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MEDIAN GRAPHS AND HELLY HYPERGRAPHS

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Median graphs and Helly hypergraphs*)

by

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ABSTRACT

One-to-one correspondences are established between the following combinatorial structures: (i) median interval structures (or median segments, introduced by SHOLANDER); (ii) maximal Helly hypergraphs such that with each edge also its complement is in the hypergraph; and (iii) median graphs (connected graphs such that for any three vertices u, v, w there is exactly one vertex x such that d(u,v) = d(u,x) + d(x,v), d(v,w) = d(v,x) + d(x,w) and d(w,u) = d(w,x) + d(x,u), where d is the distance function of the graph).

KEY WORDS & PHRASES: *median, interval structure, Helly hypergraph, copair, median graph.*

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0. INTRODUCTION

In this paper one-to-one correspondences will be established between three at first sight fairly distinct concepts. These concepts are:

(i) *median interval structures* introduced by M. SHOLANDER [7], [8] under the name of median segments (cf. 1.1);

(ii) *maximal Helly copair hypergraphs* (i.e. simple Helly hypergraphs, the edge-set of which contains with each edge its complement, and which are maximal with respect to this property; see 1.2); and

(iii) *median graphs*, introduced in section 1.3.

The one-to-one correspondences are established in section 2.

In section 3 is elaborated how to construct a maximal Helly copair hypergraph from a median graph, using results of SHOLANDER [9].

With minor adaptations we adopt the terminology of BERGE [1] on hypergraphs, of WILSON [10] on graphs and of BIRKHOFF [2] on lattice theory.

1. DEFINITIONS AND PRELIMINARIES

Throughout this paper V denotes a fixed finite set.

1.1. <u>INTERVAL STRUCTURES</u>. A function $I:V \times V \to \mathcal{P}(V)$ is called an *interval structure* on V if

$$\text{(I1)}\quad x,y \in I(u,v) \quad \text{iff} \quad I(x,y) \subset I(u,v) \quad (x,y,u,v \in V),$$

$$\text{(I2)}\quad I(u,v) \cap I(v,w) \cap I(w,u) \neq \emptyset \qquad (u,v,w \in V).$$

Each set I(u,v) is called an *interval*. A subset U of V is I-*convex* if for all u, v $\in$ V the interval I(u,v) is contained in U. The notion of interval structure was introduced in [3]. Examples of interval structures on V can be obtained from trees with vertex-set V (then take $I(u,v) = \{w \in V \mid w \text{ lies on the shortest } u,v\text{-path}\}$), and from lattices $(V,\leq)$ (in this case $I(u,v) = \{w \in V \mid u \wedge v \leq w \leq u \vee v\}$).

If I satifies condition (I1) and the following condition

$$\text{(I2')}\quad |I(u,v) \cap I(v,w) \cap I(w,u)| = 1 \quad (u,v,w \in V),$$

then I is called a *median interval structure* on V. Interval stuctures obtained from trees as indicated above are median interval structures. An interval structure obtained from a lattice is a median interval structure iff the lattice is distributive (cf. [2]). SHOLANDER [8] has given the following characterization of median interval structures (he used the term median segments):

THEOREM 1. (SHOLANDER [8]) *A function* $I:V \times V \to P(V)$ *is a median interval structure on* V *iff*

$$\textit{if} \quad w \in I(u,v) \quad \textit{then} \quad I(u,w) \subset I(u,v) \cap I(v,u) \qquad (u,v \in V),$$
$$|I(u,v) \cap I(v,w) \cap I(w,u)| = 1 \qquad (u,v,w \in V),$$
$$I(v,v) = \{v\} \qquad (v \in V).$$

1.2. HYPERGRAPHS. In this paper a *hypergraph* $H = (V,E)$ consists of a *vertex-set* V and a family $E \subset P(V)$ of nonvoid subsets of V, the members of which are called *edges*. Occasionally we will write E instead of (V,E).

A hypergraph is a *Helly hypergraph* if it satisfies the *Helly property*, i.e. every subfamily of E, any two members of which meet, has a non-empty intersection. For vertices u and v of the hypergraph (V,E) define

$$I_E(u,v) = \cap \{B \in E \mid u,v \in B\}.$$

A theorem of P.C. GILMORE (see [5], or [1] p. 396) can be formulated as follows:

THEOREM 2. (GILMORE) *A hypergraph* (V,E) *satisfies the Helly property iff* I_E *is an interval structure on* V.

As a consequence of GILMORE's theorem we have: *Let* I *be an interval structure on* V. *Any family* E *of nonvoid* I*-convex subsets of* V *satisfies the Helly property.*

A hypergraph (V,E) with the property that $V\backslash B \in E$ for all $B \in E$ will be called a *copair hypergraph*. We call the set $\{B,V\backslash B\}$ a copair of V and $\{\phi,V\}$ the trivial copair. A Helly copair hypergraph of course is a copair hypergraph, which satisfies the Helly property. Finally a *maximal Helly copair hypergraph* (V,E) is a Helly copair hypergraph such that: if $\{A,V\backslash A\}$

is a non-trivial copair and $E \cup \{A, V\setminus A\}$ satifies the Helly property then $A \in E$.

A hypergraph (V,E) is said to *separate vertices* if for any two distinct vertices $u,v \in V$ there exists an edge $A \in E$ such that $u \in A$ and $v \notin A$.

LEMMA 3. *Let* (V,E) *be a Helly copair hypergraph. Then* (V,E) *is maximal iff* (V,E) *separates vertices.*

PROOF. Note that (V,E) separates vertices iff $I_E(v,v) = \{v\}$ for all $v \in V$.

Assume that E does not separate vertices. That is there exists a vertex $v \in V$ such that $I_E(v,v)$ contains besides v another vertex. Using GILMORE's theorem it can be verified that in this case $E \cup \{\{v\}, V\setminus\{v\}\}$ satisfies the Helly property. Therefore E is not maximal.

To prove sufficiency of vertex separation let $\{A, V\setminus A\}$ be a non-trivial copair of V not in E. Take a vertex $u \in A$ and a vertex $v \in V\setminus A$ such that $|\ I_E(u,v)\ |$ is as small as possible. We assert that $I_E(u,v) \cap A = \{u\}$ and $I_E(u,v)\setminus A = \{v\}$.

For suppose $I_E(u,v) \cap A \neq \{u\}$ and let $w \in I_E(u,v) \cap A$ with $w \neq u$. Since E separates vertices, there exists an edge $C \in E$ such that $w \in C$ and $u \notin C$. Then we have that $v \in C$. So $u \notin I_E(w,v) \subset I_E(u,v)$, contradicting the minimality of $I_E(u,v)$. In the same way we prove $I_E(u,v)\setminus A = \{v\}$. Hence $I_E(u,v) = \{u,v\}$.

Let $B \in E$ be an edge such that $v \in B$ and $u \notin B$. Then $A \cap B \neq \emptyset$ or $(V\setminus A) \cap (V\setminus B) \neq \emptyset$, since $A \notin \{B, V\setminus B\} \subset E$; say $A \cap B \neq \emptyset$. Now the set of edges, which contain both u and v, together with A and B forms a family of subsets of V, any two members of which meet. The intersection of this family equals

$$I_E(u,v) \cap A \cap B = \{u,v\} \cap A \cap B,$$

which clearly is empty. Thus $E \cup \{A, V\setminus A\}$ does not satisfy the Helly property. $\square$

COROLLARY 4. *Let* (V,E) *be a maximal Helly copair hypergraph. Then*

$$|E| \geq 2\lceil {}^2\log|V| \rceil.$$

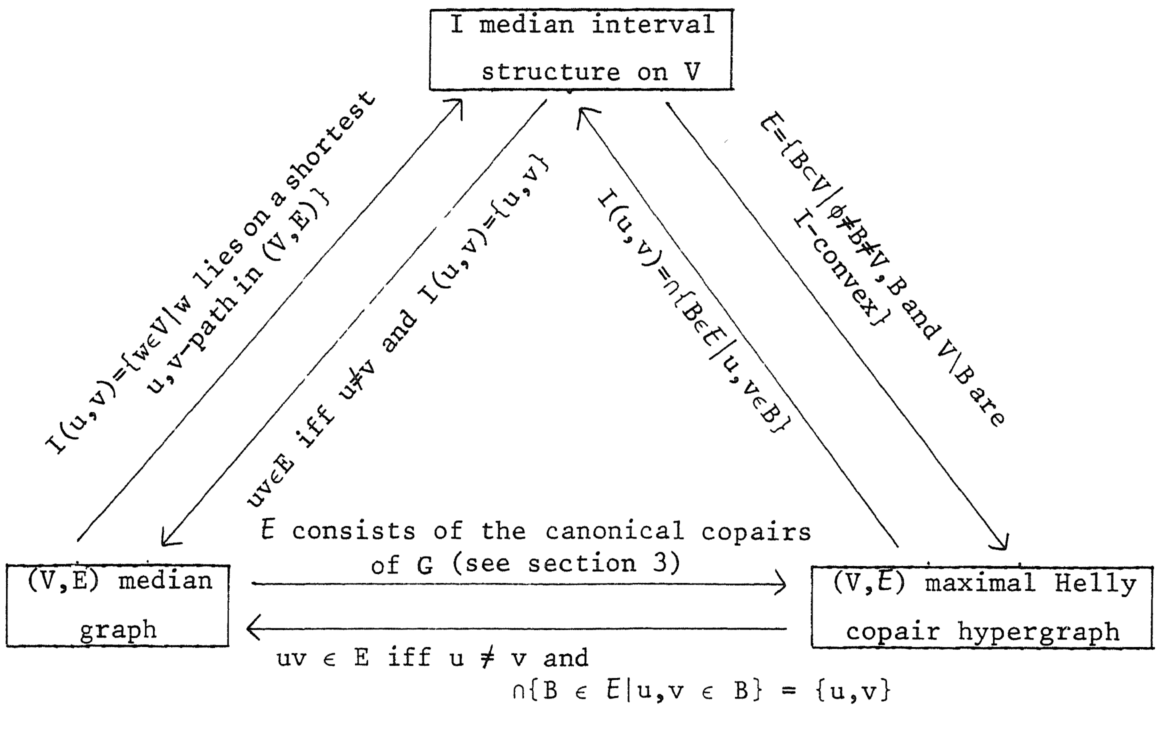
1.3 <u>MEDIAN GRAPHS</u>. Let G be a simple loopless graph with vertex-set V and distance function d. G will be called a *median graph* if it is connected and satisfies the *graph median property*, i.e. for any $u,v,w \in V$ there exists precisely one vertex $x \in V$, called the *graph median* of u,v and w, such that

$$\begin{cases} d(u,x) + d(x,v) = d(u,v) \\ d(v,x) + d(x,w) = d(v,w) \\ d(w,x) + d(x,u) = d(w,u). \end{cases}$$

Note that all trees and the n-cubes are median graphs. It is easy to see that each median graph is bipartite.

2. THE THEOREM

<u>THEOREM 5.</u> *There exists a one-to-one correspondence between the median interval structures on V, the maximal Helly copair hypergraphs with vertex-set V and the median graphs with vertex-set V. The one-to-one correspondences are indicated in the following diagram, which commutes in all directions.*



The proof of the theorem amounts to the following propositions (the direct correspondence between median graphs and maximal Helly copair hypergraphs will be explained in section 3).

For vertices u and v of the graph G = (V,E) define

$$I_G(u,v) = \{w \in V \mid w \text{ lies on a shortest } u,v\text{-path in } G\}.$$

PROPOSITION 6. *Let* G = (V,E) *be a median graph. Then* I_G *is a median interval structure on* V.

PROOF. I_G satisfies the conditions mentioned in theorem 1. □

PROPOSITION 7. *Let* I *be a median interval structure on* V. *Define the graph* G_I *with vertex-set* V *by*

$$uv \in E(G_I) \textit{ iff } u \neq v \textit{ and } I(u,v) = \{u,v\} \qquad (u,v \in V).$$

Then G_I *is a median graph.*

PROOF. We will prove that G_I is connected and that $I_{G_I} = I$. Then clearly G_I is a median graph.

First observe that for u,v,w ∈ V we have

$$w \in I(u,v) \text{ iff } I(u,w) \cap I(w,v) = \{w\}.$$

Thus for $w \in I(u,v)\setminus\{u,v\}$ holds $u \notin I(w,v) \subset I(u,v)$ and $v \notin I(u,w) \subset I(u,v)$. Using this it is easily verified by induction on $|I(u,v)|$ that I(u,v) induces a connected subgraph of G_I for all u,v ∈ V. Hence G_I is connected.

To prove that $I(u,v) = I_{G_I}(u,v)$ for all u,v ∈ V we use induction on d(u,v). Clearly $I(u,v) = I_{G_I}(u,v)$ for all u,v ∈ V with $d(u,v) \leq 1$. So take vertices u,v ∈ V with $d(u,v) > 1$.

Let $w \in I_{G_I}(u,v)\setminus\{u,v\}$. Then $d(u,w) < d(u,v)$ and $d(w,v) < d(u,v)$, so $I_{G_I}(u,w) = I(u,w)$ and $I_{G_I}(w,v) = I(w,v)$. Since clearly $I_{G_I}(u,w) \cap I_{G_I}(w,v) = \{w\}$, we have $w \in I(u,v)$ and thus $I_{G_I}(u,v) \subset I(u,v)$.

Assume $I(u,v)\setminus I_{G_I}(u,v) \neq \emptyset$.

For any vertex $w \in I(u,v)\setminus I_{G_I}(u,v)$ we must have $I(u,w) \cap I_{G_I}(u,v)=\{u\}$, and similarly $I(w,v) \cap I_{G_I}(u,v) = \{v\}$. For if $w' \in I(u,w) \cap I_{G_I}(u,v)$, with $w' \neq u$, then $w \in I(w',v)$ and by the induction hypothesis $I(w',v) = I_{G_I}(w',v) \subset I_{G_I}(u,v)$. Hence $w \in I_{G_I}(u,v)$, contradicting the choice of w.

Since $I(u,v)$ induces a connected subgraph of G_I, there exists a path P from u to v, all the internal vertices of which lie in $I(u,v)\backslash I_{G_I}(u,v)$. Clearly the length of P exceeds $d(u,v)$ so P has at least two distinct internal vertices, say x and y.

Since $d(u,v) \geq 2$, there exists a vertex $z \in I_{G_I}(u,v)\backslash\{u,v\}$. By the induction hypothesis we have $I(u,z) = I_{G_I}(u,z)$ and $I(z,v) = I_{G_I}(z,v)$. Now

$$u \in I(u,z) \cap I(u,x) = I_{G_I}(u,z) \cap I(u,x) \subset I_{G_I}(u,v) \cap I(u,x)=\{u\}.$$

So $u \in I(z,x)$. Similarly $v \in I(z,x)$ and thus $I(u,v) \subset I(z,x) \subset I(u,v)$. In the same way it follows that $I(u,v) = I(z,y)$. But then

$$x,y \in I(x,y) = I(z,x) \cap I(x,y) \cap I(y,z),$$

contradicting the fact that I is a median interval structure. Conclusion: $I(u,v) = I_{G_I}(u,v)$. □

In the proof of the preceding proposition we have seen that for a median interval structure I holds: $I_{G_I} = I$. Furthermore from propositions 6 and 7 follows immediately that, when G is a median graph, we have $G_{I_G} = G$.

PROPOSITION 8. *Let* (V,E) *be a maximal Helly copair hypergraph. Then* I_E *is a median interval structure on* V.

PROOF. Assume that there exist vertices $u,v,w \in V$ such that $x,y \in I_E(u,v) \cap I_E(v,w) \cap I_E(w,u)$ for vertices $x,y \in V$, with $x \neq y$. According to lemma 3 there is an edge $B \in E$ such that $x \in B$ and $y \notin B$. Then one of the edges B and $V \setminus B$, say B, must contain at least two of the three vertices u,v and w, say u and v. But then $y \notin I_E(u,v)$. Contradiction. □

PROPOSITION 9. *Let* I *be a median interval structure on* V *and let*

$$E_I = \{B \subset V \mid \phi \neq B \neq V,\ B \textit{ and } V\backslash B \textit{ are } I\textit{-convex}\}.$$

Then $(V,\mathcal{E}_I)$ *is a maximal Helly copair hypergraph.*

PROOF. Clearly $(V,\mathcal{E}_I)$ is a Helly copair hypergraph. By lemma 3 it suffices to show that $\mathcal{E}_I$ separates vertices. So suppose that for vertices $u,v \in V$, with $u \neq v$, there is no edge B such that $u \in B$ and $v \notin B$. Assume futhermore that u and v are such that $|I(u,v)|$ is as small as possible.

We first prove that $I(u,v) = \{u,v\}$. Suppose $w \in I(u,v)\setminus\{u,v\}$. Since $|I(u,w)| < |I(u,v)|$, there exists an edge A such that $u \in A$ and $w \notin A$. It follows that $v \in A$ (u and v cannot be separated). So $w \in I(u,v) \subset A$, for A is I-convex, contradicting $w \notin A$. Therefore $I(u,v) = \{u,v\}$.

Now let $B = \{z \in V | v \notin I(u,z)\}$.

Then $V \setminus B = \{z \in V | u \notin I(z,v)\}$, since $I(u,z) \cap I(z,v) \cap \{u,v\}$ is a singleton. We assert that B and $V \setminus B$ are I-convex, that is $B \in \mathcal{E}_I$. Since $u \in B$ and $v \notin B$ this contradicts our assumption that $\mathcal{E}_I$ does not separate vertices.

We only prove that B is I-convex (the I-convexity of $V \setminus B$ can be treated similarly).

Note that for each $z \in B$ we have $I(u,z) \subset B$, since $v \notin I(u,z)$. Let $x,y \in B$ and suppose $I(x,y) \not\subset B$. Take $w \in I(x,y)\setminus B$. Since $I(u,x) \subset I(v,x)$ and $I(u,y) \subset I(v,y)$ we have that

$$\{z\} = I(u,x) \cap I(x,y) \cap I(y,u) = I(v,x) \cap I(x,y) \cap I(y,v)$$

for some $z \in B$. Now also

$$\{z\} \subset I(z,w) \cap I(z,v) \subset I(x,y) \cap I(x,v) \cap I(y,v) = \{z\},$$

since $z,w \in I(x,y)$ and $z \in I(u,x) \cap I(u,y) \subset I(v,x) \cap I(v,y)$. This implies $z \in I(w,v)$ according to the observation made at the beginning of the proof of proposition 7. So $I(z,v) \subset I(w,v)$. But, since $w \notin B$, $u \notin I(w,v)$ and thus $u \notin I(z,v)$, that is $z \in V \setminus B$, contradicting the fact that $z \in B$. □

From propositions 8 and 9 we deduce: let I be a median interval structure on V, then $I_{\mathcal{E}_I} = I$; and let $(V,\mathcal{E})$ be a maximal Helly copair hypergraph, then $\mathcal{E}_{I_\mathcal{E}} = \mathcal{E}$.

3. MEDIAN GRAPHS AND HELLY HYPERGRAPHS

In this section the direct correspondence between median graphs and maximal Helly copair hypergraphs with vertex-set V, mentioned in the theorem, is further elaborated.

3.1. MEDIAN SEMILATTICES. Let $(V,\leq)$ be a partially ordered set (poset). v is said to *cover* u $(u,v \in V)$, if $u \leq v$ and there is no $w \in V$ such that $u < w < v$. A *semilattice* $(V,\leq)$ is a poset, in which any two elements u,v have a greatest lower bound $u \wedge v$. For $u,v \in V$ set $[u,v] = \{w \in V | u \leq w \leq v\}$. The semilattice $(V,\leq)$ is called *distributive* if $([u,v],\leq)$ is a distributive lattice for all $u,v \in V$. The semilattice is said to satisfy the *coronation property* if for any three elements $u,v,w \in V$, such that the three least upper bounds $u \vee v$, $v \vee w$, $w \vee u$ exist, there exists a least upper bound $u \vee v \vee w$.

A *median semilattice* is a distributive semilattice, which satisfies the coronation property. This concept was introduced by SHOLANDER [9].

On a median semilattice $(V,\leq)$ the ternary operation $(u,v,w) = (u \wedge v) \vee (u \wedge w) \vee (w \wedge u) \in V$ can be defined, called the *median* of u,v and w (SHOLANDER [9] also characterized medians).

We review some results of SHOLANDER [9] reformulating them in our terminology:

(A) *Each median semilattice* $(V,\leq)$ *yields a median interval structure* $I_{\leq}$ *on* V, *where*

$$I_{\leq}(u,v) = \{w | w \text{ is the median of } u,v,w\} \qquad (u,v \in V).$$

(B) *Let* I *be a median interval structure on* V *and* $u \in V$. *Define an ordering* $\leq_{I,u}$ *on* V *by*

$$v \leq_{I,u} w \text{ iff } v \in I(u,w) \qquad (v,w \in V).$$

Then $(V,\leq_{I,u})$ *is a median semilattice. Furthermore the correspondences given in* (A) *and* (B) *commute.*

(C) *Let* (V,≤) *be a median semilattice. Then* (V,≤) *can be embedded in a Boolean algebra by an order preserving mapping, which also perserves the covering relation in* (V,≤).

3.2 CUTSET COLOURINGS. A *cutset colouring* of a connected graph is a colouring of the edges in such a way that the edges of any colour form a matching as well as a cutset (i.e. a minimal disconnecting edge-set). If we want to establish a cutset colouring of a graph we are forced to colour non-adjacent edges in each circuit of length four with the same colour. So the n-cube admits a cutset colouring with n colours, which is uniquely determined up to the labelling of the colours. Deleting the edges with a given colour from the n-cube breaks the graph up into two components, which both are (n-1)-cubes.

Note that not all connected graphs admit a cutset colouring. Necessary conditions for the existence of a cutset colouring of the edges of a connected graph are for instance that the graph is simple, loopless and bipartite and that it does not contain $K_{2,3}$ as a subgraph.

3.3 MEDIAN GRAPHS AND MAXIMAL HELLY COPAIR HYPERGRAPHS. The *diagraph* of a poset (V,≤) is the graph with vertex-set V, in which two vertices are joined by an edge iff one of the two covers the other in the poset. Clearly, the diagraph of the Boolean algebra on 2^n elements is the n-cube. As a consequence of (A) and (B) and propositions 6 and 7 we have

PROPOSITION 10. *Let* G *be a graph. Then* G *is a median graph iff* G *is the diagraph of a median semilattice.*

PROPOSITION 11. *Let* G *be a graph. Then* G *is a median graph iff* G *is a connected induced subgraph of an* n*-cube such that with any three vertices of* G *their graph median in the* n*-cube also is a vertex of* G.

PROOF. The only if part follows from proposition 10 and (C).

The if part follows as soon as we have proved that the distance in G between two vertices equals their distance in the n-cube. Let d be the distance function of G and e that of the n-cube. Assume that there are vertices u,v of G with d(u,v) ≠ e(u,v) and let k:= d(u,v) be as small as possible.

Note that $k > 2$.

Let w be a vertex of G with $d(u,w) = 2$ and $d(w,v) = k - 2$. Then $e(u,w) = 2$ and $e(w,v) = k - 2$. Let z be the graph median of u,v and w in the n-cube. Thus z is a vertex of G.

If $z = w$ then $e(u,v) = e(u,w) + e(w,v) = 2 + k - 2$. So $z \neq w$. But then, since $e(u,w) = 2 = d(u,w)$, z is a common neighbour of u and w. Now $e(z,v) = e(w,v) - e(w,z) = k - 2 - 1 = k - 3$. Thus $d(u,v) \leq d(u,z)+d(z,v) = 1 + e(z,v) = k - 2 < k$, which is a contradiction. □

Let G be a median graph with vertex-set V. Embed G in an n-cube K with n as small as possible. Since G is connected G has at least one edge of each colour from the cutset colouring of K.

The cutset colouring of K induces an edge colouring of G. According to proposition 10 with any two vertices u and v of G a shortest u,v-path of u lies entirely in G. So the induced edge colouring of G in fact is a cutset colouring. Any cutset from this colouring induces a copair of V: after deleting the cutset from G the graph breaks up into two components, the vertex-sets of which form the complementary subsets of the copair. In this way the cutset colouring of G induces a copair hypergraph $(V,\mathcal{E}_G)$. Since G is an induced subgraph of K it follows that $\mathcal{E}_G$ consists of I_G-convex subsets of V. Besides it follows that $\mathcal{E}_G$ separates vertices. And thus according to lemma 3 $(V,\mathcal{E}_G)$ is a maximal Helly copair hypergraph. Furthermore uv is an edge in G iff $u \neq v$ and $\cap \{B \in \mathcal{E}_G | u,v \in B\} = \{u,v\}$. That is $G_{I_{\mathcal{E}_G}} = G$.

Starting with a maximal Helly copair hypergraph $(V,\mathcal{E})$ then $G_{\mathcal{E}} = G_{I_{\mathcal{E}}}$ is a median graph with vertex-set V. Moreover $\mathcal{E}$ consists of $I_{G_{\mathcal{E}}}$-convex subsets of V. But also $\mathcal{E}_{G_{\mathcal{E}}}$ is a Helly copair hypergraph consisting of $I_{G_{\mathcal{E}}}$-convex subsets of V. Since both $\mathcal{E}$ and $\mathcal{E}_{G_{\mathcal{E}}}$ are maximal, we have that $\mathcal{E} = \mathcal{E}_{G_{\mathcal{E}}}$.

The preceding observations imply that a median graph G, with vertex-set V, admits only one cutset colouring which induces a maximal Helly copair hypergraph. Let us call the copairs of V induced by this cutset colouring of G the *canonical copairs* of G. (In fact it can be proved that up to the labelling of the colours a median graph admits exactly one cutset colouring of its edges, cf. [6].)

Recapitulating we have proved:

PROPOSITION 12. *The hypergraph* (V,E) *is a maximal Helly copair hypergraph iff* E *consists of the canonical copairs of a median graph with vertex-set* V.

3.4 CONCLUDING REMARKS. Let G be a connected graph with n vertices, which admits a cutset colouring. Since each cutset contains edges of a spanning tree, the number of colours in the cutset colouring is at most n - 1.

LEMMA 13. *Let* G *be a connected graph with* n *vertices admitting a cutset colouring. Then the number of colours in the cutset colouring is* n - 1 *iff* G *is a tree.*

PROOF. The if part of this lemma is trivial. To prove the only if part let T be a spanning tree of G. Then T has n - 1 edges, so the edges of T all have different colours. Thereby every edge of T determines exactly one cutset of the colouring. Assume that there is an edge joining u and v in G, which is not in T. The u,v-path in T must contain at least two edges, say $f_1, f_2, \ldots$ But then the edge uv is in the cutset determined by f_1 and in the cutset determined by f_2, which is a contradiction. □

The term *maximum* will be used in the sense of: with a maximal number of edges.

PROPOSITION 14. *The hypergraph* (V,E) *is a maximum Helly copair hypergraph iff* E *consists of the canonical copairs of a tree with vertex-set* V.

COROLLARY 15. *Let* (V,E) *be a Helly copair hypergraph. Then*

$$|E| \leq 2(|V| - 1).$$

COROLLARY 16. (E.C. MILNER, cf. [4]). *Let* (V,E) *be a Helly hypergraph. Then*

$$|E| \leq 2^{|V| - 1} + |V| - 1.$$

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