3.2) \quad \sigma > 1 - \frac{\log 2}{\log(1 + \frac{1}{N-\frac{1}{2}})}, \quad \operatorname{Im}(s) = t \geq 0$$

is a zero-free region for $\zeta_N(s)$.

For a more precise formulation see theorem 2.2.

Section 3 deals with the multiplicity of the zeros of $\zeta_N(s)$. It will be shown that for every $N \geq 2$ there exists a number $\omega(N)$ such that the multiplicity of any zero of $\zeta_N(s)$ does not exceed $\omega(N)$.

In particular it will be proved that for $N = 2(1)4$ we may take $\omega(N) = 1$ so that for these values of N all zeros of $\zeta_N(s)$ are simple.

It is *conjectured* that for all $N \geq 2$ all zeros of $\zeta_N(s)$ are simple.

Section 4 deals with the location of the "lowest" zero of $\zeta_N(s)$.

In SPIRA [5] it was shown that if N is large enough then the lowest zero of $\zeta_N(s)$ is simple and lies either in the rectangle

$$(0.4) \quad \left\{ \begin{array}{l} 1 - \frac{c_1}{\log^3 N} \leq \sigma \leq 1 + \frac{c_2}{\log^3 N}, \\ \frac{2\pi}{\log N} - \frac{c_3}{\log^2 N} \leq t \leq \frac{2\pi}{\log N} \end{array} \right.$$

or in the triangle

$$(0.5) \quad \left\{ \begin{array}{l} 1 \leq \sigma \leq 1 + \frac{c_4}{\log^3 N}, \\ \frac{2\pi}{\log N} \leq t \leq \frac{2\pi}{\log N} + \frac{\sigma-1}{\log^2 N}. \end{array} \right.$$

For the positive constants c_1, c_2, c_3 and c_4 , see SPIRA [5].

We will show that if N is large enough then the region (0.5) does *not* contain any zero of $\zeta_N(s)$.

For more results on this subject see LEVINSON [1].

1. In [2] and [3] it was shown that for $N = 2(1)6$ and $N = 8(1)10$ there exist positive constants $p(N)$ such that

$$(1.1) \quad R_N(t) \stackrel{\text{def}}{=} \operatorname{Re} \zeta_N(1+it) \geq p(N), \quad (t \in \mathbb{R}).$$

Defining

$$(1.2) \quad R_N(\sigma, t) = \operatorname{Re} \zeta_N(\sigma+it) = \sum_{n=1}^N \frac{\cos(t \log n)}{n^\sigma}$$

we clearly have

$$(1.3) \quad R_N(\sigma, 0) = \sum_{n=1}^N n^{-\sigma} > 0, \quad (\sigma \in \mathbb{R})$$

and

$$(1.4) \quad \begin{aligned} R_N(\sigma, t) &\geq 1 - \sum_{n=2}^N n^{-\sigma} > 2 - \zeta(\sigma) \geq \\ &\geq 2 - \zeta(2) = 2 - \frac{\pi^2}{6} > 0, \quad (\sigma \geq 2; t \in \mathbb{R}). \end{aligned}$$

Next observe that $R_N(t)$ is an almost-periodic function so that for every $\varepsilon > 0$ there exists an increasing positive sequence $\{T_k\}_{k=1}^{\infty}$ such that $T_k \rightarrow \infty$ as $k \rightarrow \infty$ and

$$(1.5) \quad R_N(T_k) > R_N(0) - \varepsilon.$$

If we choose ε small enough it follows that in the sum

$$(1.6) \quad R_N(T_k) = \sum_{n=1}^N \frac{1}{n} \cos(T_k \log n)$$

all cosines must be close to 1 (and hence positive) so that also

$$(1.7) \quad R_N(\sigma, T_k) = \sum_{k=1}^N \frac{1}{n^{\sigma}} \cos(T_k \log n) > 0, \quad (\sigma \in \mathbb{R}).$$

Finally observe that $R_N(\sigma, t)$ is a harmonic function so that by the minimum principle for harmonic functions we easily obtain the following

THEOREM 1.1. For $N = 2(1)6$ and $N = 8(1)10$ we have

$$(1.8) \quad R_N(\sigma, t) > 0, \quad (\sigma \geq 1, t \geq 0).$$

As a simple consequence we have

THEOREM 1.2. For $N = 2(1)6$ and $N = 8(1)10$ we have

$$(1.9) \quad \zeta_N(s) \neq 0 \quad \text{for} \quad \sigma \geq 1.$$

Similarly one may prove

THEOREM 1.3. If $R_N(\sigma_0, t) > 0$, ($t \in \mathbb{R}$) then $\zeta_N(s) \neq 0$ for $\sigma \geq \sigma_0$.

The last theorem can easily be extended to functions $D_N(s) = \sum_{n=1}^N a_n n^{-s}$ with $a_1 > 0$ and $a_n \geq 0$ for $1 < n \leq N$.

REMARK. In order to prove theorem 1.1 it suffices to show that

$$R_N(\sigma, t) > 0, \quad (1 \leq \sigma \leq 2, t \geq 0)$$

for all N stated in the theorem. Indeed, by (1.4) we already know that

$$R_N(\sigma, t) > 0, \quad (\sigma \geq 2; t \in \mathbb{R}).$$

Observing that

$$(1.10) \quad \frac{\partial R_N(\sigma, t)}{\partial \sigma} = - \sum_{n=2}^N \frac{\log n}{n^\sigma} \cos(t \log n)$$

we have

$$(1.11) \quad \left| \frac{\partial R_N(\sigma, t)}{\partial \sigma} \right| \leq \sum_{n=2}^N \frac{\log n}{n^\sigma} \leq \sum_{n=2}^N \frac{\log n}{n}$$

for $\sigma \geq 1$ and all $t \in \mathbb{R}$.

Hence, by the maximal slope principle, if

$$(1.12) \quad \sum_{n=2}^N \frac{\log n}{n} < \sum_{n=1}^N \frac{1}{n} - \varepsilon = R_N(0) - \varepsilon,$$

then

$$(1.13) \quad \begin{aligned} R_N(\sigma, T_k) &\geq R_N(T_k) - (\sigma-1) \sum_{n=2}^N \frac{\log n}{n} > \\ &> R_N(0) - \varepsilon - \sum_{n=2}^N \frac{\log n}{n}, \quad (1 \leq \sigma \leq 2). \end{aligned}$$

Since $\varepsilon > 0$ may be chosen as small as we please it suffices to show that

$$(1.14) \quad L_N \stackrel{\text{def}}{=} \sum_{n=2}^N \frac{\log n}{n} < \sum_{n=1}^N \frac{1}{n} = R_N(0).$$

It is easily verified (see table 1) that (1.14) is true indeed for $N = 2(1)10$ so that we have obtained an alternative proof of theorem 1.1.

TABLE 1

N	L_N	$R_N(0)$
2	.3466	1.5000
3	.7128	1.8333
4	1.0594	2.0833
5	1.3812	2.2833
6	1.6799	2.4500
7	1.9579	2.5929
8	2.2178	2.7179
9	2.4619	2.8290
10	2.6922	2.9290
11	2.9102	3.0199
12	3.1172	3.1032
13	3.3145	3.1801
14	3.5031	3.2516
15	3.6836	3.3182

2. THEOREM 2.1. If $\zeta_N(s) = 0$ then

$$(2.1) \quad \sigma = \operatorname{Re}(s) > 1 - \frac{\log 2}{\log(1 + \frac{2}{2N-1})}.$$

In order to prove this we first establish the following

Lemma 2.1. For all real $p \geq 1$ we have

$$(2.2) \quad \sigma_n(p) \stackrel{\text{def}}{=} \sum_{k=1}^n k^p < \frac{n^p(2n+1)^{p+1}}{(2n+1)^{p+1} - (2n-1)^{p+1}}.$$

Proof of the lemma.

It is easily verified that (2.2) is true for $n = 1$ and all $p \geq 1$.
Now assume that the lemma is true for $n = 1, \dots, N$ and all $p \geq 1$.

Then we have

$$(2.3) \quad \sigma_{N+1}(p) = (N+1)^p + \sigma_N(p) < (N+1)^p + \frac{N^p(2N+1)^{p+1}}{(2N+1)^{p+1} - (2N-1)^{p+1}},$$

so that it suffices to show that

$$(2.4) \quad (N+1)^p + \frac{N^p(2N+1)^{p+1}}{(2N+1)^{p+1} - (2N-1)^{p+1}} \leq \frac{(N+1)^p(2N+3)^{p+1}}{(2N+3)^{p+1} - (2N+1)^{p+1}}$$

for all $N \in \mathbb{N}$ and all $p \geq 1$.

Putting $\frac{1}{N} = x$ we arrive at the equivalent inequality

$$(2.5) \quad (1+x)^p + \frac{(2+x)^{p+1}}{(2+x)^{p+1} - (2-x)^{p+1}} \leq \frac{(1+x)^p(2+3x)^{p+1}}{(2+3x)^{p+1} - (2+x)^{p+1}}.$$

Replace x by $2x$ (so that from now on $0 < x \leq \frac{1}{2}$) in order to arrive at the equivalent inequality

$$(2.6) \quad (1+2x)^p + \frac{(1+x)^{p+1}}{(1+x)^{p+1} - (1-x)^{p+1}} \leq \frac{(1+2x)^p(1+3x)^{p+1}}{(1+3x)^{p+1} - (1+x)^{p+1}}.$$

After crossmultiplication and some simplification it turns out that we may just as well prove that

$$(2.7) \quad (1+2x)^p \left\{ (1+x)^{p+1} - (1-x)^{p+1} \right\} \geq (1+3x)^{p+1} - (1+x)^{p+1}$$

which is equivalent to

$$(2.8) \quad \frac{(1+x)^{p+1} - (1-x)^{p+1}}{x} \geq \frac{\left(1 + \frac{x}{1+2x}\right)^{p+1} - \left(1 - \frac{x}{1+2x}\right)^{p+1}}{\frac{x}{1+2x}}.$$

Since $x > \frac{x}{1+2x}$ for $x > 0$ it follows that the proof is complete as soon as we can show that the function

$$(2.9) \quad \varphi(x) \stackrel{\text{def}}{=} \frac{(1+x)^{p+1} - (1-x)^{p+1}}{x}$$

is increasing on the interval $0 < x \leq \frac{1}{2}$.

Observe that for $x > 0$ we have

$$(2.10) \quad \varphi'(x) = \frac{x \left\{ (p+1)(1+x)^p + (p+1)(1-x)^p \right\} - (1+x)^{p+1} + (1-x)^{p+1}}{x^2}$$

so that it suffices to show that

$$(2.11) \quad (px+x-1-x)(1+x)^p + (px+x+1-x)(1-x)^p \geq 0$$

or, equivalently, that

$$(2.12) \quad f(x) \stackrel{\text{def}}{=} (px-1)(1+x)^p + (px+1)(1-x)^p \geq 0.$$

Since $f(0) = 0$, it suffices to show that for $x > 0$

$$(2.13) \quad f'(x) = p(1+x)^p + (px-1)p(1+x)^{p-1} + p(1-x)^p - (px+1)p(1-x)^{p-1} \geq 0$$

or, equivalently, that

$$\begin{aligned}
 (2.14) \quad & (p+px+p^2x-p)(1+x)^{p-1} + (p-px-p^2x-p)(1-x)^{p-1} = \\
 & = (p+p^2)x \left\{ (1+x)^{p-1} - (1-x)^{p-1} \right\} \geq 0.
 \end{aligned}$$

Since (2.14) is clearly true for $p \geq 1$ and $0 < x < 1$ the proof of the lemma is complete.

Proof of theorem 2.1.

The case $N = 2$ being trivial we assume $N \geq 3$.

Put

$$(2.15) \quad p = \frac{\log 2}{\log(1 + \frac{2}{2N-1})} - 1$$

and assume the theorem to be false. Then there exists an $N \geq 3$ and an $s \in \mathbb{C}$ such that

$$(2.16) \quad \sigma = \operatorname{Re}(s) \leq -p$$

and

$$(2.17) \quad \zeta_N(s) = \sum_{n=1}^N n^{-s} = 0,$$

so that

$$(2.18) \quad \sum_{n=1}^{N-1} n^{-s} = -N^{-s}.$$

It follows that

$$(2.19) \quad -1 = \sum_{n=1}^{N-1} \left(\frac{N}{n}\right)^s,$$

so that

$$(2.20) \quad 1 \leq \sum_{n=1}^{N-1} \left(\frac{N}{n}\right)^{\sigma}.$$

In (2.20) we have $1 \leq n \leq N-1$ so that $\frac{N}{n} > 1$. Since $\sigma \leq -p$ we have

$$(2.21) \quad 1 \leq \sum_{n=1}^{N-1} \left(\frac{N}{n}\right)^{-p} = \sum_{n=1}^{N-1} \left(\frac{n}{N}\right)^p$$

or, equivalently,

$$(2.22) \quad 2 N^p \leq \sigma_N(p).$$

It is easily verified that $p > 1$ for $N \geq 3$. Hence, we may apply lemma 2.1 to (2.22) in order to obtain

$$(2.23) \quad 2 N^p < \frac{N^p (2N+1)^{p+1}}{(2N+1)^{p+1} - (2N-1)^{p+1}},$$

from which it is easily seen that

$$(2.24) \quad p + 1 < \frac{\log 2}{\log(1 + \frac{2}{2N-1})},$$

contradicting the definition of p and hence proving the theorem.

REMARK. In [4] it was shown by very simple means that for $p > 0$ one has

$$(2.25) \quad \sigma_N(p) < \frac{N^p (N+1)^{p+1}}{(N+1)^{p+1} - N^{p+1}}.$$

Similarly as above, from this inequality one may derive that if $\zeta_N(s) = 0$ then

$$(2.26) \quad \sigma > 1 - \frac{\log 2}{\log(1 + \frac{1}{N})},$$

a result just a little less sharp than (2.1).

We conclude this section by proving

THEOREM 2.2. $\zeta_N(s)$ has no zeros in the domain G_N described by

$$(2.27) \quad \begin{cases} s = \sigma + a \sigma i; & a, \sigma \in \mathbb{R} \\ \sigma \leq -1 \\ |s-1| \leq \frac{2(N-1)}{1+\sqrt{1+a^2}} \end{cases}$$

PROOF. If $\zeta_N(s) = 0$ then we have

$$(2.28) \quad \begin{aligned} 0 = \zeta_N(s) &= \sum_{n=1}^N n^{-s} = \int_{1-}^{N+} x^{-s} d[x] = \\ &= \int_1^N x^{-s} dx - \int_{1-}^{N+} x^{-s} d\varphi_1(x) = \end{aligned}$$

(where $\varphi_1(x) = x - [x] - \frac{1}{2}$)

$$= \frac{N^{-s+1}-1}{-s+1} + \frac{1}{2} (N^{-s+1}) - s \int_1^N \frac{\varphi_1(x)}{x^{s+1}} dx$$

so that

$$(2.29) \quad \frac{1-N^{s-1}}{s-1} = \frac{1}{2N} + \frac{1}{2} N^{s-1} - s N^{s-1} \int_1^N \frac{\varphi_1(x)}{x^{s+1}} dx.$$

It follows that we must have

$$(2.30) \quad \begin{aligned} \frac{1-N^{\sigma-1}}{|s-1|} &\leq \frac{1}{2N} + \frac{1}{2} N^{\sigma-1} + |s| N^{\sigma-1} \int_1^N \frac{|\varphi_1(x)|}{x^{\sigma+1}} dx \\ &\leq \frac{1}{2N} + \frac{1}{2} N^{\sigma-1} + |s| N^{\sigma-1} \cdot \frac{1}{2} \frac{N^{-\sigma}-1}{-\sigma} \\ &= \frac{1}{2N} + \frac{1}{2} N^{\sigma-1} + \frac{|s|}{-2\sigma} \left(\frac{1}{N} - N^{\sigma-1} \right). \end{aligned}$$

Now let $s = \sigma + a \sigma i$ with $\sigma \leq -1$ and $a \in \mathbb{R}$. Then we obtain

$$(2.31) \quad \frac{1-N^{-2}}{|s-1|} > \frac{1}{2N} + \frac{1}{2N^2} + \frac{|s|}{2N|\sigma|}$$

so that

$$(2.32) \quad \frac{N^2-1}{|s-1|} > \frac{N+1}{2} + \frac{N}{2} \sqrt{1+a^2} < \frac{N+1}{2} (1+\sqrt{1+a^2}).$$

Hence

$$(2.33) \quad |s-1| > \frac{2(N-1)}{1+\sqrt{1+a^2}},$$

proving the theorem.

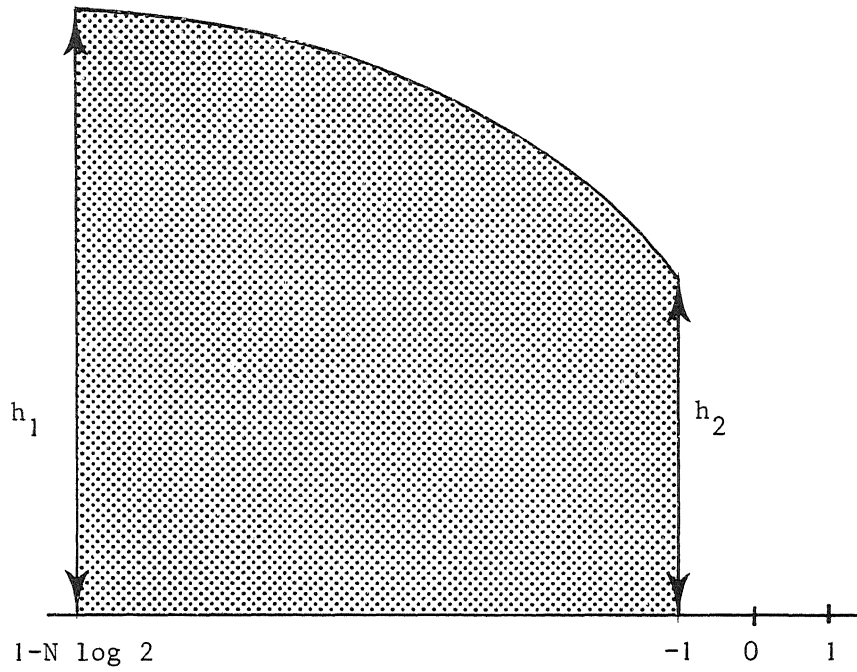
REMARK. Since

$$(2.34) \quad 1 - N \log 2 < 1 - \frac{\log 2}{\log(1 + \frac{1}{N-1})}, \quad (\forall N \in \mathbb{N})$$

we derive from theorems 2.1 and 2.2 that the intersection of the domain G_N in theorem 2.2 and the half-strip

$$(2.35) \quad \begin{cases} \sigma > 1 - N \log 2, \\ t \geq 0 \end{cases}$$

is a zero-free region for $\zeta_N(s)$. This region may be depicted as the shaded area in the figure below. One should compare these results with the empirical observations made by SPIRA [6; section 4].



From (2.7) it is easily seen that if N is large enough then

$$(2.36) \quad h_1 > \frac{1}{2} N$$

and

$$(2.37) \quad h_2 \sim \sqrt{2N}.$$

Finally we want to state that it is possible to improve theorem 2.2 in various ways. However, we will not pursue this subject here.

3. THEOREM 3.1. For $N = 2(1)4$ all zeros of $\zeta_N(s)$ are simple.

PROOF. The case $N = 2$ being trivial we first establish the case $N = 3$.

If $\zeta_3(s)$ has a multiple zero then

$$(3.1) \quad 1 + \frac{1}{2^s} + \frac{1}{3^s} = 0$$

and

$$(3.2) \quad \frac{\log 2}{2^s} + \frac{\log 3}{3^s} = 0.$$

From (3.2) it follows that

$$(3.3) \quad \left(\frac{3}{2}\right)^s = -\frac{\log 3}{\log 2}$$

so that for some $k \in \mathbb{Z}$

$$(3.4) \quad s = \frac{(2k+1)\pi i + \log \frac{\log 3}{\log 2}}{\log \frac{3}{2}}.$$

Since

$$(3.5) \quad \frac{1}{3^s} = -\frac{\frac{\log 2}{\log 3}}{2^s}$$

it also follows that

$$(3.6) \quad 1 + \frac{1}{2^s} - \frac{\frac{\log 2}{\log 3}}{2^s} = 0$$

so that

$$(3.7) \quad 2^s = -\left(1 - \frac{\log 2}{\log 3}\right)$$

from which we obtain

$$(3.8) \quad s = \frac{(2m+1)\pi i + \log\left(1 - \frac{\log 2}{\log 3}\right)}{\log 2}.$$

Since

$$(3.9) \quad \frac{\log\left(1 - \frac{\log 2}{\log 3}\right)}{\log 2} < 0$$

and

$$(3.10) \quad \frac{\log \frac{\log 3}{\log 2}}{\log \frac{3}{2}} > 0$$

we arrive at a contradiction.

We now prove the simplicity of the zeros of $\zeta_4(s)$.
If $\zeta_4(s)$ has any multiple zero then

$$(3.11) \quad 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} = 0,$$

and

$$(3.12) \quad \frac{\log 2}{2^s} + \frac{\log 3}{3^s} + \frac{\log 4}{4^s} = 0.$$

From the last equation it follows that

$$(3.13) \quad \frac{\frac{\log 2}{\log 3}}{2^s} + \frac{1}{3^s} + \frac{\frac{\log 4}{\log 3}}{4^s} = 0$$

so that

$$(3.14) \quad 1 + \frac{1}{2^s} - \frac{\frac{\log 2}{\log 3}}{2^s} - \frac{\frac{\log 4}{\log 3}}{4^s} + \frac{1}{4^s} = 0$$

or

$$(3.15) \quad 1 + \left(1 - \frac{\log 2}{\log 3}\right) \frac{1}{2^s} + \left(1 - \frac{\log 4}{\log 3}\right) \left(\frac{1}{2^s}\right)^2 = 0.$$

The discriminant of the above quadratic equation in $\frac{1}{2^s}$ is

$$(3.16) \quad D = \left(1 - \frac{\log 2}{\log 3}\right)^2 - 4\left(1 - \frac{\log 4}{\log 3}\right) > 0$$

so that $\frac{1}{2^s}$ is real. It follows that also $\frac{1}{4^s}$ and hence also $\frac{1}{3^s}$ is real.

Putting $2^s = x_2 \in \mathbb{R}$ and $3^s = x_3 \in \mathbb{R}$ we have

$$(3.17) \quad s = \frac{\log |x_2| + k\pi i}{\log 2} \text{ for some } k \in \mathbb{Z}$$

and

$$(3.18) \quad s = \frac{\log |x_3| + m\pi i}{\log 3} \text{ for some } m \in \mathbb{Z}.$$

It follows that for some $k, m \in \mathbb{Z}$

$$(3.19) \quad \frac{k}{\log 2} = \frac{m}{\log 3}$$

which is impossible unless $k = m = 0$. Hence s is real, which is clearly impossible. This contradiction proves that all zeros of $\zeta_4(s)$ are simple.

THEOREM 3.2. *For every $N \geq 2$ there is a number $\omega(N)$ such that the orders of the zeros of $\zeta_N(s)$ do not exceed $\omega(N)$.*

PROOF. From our previous work it is clear that we may take

$$\omega(2) = \omega(3) = \omega(4) = 1.$$

In general let $\zeta_N(s)$ have a zero of order $k + 1$.

Then

$$(3.20) \quad \begin{cases} \zeta_N(s) = 0, \\ \zeta'_N(s) = 0, \\ \dots\dots\dots, \\ \zeta_N^{(k)}(s) = 0. \end{cases}$$

Hence

$$(3.21) \quad \sum_{n=1}^{N-1} \frac{(\log n)^r}{n^s} = - \frac{(\log N)^r}{N^s}, \quad (0 \leq r \leq k)$$

so that

$$(3.22) \quad \frac{(\log N)^r}{N^\sigma} \leq \sum_{n=1}^{N-1} \frac{(\log n)^r}{n^\sigma}, \quad (0 \leq r \leq k)$$

or

$$(3.23) \quad 1 \leq \sum_{n=1}^{N-1} \left(\frac{\log n}{\log N} \right)^r \left(\frac{N}{n} \right)^\sigma, \quad (0 \leq r \leq k).$$

Since s is a zero of $\zeta_N(s)$, we certainly have that $\sigma < 2$. It follows that for any fixed N (3.23) can be true only if r is not too large. More explicitly, we must have, for example,

$$(3.24) \quad \begin{aligned} 1 &\leq \sum_{n=1}^{N-1} \left(\frac{\log(N-1)}{\log N} \right)^r \left(\frac{N}{n} \right)^2 = \\ &= N^2 \left(\frac{\log(N-1)}{\log N} \right)^r \sum_{n=1}^{N-1} \frac{1}{n^2} < \\ &< N^2 \frac{\pi^2}{6} \left(\frac{\log(N-1)}{\log N} \right)^r \end{aligned}$$

from which it is easily seen that we may take

$$(3.25) \quad \omega(N) = 1 + \frac{\log \frac{N^2 \pi^2}{6}}{\log \frac{\log N}{\log(N-1)}}, \quad (N \geq 3).$$

From (3.23) it is evident that one would like to have good upper bounds for the real parts of the zeros of $\zeta_N(s)$.

Defining

$$(3.26) \quad \rho_N = \sup \left\{ \sigma \in \mathbb{R} \mid \zeta_N(\sigma + it) = 0 \text{ for some } t \in \mathbb{R} \right\}$$

we have, c.f. SPIRA [6], $\rho_N < 1.85$.

For small N this upper bound may be improved as follows.

Defining

$$(3.27) \quad \sigma_N = \sup \left\{ \sigma \in \mathbb{R} \mid R_N(\sigma, t) = 0 \text{ for some } t \in \mathbb{R} \right\}$$

we clearly have

$$(3.28) \quad \rho_N \leq \sigma_N.$$

For example, let us take $N = 3$. Then

$$(3.29) \quad \begin{aligned} R_3(\sigma, t) &= 1 + \frac{1}{2^\sigma} \cos(t \log 2) + \frac{1}{3^\sigma} \cos(t \log 3) \\ &\geq 1 - \frac{1}{2^\sigma} - \frac{1}{3^\sigma} \stackrel{\text{def}}{=} f_3(\sigma). \end{aligned}$$

Since $f_3(\sigma)$ is increasing and

$$(3.30) \quad f_3(\sigma) = 0 \quad \text{for } \sigma = .787885\dots$$

it follows that

$$(3.31) \quad \rho_3 < .7879.$$

Substitution of this estimate in (3.23) yields

$$(3.32) \quad 1 \leq \sum_{n=1}^2 \left(\frac{\log n}{\log 3} \right)^r \left(\frac{3}{n} \right)^{.7879}.$$

Since this inequality does *not* hold for $r = 1$ we have an alternative proof of the fact that all zeros of $\zeta_3(s)$ are simple.

Now let us consider the case $N = 4$. For $N = 4$ we have

$$\begin{aligned}
 (3.33) \quad R_4(\sigma, t) &= 1 + \frac{1}{2^\sigma} \cos(t \log 2) + \frac{1}{3^\sigma} \cos(t \log 3) + \frac{1}{4^\sigma} \cos(2t \log 2) \\
 &\geq 1 + \frac{1}{2^\sigma} \cos u - \frac{1}{3^\sigma} + \frac{1}{4^\sigma} \cos(2u) = \\
 &= 1 - \frac{1}{3^\sigma} + \frac{1}{2^\sigma} \cos u + \frac{1}{4^\sigma} (2 \cos^2 u - 1) \geq \\
 &\geq 1 - \frac{1}{3^\sigma} - \frac{1}{4^\sigma} + \min_{|x| \leq 1} \left\{ \frac{x}{2^\sigma} + \frac{2x^2}{4^\sigma} \right\} \geq \\
 &\geq 1 - \frac{1}{3^\sigma} - \frac{1}{4^\sigma} + \min_{v \in \mathbb{R}} (v + 2v^2) = \\
 &= \frac{7}{8} - \frac{1}{3^\sigma} - \frac{1}{4^\sigma} = f_4(\sigma).
 \end{aligned}$$

Since $f_4(\sigma)$ is increasing and

$$(3.34) \quad f_4(\sigma) = 0 \text{ for } \sigma = .669081\dots$$

we find that

$$(3.35) \quad \rho_4 < .6691.$$

Substitution of (3.35) in (3.23) yields that all zeros of $\zeta_4(s)$ have multiplicity ≤ 2 .

Similarly one may show that we may take

$$(3.36) \quad \omega(5) = 5.$$

CONJECTURE. For every $N \geq 2$ all zeros of $\zeta_N(s)$ are simple.

In SPIRA [6] it was shown that $\rho_N < 1.85$. This estimate may be improved as follows.

If $\zeta_N(s) = 0$ then

$$(3.37) \quad 0 = |\zeta_N(s)| \geq 1 - \sum_{n=2}^N n^{-\sigma} > 2 - \zeta(\sigma)$$

so that

$$(3.38) \quad \zeta(\sigma) > 2.$$

Since $\zeta(s)$ is decreasing on $s > 1$ we must have

$$(3.39) \quad \sigma < p_0$$

where

$$(3.40) \quad p_0 > 1 \text{ and } \zeta(p_0) = 2$$

One may verify that $p_0 = 1.728647\dots$

4. In this section we will show that for any constant $c_4 > 0$, $\zeta_N(s)$ does *not* have a zero in the triangle

$$(4.1) \quad \begin{cases} 1 \leq \sigma \leq 1 + \frac{c_4}{\log^3 N} \\ \frac{2\pi}{\log N} \leq t \leq \frac{2\pi}{\log N} + \frac{\sigma-1}{\log^2 N} \end{cases}$$

if N is large enough. Compare SPIRA [5, p. 171]. In order to prove this we consider

$$R_N(t) = \sum_{n=1}^N \frac{1}{n} \cos(t \log n)$$

at the point $t = \frac{a\pi}{\log N}$, ($2 \leq a \leq 3$).

By means of the Euler-Maclaurin summation formula we have

$$(4.2) \quad \begin{aligned} R_N(t) &= \int_1^N \frac{1}{x} \cos(t \log x) dx - \int_1^{N+} \frac{1}{x} \cos(t \log x) d\varphi_1(x) \\ &= I_1 + I_2. \end{aligned}$$

The substitution $x = N^u$ in I_1 yields

$$(4.3) \quad \begin{aligned} I_1 &= \log N \int_0^1 \cos(t u \log N) du = \\ &= \log N \frac{\sin(t \log N)}{t \log N} = \frac{\sin a\pi}{a\pi} \log N, \end{aligned}$$

so that in view of $2 \leq a \leq 3$ we have

$$(4.4) \quad I_1 \geq 0.$$

Furthermore we have

$$(4.5) \quad \begin{aligned} I_2 &= - \frac{\varphi_1(x)}{x} \cos(t \log x) \Big|_1^{N+} + \int_1^N \varphi_1(x) d \frac{\cos(t \log x)}{x} \\ &= \frac{1}{2} + \frac{1}{2N} \cos(a\pi) - \int_1^N \varphi_1(x) \frac{t \sin(t \log x) + \cos(t \log x)}{x^2} dx. \end{aligned}$$

Now observe that

$$(4.6) \quad \begin{aligned} \left| \int_1^N \varphi_1(x) \frac{t \sin(t \log x)}{x^2} dx \right| &\leq t^2 \int_1^N \frac{1}{2} \frac{\log x}{x^2} dx \leq \\ &\leq \frac{1}{2} \left(\frac{3\pi}{\log N} \right)^2 \int_1^\infty \frac{\log x}{x^2} dx \rightarrow 0 \text{ as } N \rightarrow \infty, \end{aligned}$$

and that, uniformly in t ,

$$(4.7) \quad \int_1^N \varphi_1(x) \frac{\cos(t \log x)}{x^2} dx \rightarrow \int_1^\infty \frac{\varphi_1(x)}{x^2} dx, \quad (N \rightarrow \infty).$$

Hence

$$(4.8) \quad I_2 \rightarrow \frac{1}{2} - \int_1^\infty \frac{\varphi_1(x)}{x^2} dx, \quad (N \rightarrow \infty)$$

uniformly in t . Observing that

$$(4.9) \quad \frac{1}{2} - \int_1^\infty \frac{\varphi_1(x)}{x^2} dx = \gamma$$

where $\gamma = .577415\dots$ is Euler's constant,

it follows that

$$(4.10) \quad R_N(t) > \frac{1}{2}, \quad \left(\frac{2\pi}{\log N} \leq t \leq \frac{3\pi}{\log N} \right)$$

if N is large enough. Next we observe that

$$(4.11) \quad \frac{\partial R_N(\sigma, t)}{\partial \sigma} = - \sum_{n=2}^N \frac{\log n}{n^\sigma} \cos(t \log n)$$

so that

$$(4.12) \quad \left| \frac{\partial R_N(\sigma, t)}{\partial \sigma} \right| \leq \sum_{n=2}^N \frac{\log n}{n} \leq c_5 \log^2 N, \quad (\sigma \geq 1).$$

By the maximal slope principle we thus obtain from (4.10) and (4.12) that

$$(4.13) \quad R_N(\sigma, t) > 0$$

on the rectangle

$$(4.14) \quad \begin{cases} 1 \leq \sigma \leq 1 + \frac{1}{4 \log^2 N} \\ -\frac{2\pi}{\log N} \leq t \leq \frac{3\pi}{\log N} \end{cases}$$

If N is large enough the triangle described by (4.1) is entirely contained in the rectangle described by (4.14), proving our assertion.

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