stichting
mathematisch
centrum

MC

AFDELING ZUIVERE WISKUNDE ZW 87/76 NOVEMBER
(DEPARTMENT OF PURE MATHEMATICS)

M.R. BEST

THE EXCESS OF A HADAMARD MATRIX

2e boerhaavestraat 49 amsterdam

Printed at the Mathematical Centre, 49, 2e Boerhaavestraat, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O), by the Municipality of Amsterdam, by the University of Amsterdam, by the Free University at Amsterdam, and by industries.

AMS(MOS) subject classification scheme (1970): 05B20

The excess of a Hadamard matrix

by

M.R. Best

ABSTRACT

Let $\sigma(n)$ be the greatest possible sum of the entries of a Hadamard matrix of order n. We derive

$$n^2\ 2^{-n}\binom{n}{\frac{1}{2}n} \le \sigma(n) \le n\sqrt{n},$$

which implies

$$n\sqrt{n/2} \le \sigma(n) \le n\sqrt{n}.$$

Besides, $\sigma(n)$ is evaluated for $n \le 20$ and several other values of n.

KEY WORDS & PHRASES: *sum of the entries of a Hadamard matrix.*

1. INTRODUCTION

In [3], K.W. SCHMIDT asked for an estimate on the maximal number of ones in a Hadamard matrix of order n. At the international colloquium on combinatorial problems and graph theory held at Orsay, July 9-13, 1976, E.T.H. WANG repeated the question and presented some results obtained by Mr. Schmidt and himself.

In this note we give sharper upper and lower bounds on the maximal number of ones, which enable us to calculate the maximum explicitly for $n \leq 20$ and several other values of n.

2. DEFINITIONS AND OBSERVATIONS

A *Hadamard matrix* of order n is an $n \times n$ - $(1,-1)$-matrix whose rows are mutually orthogonal. The set of all Hadamard matrices of order n is denoted by Ω_n.

The famous conjecture on the existence of Hadamard matrices states that Hadamard matrices of order n exist if and only if $n = 1$, $n = 2$ or $n \equiv 0 \pmod 4$. The necessity of the condition is certain; the sufficiency has only been proved for $n \leq 264$. The set of all integers n for which $\Omega_n \neq \emptyset$ is denoted by N.

The *weight* $w(H)$ of a Hadamard matrix H is defined as the number of its positive entries. It turns out however that it is more natural to study the *excess* $\sigma(H)$ of H, defined as the sum of all its entries. Obviously $w(H) = \frac{1}{2}(n^2+\sigma(H))$, where n denotes the order of H.

For each $n \in N$ we define

$$w(n) = \max\{w(H) \mid H \in \Omega_n\},$$

$$\sigma(n) = \max\{\sigma(H) \mid H \in \Omega_n\}.$$

Now $w(n) = \frac{1}{2}(n^2+\sigma(n))$ for each $n \in N$.

Obviously $\sigma(1) = 1$ and $\sigma(2) = 2$, so $w(1) = 1$ and $w(2) = 3$. Throughout this paper we will assume that $n \in N$ and $n \equiv 0 \pmod 4$.

For each $H \in \Omega_n$ we denote the entry in the i-th row and the k-th column by H_{ik}, and the k-th column sum $\sum_{i=1}^{n} H_{ik}$ by s_k.

Then $s_k \equiv 0 \pmod 2$ and it follows from the orthogonality of the columns that $s_k \equiv s_\ell \pmod 4$. Therefore $\sigma(H) = \sum_{k=1}^{n} s_k \equiv 0 \pmod 4$, and so $\sigma(n) \equiv 0 \pmod 4$.

Since the Kronecker product of two Hadamard matrices is again a Hadamard matrix, and since it is easily checked that $\sigma(H_1 \otimes H_2) = \sigma(H_1)\sigma(H_2)$, it follows that $\sigma(nm) \geq \sigma(n)\,\sigma(m)$. [1)]

3. THE UPPER BOUND

Evaluating the sum of the inner products of all ordered pairs of rows in a Hadamard matrix H of order n in two ways, we obtain:

$$n^2 = \sum_{i,j=1}^{n} \sum_{k=1}^{n} H_{ik} H_{jk} = \sum_{k=1}^{n} \left(\sum_{i=1}^{n} H_{ik} \right)^2 = \sum_{k=1}^{n} s_k^2.$$

By the Cauchy-Schwarz inequality, we have

$$\sigma(H) = \sum_{k=1}^{n} s_k \leq \left(n \sum_{k=1}^{n} s_k^2 \right)^{\frac{1}{2}} = n\sqrt{n}.$$

This proves

Theorem 1. $\sigma(n) \leq n\sqrt{n}$ *and* $w(n) \leq \frac{1}{2}n(n+\sqrt{n})$.

Remark.

Often the bound can slightly be improved by taking into account that the numbers s_k are even integers, all in the same residue class modulo 4. This idea will be illustrated in the evaluation of $\sigma(12)$ (cf. section 6).

4. REGULAR HADAMARD MATRICES

The upper bound established in Theorem 1 can only be achieved if n is

---

1) This observation is due to SCHMIDT & WANG [4], who formulated it in terms of the weights.

a square and all column sums are equal. In this case the Hadamard matrix is called *regular*, and it yields a symmetric block design SBIBD(n, $\frac{1}{2}(n-\sqrt{n})$, $\frac{1}{4}(n-2\sqrt{n})$) (cf. WALLIS et al. [5], p.280).

Theorem 2.

$\sigma(n) = n\sqrt{n}$ *if and only if there exists a regular Hadamard matrix of order* n.

Proof. Follows at once from the proof of Theorem 1.

Regular Hadamard matrices are known to exist for many square orders $n = s^2$, e.g. if s is the order of a Hadamard matrix, if s+1 and s-1 are both prime powers, if BIBD(s^2-1, $\frac{1}{2}s(s-1)$, s+1, $\frac{1}{2}s$, 1) exists, or if n is the product of orders of regular Hadamard matrices (cf. WALLIS et al. [5], appendix E). Hence the existence of regular Hadamard matrices of orders 0, 1,4,16,36,64,100,144,196 (BIBD(195,91,15,7,1) exists from HANANI [2], table 5.20 and Lemma 2.10), 256, 324, 400 and many others is settled.

5. THE LOWER BOUND

Let $H \in \Omega_n$. Suppose that we were able to find a row x in $\{1,-1\}^n$ such that the inner products of x with all rows of H were all about equal to $\pm\sqrt{n}$. Then we would multiply (entrywise) all rows of H with this row x, negate all rows with negative row sums, and obtain a Hadamard matrix with each row sum about equal to $\sqrt{n}$, thus proving that the upper bound is almost sharp.

Unfortunately, we do not know how to prove the existence of such a nice row x, but it turns out that just a random row does not work too badly on average. This idea leads to the following lower bound.

Theorem 3. $\sigma(n) \geq n^2\, 2^{-n}\binom{n}{\frac{1}{2}n}$ *and* $w(n) \geq \frac{1}{2}n^2\left(1+2^{-n}\binom{n}{\frac{1}{2}n}\right)$.

Remark.

This lower bound for $\sigma(n)$ is asymptotically equivalent to $n\sqrt{\frac{2n}{\pi}}$ for $n \to \infty$, and is not less that $n\sqrt{n/2}$ for all n. Hence it is of the same order of magnitude as the upper bound $n\sqrt{n}$, but still considerably smaller.

<u>Proof</u>.

Let $H \in \Omega_n$, and let x be some row in $\{1,-1\}^n$. We construct the matrix $H_x \in \Omega_n$ by multiplying each row of H by x (entrywise), and then negating all rows with negative row sums. Now

$$\sigma(H_x) = \sum_{z \in H} |\langle x,z \rangle| .$$

[Here $z \in H$ means: z is a row of H, and $\langle x,z \rangle$ denotes the inner product of x and z.]

Obviously, there must be a choice x_0 for x such that

$$\sigma(H_{x_0}) \geq 2^{-n} \sum_{x \in \{1,-1\}^n} \sigma(H_x).$$

Hence

$$\sigma(n) \geq 2^{-n} \sum_{x \in \{1,-1\}^n} \sum_{z \in H} |\langle x,z \rangle| =$$

$$= 2^{-n} \sum_{z \in H} \sum_{i=0}^{n} \sum_{\substack{x \in \{1,-1\}^n \\ d(x,z)=i}} |n-2i| =$$

$$= 2^{-n}\, n \sum_{i=0}^{n} |n-2i| \binom{n}{i} = n^2\, 2^{-n} \binom{n}{\frac{1}{2}n} .$$

[Here $d(x,z)$ denotes the Hamming distance between x and z.] $\square$

6. $\sigma(12)$ AND $\sigma(20)$

If we apply Theorem 1 and 3 to the case $n = 12$, we obtain $33 \leq \sigma(12) \leq 41$. Since $\sigma(12) \equiv 0 \pmod 4$, either $\sigma(12) = 36$ or $\sigma(12) = 40$.

But we can improve the upper bound. From the proof of Theorem 1, there must exist even numbers $s_1, \ldots, s_{12}$, all in the same residue class modulo 4, such that $\sum_{k=1}^{12} s_k^2 = 144$ and $\sum_{k=1}^{12} s_k = \sigma(12)$.

If $s_k \equiv 0 \pmod 4$ for all k, then $\sum_{k=1}^{12} s_k$ attains its maximum - subject to the condition $\sum_{k=1}^{12} s_k^2 = 144$ - if $s_k = 0$ for three values of k, and $s_k = 4$ for the other nine values. Hence $\sum_{k=1}^{12} s_k \le 36$.

If $s_k \equiv 2 \pmod 4$ for all k, then $\sum_{k=1}^{12} s_k$ attains its maximum if $s_k = 2$ for nine values of k, and $s_k = 6$ for the other three values. Hence again $\sum_{k=1}^{12} s_k \le 36$.

This proves

<u>Theorem 4</u>. $\sigma(12) = 36$ *and* $w(12) = 90$.

In the case n = 20, a similar argument only yields $\sigma(20) \in \{72,76,80,84\}$. However, the Hadamard matrices of order 20 are explicitly known: they occur in only three non-isomorphic variants (cf. HALL [1]). Exhaustive search by computer learns that no Hadamard matrix of order 20 and excess 84 exists, but that excess 80 does occur, e.g.

$$\sigma \begin{bmatrix}
\texttt{------++++-+++++++++} \\
\texttt{+-+--+---+++-++++--+} \\
\texttt{-++-++++++++-+----++} \\
\texttt{++--+-+---++-+-++++-} \\
\texttt{+++--+++--+-+-+-++++} \\
\texttt{+-++---++-+--+-+-+++} \\
\texttt{+--++-+-+++---+-+-++} \\
\texttt{-+++----+-+++++-+-+-} \\
\texttt{-++++-+--++-++++-+-+} \\
\texttt{--++++++-----++++-+-} \\
\texttt{-+-+++-++-++--++++-+} \\
\texttt{--+-++--+++-+--++++-} \\
\texttt{+-+-+-+++-+++-++----} \\
\texttt{++-+-++++++-++-++---} \\
\texttt{+++-+--+++---++-++--} \\
\texttt{++++-++-++-+--++-++-} \\
\texttt{+++++--+-+-++--++-++} \\
\texttt{+-+++++-+--+++--++-+} \\
\texttt{+--+++-+-++++++--++-} \\
\texttt{++--++--+---++++--++}
\end{bmatrix} = 80.$$

(The matrix has been derived from the quadratic residues modulo 19.)

This proves

<u>Theorem 5</u>. $\sigma(20) = 80$ *and* $w(20) = 240$.

7. NUMERICAL DATA AND A PRELIMINARY CONJECTURE

For $n \le 20$, the value of $\sigma(n)$ is known by now:

$$\begin{aligned}
\sigma(\ 0) &= \ 0\\
\sigma(\ 1) &= \ 1\\
\sigma(\ 2) &= \ 2\\
\sigma(\ 4) &= \ 8\\
\sigma(\ 8) &= 20\\
\sigma(12) &= 36\\
\sigma(16) &= 64\\
\sigma(20) &= 80.
\end{aligned}$$

The above data suggest that $\sigma(n)$ is divisible by $\frac{1}{2}n$. A formula that satisfies this property might be

$$\sigma(n) = \begin{cases} \frac{1}{2}n[2\sqrt{n}] & \text{if } n \equiv 0 \pmod 8 \\ n[\sqrt{n}] & \text{otherwise.} \end{cases}$$

REFERENCES

[1] HALL JR., MARSHALL, *Hadamard matrices of order* 20, Technical Report no. 32-761, Jet Propulsion Labaratory, California Institute of Technology, Pasadena, California, 1965.

[2] HANANI, H., *Balanced incomplete block designs and related designs*, Discrete Math., 11 (1975), 255-369.

[3] SCHMIDT, K.W., Problem 863, Math. Mag., 46 (1973), 103.

[4] SCHMIDT, K.W. & E.T.H. WANG, *The weights of Hadamard Matrices*, to appear.

[5] WALLIS, W.D., A.P. STREET & J.S. WALLIS, *Combinatorics: Room Squares, Sum-Free Sets, Hadamard Matrices*, Lecture Notes in Mathematics, Springer-Verlag, 1972.