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CONVEX APPROXIMATION OF INTEGRALS

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Convex approximation of integrals

by

J. van de Lune & M. Voorhoeve

ABSTRACT

For continuous f, the integral $\int_a^b f(x)\,dx$ is canonically approximated by the trapezoidal sums

$$T_n(f;a,b) = \frac{1}{n}\{-\frac{1}{2} f(a) + \sum_{k=0}^{n} f(a+k(b-a)/n) - \frac{1}{2} f(b)\}.$$

In this paper we establish some criteria for these sums to be convex (in n).

KEY WORDS & PHRASES: *convexity, approximations.*

0. INTRODUCTION

Let $f\colon [a,b] \to \mathbb{R}$ be continuous. We define the n-th canonical trapezoidal approximation $T_n(f;a,b)$ of $\int_a^b f(x)\,dx$ by

$$T_n(f;a,b) = \frac{1}{n}\{-\frac{1}{2} f(a) + \sum_{k=0}^{n} f(a+k(b-a)/n) - \frac{1}{2} f(b)\}.$$

In this paper we investigate the sums $T_n(x^s;a,b)$ for $s \in \mathbb{R}$. The first named author showed in [2] that the sequence $\{T_n(x^s;0,1)\}_{n=1}^{\infty}$ is decreasing for any fixed $s > 1$. This is equivalent to the inequality

$$\sum_{k=1}^{n} k^s > \frac{1}{2} \frac{n^{s+1}(n+1)^s + n^s(n+1)^{s+1}}{(n+1)^{s+1} - n^{s+1}}, \qquad (s > 1).$$

In the first part of this paper we show that for fixed $m \in \mathbb{N}$ the sequence $\{T_n(x^m;0,1)\}_{n=1}^{\infty}$ is convex, i.e.

$$2T_n(x^m;0,1) \le T_{n-1}(x^m;0,1) + T_{n+1}(x^m;0,1).$$

This immediately implies that the sequence $\{T_n(f;0,b)\}_{n=1}^{\infty}$ is convex if the Taylor expansion of f around the origin converges in $[0,b]$ and has non-negative coefficients. The convexity of the sequence $\{T_n(x^m;0,1)\}_{n=1}^{\infty}$ is proved by defining a suitable function $\phi(y)$ such that

$$\phi(n) = T_n(x^m;0,1)$$

and checking that $\phi''(y) > 0$ for $y > 0$, so that ϕ is convex.

In the second part of this paper we prove that for fixed $s < 0$ the sequence $\{T_n(x^s;a,b)\}_{n=1}^{\infty}$ is logarithmically convex, i.e.

$$T_n^2(x^s;a,b) \le T_{n-1}(x^s;a,b)T_{n+1}(x^s;a,b), \qquad (0 < a < b;\ s < 0).$$

The essential step of this prove lies in establishing the convexity of the function

$$\log\left(\frac{1}{x}\,\frac{e^{\frac{1}{x}}+1}{e^{\frac{1}{x}}-1}\right) \qquad \text{for } x>0,$$

which implies the log-convexity of $\{T_n(e^{\lambda x};a,b)\}_{n=1}^{\infty}$ for all $\lambda \in \mathbb{R}$, $a<b$.

1. CONVEX APPROXIMATION OF $\int_0^1 x^m dx$, $(m \in \mathbb{N})$.

1.1. Preliminaries; statement of the Theorem

Let $f\colon [0,1] \to \mathbb{R}$ be twice differentiable with continuous second derivative. Then we have by the Euler-Maclaurin summation formula

$$T_n(f) \overset{\text{def}}{=\!=} T_n(f;0,1) = \frac{1}{n}\{-\frac{1}{2}f(0) + \sum_{k=0}^{n} f(\frac{k}{n}) - \frac{1}{2}f(1)\} =$$
$$= \int_0^1 f(x)dx + \frac{1}{n}\int_0^n (x-[x]-\frac{1}{2})df(\frac{x}{n}).$$

Let the function $\theta(t)$ be defined by

(1) $$\theta(t) = -\int_0^t (x-[x]-\frac{1}{2})dx, \qquad t \in \mathbb{R}.$$

Since $\theta(t)=0$ for $t \in \mathbb{Z}$ we can write

$$T_n(f) = \int_0^1 f(x)dx - \frac{1}{n^2}\int_0^n f'(\frac{x}{n})d\theta(x) =$$
$$= \int_0^1 f(x)dx + \frac{1}{n^3}\int_0^n f''(\frac{x}{n})\theta(x)dx$$

Now define

$$\phi_f(t) = \frac{1}{t^3}\int_0^t f''(\frac{x}{t})\theta(x)dx, \qquad t>0.$$

If f is four times differentiable and if $f''(1) = f^{(3)}(1) = 0$, then $\phi_f(t)$ has a continuous second derivative for $t>0$, satisfying

$$\phi_f''(t) = \frac{1}{t^4}\int_0^1 (12f''(u) + 8uf^{(3)}(u) + u^2 f^{(4)}(u))\theta(tu)du.$$

Let $m \in \mathbb{N}$, $m \geq 5$ and put

$$g_m(x) = (1-x)^{m-1}.$$

Note that, by symmetry, $T_n(m) \overset{\text{def}}{=\!=} T_n(x^{m-1}) = T_n(g_m(x))$, so that

$$T_n(m) = \frac{1}{m} + \frac{1}{n^3}\int_0^n g_m''(\tfrac{x}{n})\theta(x)dx.$$

Since $g_m''(1) = g_m^{(3)}(1) = 0$, the corresponding function $\phi_m(t) \overset{\text{def}}{=\!=} \phi_{g_m}(t)$ satisfies

$$\text{(2)} \qquad \frac{t^4\,\phi_m''(t)}{(m-1)(m-2)} = \int_0^1 \{(m^2+m)u^2 - 8mu + 12\}(1-u)^{m-5}\theta(tu)du.$$

We intend to prove

<u>THEOREM 1</u>. *For every* $m \in \mathbb{N}$, *the sequence* $\{T_n(x^{m-1};0,1)\}_{n=1}^{\infty}$ *is convex.*

We shall prove this theorem by showing that the right-hand side of (2), and thus $\phi_m''(t)$, is positive for $m \geq 9$ and $t > 0$. Since by Taylor's theorem

$$\phi_m(n+1) + \phi_m(n-1) = 2\phi_m(n) + \tfrac{1}{2}(\phi_m''(t_1) + \phi_m''(t_2)),$$

where $t_1 \in (n-1,n)$ and $t_2 \in (n,n+1)$, this implies Theorem 1 for $m \geq 9$. For $m = 1,\ldots,8$ we express $T_n(m)$ by means of the Bernoulli polynomials (Compare for example [1]):

$$T_n(m) = \frac{1}{m}\sum_{0\leq k<\frac{1}{2}n} \binom{m}{2k} B_{2k} n^{-2k}.$$

For $m = 1,\ldots,8$, the theorem can be verified directly by this formula. So it is sufficient to show that for $t > 0$ and $m \geq 9$

(3) $$\int_0^1 \{(m^2+m)u^2 - 8mu + 12\}(1-u)^{m-5}\theta(tu)du > 0.$$

1.2. Some Lemma's

LEMMA 1. *Let* $\theta(t)$ *be defined by* (1). *Then*

a) θ *is periodic with period* 1.

b) $\theta(t) = \frac{1}{2}t(1-t)$ *for* $0 \le t < 1$.

c) $\theta(t) \le \frac{1}{8}$ *for all* $t \in \mathbb{R}$; $\theta(t) \le \frac{1}{16}t$ *for* $t \ge 2$.

d) $\int_0^n(\theta(t) - \frac{1}{12})dt = 0$ *for* $n \in \mathbb{Z}$.

e) $\int_0^x(\frac{1}{12} - \theta(t))dt \le \frac{\sqrt{3}}{216} < \frac{1}{120}$.

PROOF. By straightforward verification from (1). □

LEMMA 2. *If* $0 \le a \le \frac{1}{2}$ *and* $0 \le t \le 2$, *then* $\theta(at) \ge \frac{1}{2}t\theta(2a)$.

PROOF. Since $0 \le at \le 2a \le 1$, we have $\theta(at) = \frac{1}{2}at(1-at)$ and $\theta(2a) = a(1-2a)$. Since $0 \le t \le 2$, we then have

$$\theta(at) = \frac{1}{2}at(1-at) \ge \frac{1}{2}at(1-2a) = \frac{1}{2}t\theta(2a). \qquad \square$$

LEMMA 3. *If* $0 \le a \le \frac{1}{3}$ *and* $2 \le t \le 6$, *then* $\theta(at) \le \frac{1}{2}t\theta(2a)$.

PROOF. If $at < 1$ we have by $t \ge 2$

$$\theta(at) = \frac{1}{2}at(1-at) \le \frac{1}{2}at(1-2a) = \frac{1}{2}t\theta(2a).$$

If $at \ge 1$, then since $1 \le at \le 2$ and $0 \le 2a < 1$

$$t\theta(2a) - 2\theta(at) = at(1-2a) - (at-1)(2-at)$$

$$= (at)^2 - 2(1+a)at + 2$$

$$\ge (at)^2 - 2(1+a)at + (1+a)^2 \ge 0. \qquad \square$$

LEMMA 4. *If* $a \ge \frac{1}{4}$ *and* $0 \le x \le 2$, *then*

$$\chi_a(x) \overset{\text{def}}{=\!=} \int_0^x \theta(at)dt \geq x^2/32.$$

PROOF. Suppose $0 \leq ax \leq 1$. Then we have

$$\chi_a(x) = \int_0^x \theta(at)dt = \frac{1}{a}\int_0^{ax} \theta(u)du = \frac{1}{a}\int_0^{ax} \frac{1}{2}u(1-u)du =$$

$$= \frac{1}{12}x(3ax - 2(ax)^2) = \frac{1}{12}x^2a(3-2ax) \geq \frac{1}{12}ax^2.$$

So the lemma holds if $a \geq 3/8$. But if $a < 3/8$ we have that $ax \leq 6/8$, so $(3-2ax) \geq 3/2$. Hence

$$\int_0^x \theta(at)dt = \frac{1}{12}ax^2(3-2ax) \geq \frac{1}{48}x^2\frac{3}{2} = \frac{1}{32}x^2.$$

Suppose that $ax > 1$. Then by lemma 1(d) and 1(e)

$$\int_0^x \theta(at)dt = \frac{x}{12} + \frac{1}{a}\int_0^{ax}(\theta(u) - \frac{1}{12})du = \frac{x}{12} - \frac{1}{a}\int_{[ax]}^{ax}(\frac{1}{12} - \theta(u))du$$

$$\geq \frac{x}{12} - \frac{1}{120a} \geq x(\frac{1}{12} - \frac{1}{120}) \geq x^2/32,$$

since $x \leq 2$. □

LEMMA 5. *For* $m \geq 9$ *we have*

$$I(m) \overset{\text{def}}{=\!=} \int_0^6 t(t-2)(t-6)(1-t/m)^{m-5}dt > 0.$$

PROOF. Integration by parts reveals that

$$I(m) = \frac{m^2}{(m-3)(m-4)}\left\{12 - 24(1-\frac{6}{m})^{m-3} - \frac{16m}{m-2} - \frac{20m}{m-2}(1-\frac{6}{m})^{m-2} + \right.$$

$$\left. + \frac{6m^2}{(m-1)(m-2)} - \frac{6m^2}{(m-1)(m-2)}(1-\frac{6}{m})^{m-1}\right\}.$$

By direct calculation one may verify that $I(m) > 0$ for $m = 9,\ldots,20$. Since $(1-6/m)^m$ increases to its limit e^{-6} we have

$$I(m) > \frac{m^2}{(m-3)(m-4)}\left\{2-\frac{14}{m-1}-\frac{8}{(m-1)(m-2)}+\right.$$

$$\left.-\left(\frac{24m^3}{(m-6)^3}+\frac{20m^3}{(m-2)(m-6)^2}+\frac{6m^3}{(m-1)(m-2)(m-6)}\right)e^{-6}\right\}.$$

Since the form in curly brackets { } is monotonically increasing in m and is positive for $m = 21$, the proof is complete.

1.3. Proof of the Theorem

Put $h_m(t) \stackrel{\text{def}}{=\!=} (t-2)(t-6)(1-t/m)^{m-5}$. We shall prove that for $a > 0$ and $m \geq 9$

$$\int_0^6 h_m(t)\theta(at)dt > 0. \tag{4}$$

Since $h_m(t) \geq 0$ for $t \geq 6$ and since $\theta(t) \geq 0$ for all t, (4) implies

$$\int_0^m h_m(t)\theta(at)dt > 0,$$

so that, putting $u = t/m$ and $y = am$,

$$\int_0^1 (m^2y^2 - 8mu + 12)(1-u)^{m-5}\theta(uy)du > 0,$$

which implies (3) and the Theorem. Hence it is sufficient to show (4). Now suppose that $0 < a < \frac{1}{4}$. By Lemmas 2 and 3 we have

$$\int_0^6 h_m(t)\theta(at)dt \geq \left\{\int_0^2 + \int_2^6\right\} h_m(t)\ \frac{1}{2}\ t\theta(2a)dt =$$

$$= \frac{1}{2}\ \theta(2a)\int_0^6 th_m(t)dt > 0,$$

by Lemma 5.

So let $a \geq \frac{1}{4}$ and as before, put $\chi_a(x) = \int_0^x \theta(at)dt$. Since $\chi_a(0) = 0 = h_m(2)$, we have that

$$I_1(m) \overset{\text{def}}{=\!=} \int_0^2 h_m(t)\theta(at)dt = \int_0^2 h_m(t)d\chi_a(t) = -\int_0^2 \chi_a(t)d\,h_m(t).$$

Observe that $h_m(t)$ is decreasing for $0 \leq t \leq 2$. We thus have by Lemma 4

$$I_1(m) \geq -\int_0^2 \frac{t^2}{32}\,dh_m(t) = \int_0^2 h_m(t)\,\frac{t}{16}\,dt.$$

Since $h_m(t) \leq 0$ for $2 \leq t \leq 6$ we have by Lemma 1

$$I_2(m) \overset{\text{def}}{=\!=} \int_2^6 h_m(t)\theta(at)dt \geq \int_2^6 h_m(t)\,\frac{t}{16}\,dt.$$

Hence, by Lemma 5

$$\int_0^6 h_m(t)\theta(at)dt = I_1(m) + I_2(m) \geq \frac{1}{16}\int_0^6 t\,h_m(t)dt > 0$$

for $m \geq 9$, completing the proof of Theorem 1.

1.4. An inequality for $T_n(m)$; conclusion

Theorem 1 reads

$$T_{n-1}(m) + T_{n+1}(m) \geq 2\,T_n(m), \qquad (m,n \in \mathbb{N};\ n \geq 2).$$

Since

$$(n+1)^m T_{n+1}(m) = n^m T_n(m) + \frac{1}{2}\,n^{m-1} + \frac{1}{2}\,(n+1)^{m-1},$$

we can write the above inequality as

$$T_n(m)\left(\left(\frac{n}{n-1}\right)^m - 2 + \left(\frac{n}{n+1}\right)^m\right) \geq \frac{1}{2}\,\frac{n^{m-1}}{(n-1)^m} - \frac{1}{2}\,\frac{n^{m-1}}{(n+1)^m} + \frac{1}{n^2 - 1},$$

or

$$T_n(m) \geq \frac{1}{2} \frac{n^{m-1}(n+1)^m + 2(n+1)^{m-1}(n-1)^{m-1} + n^{m-1}(n-1)^m}{(n+1)^m n^m - 2(n+1)^m(n-1)^m + n^m(n-1)^m} .$$

The method in this section can be applied to functions x^s with $s \geq 9$. For $s < 9$ and $s \notin \mathbb{N}$ we cannot prove anything. However, numerical evidence supports the following stronger conjecture.

Conjecture. For any fixed real $s > 1$ the sequence $\{T_n(x^s;0,1)\}_{n=1}^{\infty}$ is logarithmically convex.

2. LOGARITHMICALLY CONVEX APPROXIMATION OF $\int_\alpha^\beta x^{-s}dx$, $s > 0$.

2.1. Preliminaries

A sequence $\{a_n\}_{n=1}^{\infty}$ is called logarithmically convex (or log-convex) if $a_n \geq 0$ for all $n \in \mathbb{N}$ and if $a_n^2 \leq a_{n-1}a_{n+1}$ for all $n \geq 2$. If $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are log-convex then $\{pa_n\}_{n=1}^{\infty}$, $(p > 0)$, $\{a_n b_n\}_{n=1}^{\infty}$ and $\{a_n+b_n\}_{n=1}^{\infty}$ are log-convex. The first two results are trivial, the last one is proved by means of the Cauchy-Schwartz inequality. Moreover we have

LEMMA 2.1. *Let* $\{A_n(t)\}_{n=1}^{\infty}$ *be a log-convex sequence for each* $t \in [\alpha,\beta]$. *If* $p(t) \geq 0$, *then the sequence* $\{b_n\}_{n=1}^{\infty}$, *given by*

$$b_n = \int_\alpha^\beta p(t)A_n(t)dt, \qquad (n = 1,2,\ldots)$$

is log-convex.

PROOF. Write $a_n(t) = \sqrt{A_n(t)}$. We have

$$b_n^2 = \left(\int_\alpha^\beta p(t)A_n(t)dt\right)^2 \leq \left(\int_\alpha^\beta p(t)a_{n-1}(t)a_{n+1}(t)dt\right)^2 \leq$$

$$\leq \int_\alpha^\beta p(t)a_{n-1}^2(t)dt \int_\alpha^\beta p(t)a_{n+1}^2(t)dt = b_{n-1}b_{n+1}. \qquad \square$$

2.2. Convexity of $\{T_n(e^{-\lambda x};\alpha,\beta)\}_{n=1}^{\infty}$

The following lemma is essential.

LEMMA 2.3. *The function*

$$K(x) = \frac{1}{x}\,\frac{e^{\frac{1}{x}}+1}{e^{\frac{1}{x}}-1}, \qquad x \in \mathbb{R}^+$$

satisfies

$$(\log K(x))'' \geq 0.$$

PROOF. Define $\phi(x) = \log K(x)$. Observe that

$$\phi''(x) = \frac{1}{x^2} - \frac{4}{x^3(e^{\frac{1}{x}} - e^{-\frac{1}{x}})} + \frac{2(e^{\frac{1}{x}} + e^{-\frac{1}{x}})}{x^4(e^{\frac{1}{x}} - e^{-\frac{1}{x}})^2}.$$

Setting $u = \frac{1}{x}$ we need to show that for $u > 0$

$$1 - \frac{4u}{e^u - e^{-u}} + \frac{2u^2(e^u + e^{-u})}{(e^u - e^{-u})^2} > 0$$

or, equivalently, that

$$e^{4u} - 2e^{2u} + 1 - 4u(e^{3u} - e^u) + 2u^2(e^{3u} + e^u) > 0. \tag{6}$$

The left-hand side is an entire function of u with power series expansion

$$\sum_{n=0}^{\infty} c_n u^n,$$

say.

Now observe that $c_0 = c_1 = 0$ and that for $n \geq 2$

$$c_n = \frac{1}{n!}\,(4^n - 2^{n+1} - 4n3^{n-1} + 4n + 2n(n-1)3^{n-2} + 2n(n-1)).$$

Hence $c_2 = 0$, $c_3 = 0$, $c_4 = 2$, $c_5 = 4$, $c_6 = 77/18$. For $n \geq 7$ we have

$$n!c_n > -4n3^{n-1} + 2n(n-1)3^{n-2} = 2n(n-7)3^{n-2} \geq 0,$$

so that $c_n \geq 0$ for $n = 0,1,2,\ldots$. This proves (6) and the lemma. □

We now prove

<u>THEOREM 2</u>. *Let* $\lambda \in \mathbb{R}$ *be fixed and let* $(a,b) \subset \mathbb{R}$. *Then the sequence*

$$\{T_n(e^{\lambda x};a,b)\}_{n=1}^{\infty}$$

is logarithmically convex (in n*).*

<u>PROOF</u>. Put $\Delta = b-a$. We have

$$T_n(e^{\lambda x};a,b) = \frac{1}{2n}\sum_{k=0}^{n-1}\{e^{\lambda(a+k\Delta/n)} + e^{\lambda(a+(k+1)\Delta/n)}\} =$$

$$= \frac{1}{2}\, e^{\lambda a}\, \frac{e^{\lambda\Delta}-1}{\lambda\Delta}\, \frac{\lambda\Delta}{n}\, \frac{e^{\lambda\Delta/n}+1}{e^{\lambda\Delta/n}-1}\,.$$

Since $\frac{1}{2}\, e^{\lambda a}(e^{\lambda\Delta}-1)/\lambda\Delta$ is positive, we must show that the sequence

$$\{K(\tfrac{n}{\lambda\Delta})\}_{n=1}^{\infty}$$

is log-convex. For $\lambda > 0$ this follows from Lemma 2.3. For $\lambda < 0$ observe that $K(x) = K(-x)$. For $\lambda = 0$ the theorem is trivial. □

2.2. <u>The main Theorem</u>

<u>THEOREM 3</u>. *Let* $s > 0$ *be fixed and let* $b > a > 0$. *Then the sequence*

$$\{T_n(x^{-s};a,b)\}_{n=1}^{\infty}$$

is logarithmically convex.

PROOF. For $s > 0$ and $x > 0$ we have

$$\Gamma(s) = \int_0^\infty e^{-u}u^{s-1}du = x^s \int_0^\infty e^{-xt}t^{s-1}dt.$$

so that

$$x^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty e^{-xt}t^{s-1}dt.$$

Since T_n acts as a linear operator, we have for $0 < a < b$

$$T_n(x^{-s};a,b) = \frac{1}{\Gamma(s)} \int_0^\infty T_n(e^{-xt};a,b)t^{s-1}dt.$$

Since each sequence $T_n(e^{-xt};a,b)$ is log-convex, the theorem follows directly from Lemma 2.1. □

Theorem 2.1 can be generalized as follows. Let $\{c_k\}_{k=1}^\infty$ be a sequence of real numbers such that

$$f(x) = \sum_{k=1}^\infty c_k x^{-k}$$

is convergent for $x \in [a,b]$. Then

$$T_n(f;a,b) = \sum_{k=1}^\infty c_k T_n(x^{-k};a,b) = \sum_{k=1}^\infty \frac{c_k}{\Gamma(k)} \left(\int_0^\infty T_n(e^{-xt};a,b)t^{k-1}dt \right)$$

$$= \int_0^\infty g(t)\, T_n(e^{-xt};a,b)dt,$$

where $g(t) = \sum_{k=1}^\infty \frac{c_k}{\Gamma(k)} t^{k-1}$. If $g(t)$ converges for $t \in \mathbb{R}^+$ and is non-negative on $\mathbb{R}^+$, then it follows that $T_n(f)$ is log-convex.

EXAMPLE. Let $f(x) = \sum_{k=1}^{2m+1} (-1)^{k+1}x^{-k}$. Then $g(t) = 1 - \frac{t}{1!} + \frac{t^2}{2!} - \dots + \frac{t^{2m}}{2m!}$.

Since $e^{-t} = g(t) - \frac{t^{2m+1}}{(2m+1)!} e^{-\eta t}$ for some $\eta \in (0,1)$ by Taylor's theorem, we find that $g(t) > 0$ for all $t \in \mathbb{R}^+$. So $\{T_n(f;\alpha,\beta)\}_{n=1}^{\infty}$ is log-convex. The above argument can be directly extended to functions of the form

$$f(x) = \sum_{k=1}^{\infty} c_k x^{-s_k},$$

where $0 < s_1 < s_2 < \ldots$ are real numbers, the c_k's satisfying similar conditions as above. The reader will have no difficulties in constructing an integral analogue of the above generalization of theorem 2.1.

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