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MEDIAN GRAPHS AND HELLY HYPERGRAPHS

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Median graphs and Helly hypergraphs*)

by

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ABSTRACT

One-to-one correspondences are established between the following combinatorial structures: (i) median interval structures (or median segments, introduced by SHOLANDER); (ii) maximal Helly hypergraphs such that with each edge also its complement is in the hypergraph; and (iii) median graphs (connected graphs such that for any three vertices u, v, w there is exactly one vertex x such that \( d(u,v) = d(u,x) + d(x,v) \), \( d(v,w) = d(v,x) + d(x,w) \) and \( d(w,u) = d(w,x) + d(x,u) \), where d is the distance function of the graph).

KEY WORDS & PHRASES: median, interval structure, Helly hypergraph, copair, median graph.

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0. INTRODUCTION

In this paper one-to-one correspondences will be established between three at first sight fairly distinct concepts. These concepts are:
(i) median interval structures introduced by M. SHOLANDER [7], [8] under the name of median segments (cf. 1.1);
(ii) maximal Helly copair hypergraphs (i.e. simple Helly hypergraphs, the edge-set of which contains with each edge its complement, and which are maximal with respect to this property; see 1.2); and
(iii) median graphs, introduced in section 1.3.

The one-to-one correspondences are established in section 2.

In section 3 is elaborated how to construct a maximal Helly copair hypergraph from a median graph, using results of SHOLANDER [9].


1. DEFINITIONS AND PRELIMINARIES

Throughout this paper V denotes a fixed finite set.

1.1. INTERVAL STRUCTURES. A function $I: V \times V \rightarrow P(V)$ is called an interval structure on V if

\begin{align*}
(I1) & \quad x, y \in I(u, v) \iff I(x, y) \subseteq I(u, v) \quad (x, y, u, v \in V), \\
(I2) & \quad I(u, v) \cap I(v, w) \cap I(w, u) \neq \emptyset \quad (u, v, w \in V).
\end{align*}

Each set $I(u, v)$ is called an interval. A subset $U$ of $V$ is I-convex if for all $u, v \in V$ the interval $I(u, v)$ is contained in $U$. The notion of interval structure was introduced in [3]. Examples of interval structures on $V$ can be obtained from trees with vertex-set $V$ (then take $I(u, v) = \{w \in V \mid w \text{ lies on the shortest } u, v\text{-path}\}$), and from lattices $(V, \preceq)$ (in this case $I(u, v) = \{w \in V \mid u \preceq v \preceq w \preceq u \vee v\}$).

If $I$ satisfies condition (I1) and the following condition

\begin{align*}
(I2') & \quad |I(u, v) \cap I(v, w) \cap I(w, u)| = 1 \quad (u, v, w \in V),
\end{align*}


then I is called a median interval structure on V. Interval structures obtained from trees as indicated above are median interval structures. An interval structure obtained from a lattice is a median interval structure iff the lattice is distributive (cf. [2]). SHOLANDER [8] has given the following characterization of median interval structures (he used the term median segments):

**THEOREM 1.** (SHOLANDER [8]) A function $I : V \times V \to P(V)$ is a median interval structure on $V$ iff

$$
\begin{align*}
\text{if } w \in I(u,v) \text{ then } I(u,w) &\subseteq I(u,v) \cap I(v,u) \quad (u,v \in V), \\
|I(u,v) \cap I(v,w) \cap I(w,u)| &\geq 1 \quad (u,v,w \in V), \\
I(v,v) &\subseteq \{v\} \quad (v \in V).
\end{align*}
$$

1.2. HYPERGRAPHS. In this paper a hypergraph $H = (V,E)$ consists of a vertex-set $V$ and a family $E \subseteq P(V)$ of nonvoid subsets of $V$, the members of which are called edges. Occasionally we will write $E$ instead of $(V,E)$.

A hypergraph is a Helly hypergraph if it satisfies the Helly property, i.e. every subfamily of $E$, any two members of which meet, has a non-empty intersection. For vertices $u$ and $v$ of the hypergraph $(V,E)$ define

$$
I(u,v) = \cap \{B \in E \mid u,v \in B\}.
$$

A theorem of P.C. GILMORE (see [5], or [1] p. 396) can be formulated as follows:

**THEOREM 2.** (GILMORE) A hypergraph $(V,E)$ satisfies the Helly property iff $I_E$ is an interval structure on $V$.

As a consequence of GILMORE's theorem we have: Let $I$ be an interval structure on $V$. Any family $E$ of nonvoid I-convex subsets of $V$ satisfies the Helly property.

A hypergraph $(V,E)$ with the property that $V \setminus B \in E$ for all $B \in E$ will be called a copair hypergraph. We call the set $\{B, V \setminus B\}$ a copair of $V$ and $\{\emptyset, V\}$ the trivial copair. A Helly copair hypergraph of course is a copair hypergraph, which satisfies the Helly property. Finally a maximal Helly copair hypergraph $(V,E)$ is a Helly copair hypergraph such that: if $\{A, V \setminus A\}$
is a non-trivial copair and $E \cup \{A, V\setminus A\}$ satisfies the Helly property then $A \in E$.

A hypergraph $(V,E)$ is said to separate vertices if for any two distinct vertices $u, v \in V$ there exists an edge $A \in E$ such that $u \in A$ and $v \notin A$.

**Lemma 3.** Let $(V,E)$ be a Helly copair hypergraph. Then $(V,E)$ is maximal iff $(V,E)$ separates vertices.

**Proof.** Note that $(V,E)$ separates vertices iff $I_E(v,v) = \{v\}$ for all $v \in V$.

Assume that $E$ does not separate vertices. That is there exists a vertex $v \in V$ such that $I_E(v,v)$ contains besides $v$ another vertex. Using GILMORE's theorem it can be verified that in this case $E \cup \{(v), V\setminus\{v\}\}$ satisfies the Helly property. Therefore $E$ is not maximal.

To prove sufficiency of vertex separation let $(A,V\setminus A)$ be a non-trivial copair of $V$ not in $E$. Take a vertex $u \in A$ and a vertex $v \in V\setminus A$ such that $|I_E(u,v)|$ is as small as possible. We assert that $I_E(u,v) \cap A = \{u\}$ and $I_E(u,v) \setminus A = \{v\}$.

For suppose $I_E(u,v) \cap A \neq \{u\}$ and let $w \in I_E(u,v) \cap A$ with $w \neq u$. Since $E$ separates vertices, there exists an edge $C \in E$ such that $w \in C$ and $u \notin C$. Then we have that $v \in C$. So $u \notin I_E(w,v) \subset I_E(u,v)$, contradicting the minimality of $I_E(u,v)$. In the same way we prove $I_E(u,v) \setminus A = \{v\}$. Hence $I_E(u,v) = \{u,v\}$.

Let $B \in E$ be an edge such that $v \in B$ and $u \notin B$. Then $A \cap B \neq \emptyset$ or $(V\setminus A) \cap (V\setminus B) \neq \emptyset$, since $A \notin \{B, V\setminus B\} \subseteq E$; say $A \cap B \neq \emptyset$. Now the set of edges, which contain both $u$ and $v$, together with $A$ and $B$ forms a family of subsets of $V$, any two members of which meet. The intersection of this family equals

$$I_E(u,v) \cap A \cap B = \{u,v\} \cap A \cap B,$$

which clearly is empty. Thus $E \cup \{A, V\setminus A\}$ does not satisfy the Helly property.

**Corollary 4.** Let $(V,E)$ be a maximal Helly copair hypergraph. Then

$$|E| \geq 2^{\lceil 2\log |V| \rceil}.$$
1.3 MEDIAN GRAPHS. Let $G$ be a simple loopless graph with vertex-set $V$ and distance function $d$. $G$ will be called a median graph if it is connected and satisfies the graph median property, i.e. for any $u,v,w \in V$ there exists precisely one vertex $x \in V$, called the graph median of $u,v$ and $w$, such that

$$
\begin{align*}
&d(u,x) + d(x,v) = d(u,v) \\
&d(v,x) + d(x,w) = d(v,w) \\
&d(w,x) + d(x,u) = d(w,u).
\end{align*}
$$

Note that all trees and the n-cubes are median graphs. It is easy to see that each median graph is bipartite.

2. THE THEOREM

THEOREM 5. There exists a one-to-one correspondence between the median interval structures on $V$, the maximal Helly copair hypergraphs with vertex-set $V$ and the median graphs with vertex-set $V$. The one-to-one correspondences are indicated in the following diagram, which commutes in all directions.
The proof of the theorem amounts to the following propositions (the direct correspondence between median graphs and maximal Helly copair hypergraphs will be explained in section 3).

For vertices $u$ and $v$ of the graph $G = (V,E)$ define

$$I_G(u,v) = \{ w \in V \mid w \text{ lies on a shortest } u,v\text{-path in } G \}.$$ 

**Proposition 6.** Let $G = (V,E)$ be a median graph. Then $I_G$ is a median interval structure on $V$.

**Proof.** $I_G$ satisfies the conditions mentioned in theorem 1. 

**Proposition 7.** Let $I$ be a median interval structure on $V$. Define the graph $G_I$ with vertex-set $V$ by

$$uv \in E(G_I) \text{ iff } u \neq v \text{ and } I(u,v) = \{u,v\} \quad (u,v \in V).$$

Then $G_I$ is a median graph.

**Proof.** We will prove that $G_I$ is connected and that $I_{G_I} = I$. Then clearly $G_I$ is a median graph.

First observe that for $u,v,w \in V$ we have

$$w \in I(u,v) \text{ iff } I(u,w) \cap I(w,v) = \{w\}.$$ 

Thus for $w \in I(u,v) \setminus \{u,v\}$ holds $u \notin I(w,v) \subset I(u,v)$ and $v \notin I(u,w) \subset I(u,v)$. Using this it is easily verified by induction on $|I(u,v)|$ that $I(u,v)$ induces a connected subgraph of $G_I$ for all $u,v \in V$. Hence $G_I$ is connected.

To prove that $I(u,v) = I_{G_I}(u,v)$ for all $u,v \in V$ we use induction on $d(u,v)$. Clearly $I(u,v) = I_{G_I}(u,v)$ for all $u,v \in V$ with $d(u,v) = 1$. So take vertices $u,v \in V$ with $d(u,v) > 1$.

Let $w \in I_{G_I}(u,v) \setminus \{u,v\}$. Then $d(u,w) < d(u,v)$ and $d(w,v) < d(u,v)$, so $I_{G_I}(u,w) = I(u,w)$ and $I_{G_I}(w,v) = I(w,v)$. Since clearly $I_{G_I}(u,w) \cap I_{G_I}(w,v) = \{w\}$, we have $w \in I(u,v)$ and thus $I_{G_I}(u,v) \subset I(u,v)$.

Assume $I(u,v) \setminus I_{G_I}(u,v) \neq \emptyset$.

For any vertex $w \in I(u,v) \setminus I_{G_I}(u,v)$ we must have $I(u,w) \cap I_{G_I}(u,v) = \{u\}$,
and similarly $I(w,v) \cap I_{G_I}(u,v) = \{v\}$. For if $w' \in I(u,w) \cap I_{G_I}(u,v)$, with $w' \neq u$, then $w \in I(w',v)$ and by the induction hypothesis $I(w',v) = I_{G_I}(w',v) \subset I_{G_I}(u,v)$. Hence $w \in I_{G_I}(u,v)$, contradicting the choice of $w$.

Since $I(u,v)$ induces a connected subgraph of $G_I$, there exists a path $P$ from $u$ to $v$, all the internal vertices of which lie in $I(u,v)\setminus I_{G_I}(u,v)$. Clearly the length of $P$ exceeds $d(u,v)$ so $P$ has at least two distinct internal vertices, say $x$ and $y$.

Since $d(u,v) \geq 2$, there exists a vertex $z \in I_{G_I}(u,v)\setminus\{u,v\}$. By the induction hypothesis we have $I(u,z) = I_{G_I}(u,z)$ and $I(z,v) = I_{G_I}(z,v)$. Now

$$u \in I(u,z) \cap I(u,x) = I_{G_I}(u,z) \cap I(u,x) \subset I_{G_I}(u,v) \cap I(u,x)=\{u\}.$$

So $u \in I(z,x)$. Similarly $v \in I(z,x)$ and thus $I(u,v) \subset I(z,x) \subset I(u,v)$.

In the same way it follows that $I(u,v) = I(z,y)$. But then

$$x,y \in I(x,y) = I(z,x) \cap I(x,y) \cap I(y,z),$$

contradicting the fact that $I$ is a median interval structure. Conclusion: $I(u,v) = I_{G_I}(u,v)$.

In the proof of the preceding proposition we have seen that for a median interval structure $I$ holds: $I_{G_I} = I$. Furthermore from propositions 6 and 7 follows immediately that, when $G$ is a median graph, we have $G_{I_G} = G$.

**Proposition 8.** Let $(V,E)$ be a maximal Helly bipartite hypergraph. Then $I_E$ is a median interval structure on $V$.

**Proof.** Assume that there exist vertices $u,v,w \in V$ such that $x,y \in I_E(u,v) \cap I_E(v,w) \cap I_E(w,u)$ for vertices $x,y \in V$, with $x \neq y$. According to lemma 3 there is an edge $B \in E$ such that $x \in B$ and $y \notin B$. Then one of the edges $B$ and $V \setminus B$, say $B$, must contain at least two of the three vertices $u,v$ and $w$, say $u$ and $v$. But then $y \notin I_E(u,v)$. Contradiction.

**Proposition 9.** Let $I$ be a median interval structure on $V$ and let

$$E_I = \{B \subset V| \emptyset \neq B \neq V, B \text{ and } V \setminus B \text{ are } I\text{-convex}\}.$$
Then \((V,E_I^1)\) is a maximal Helly copair hypergraph.

**PROOF.** Clearly \((V,E_I^1)\) is a Helly copair hypergraph. By lemma 3 it suffices to show that \(E_I^1\) separates vertices. So suppose that for vertices \(u,v \in V\), with \(u \neq v\), there is no edge \(B\) such that \(u \in B\) and \(v \notin B\). Assume further-more that \(u\) and \(v\) are such that \(|I(u,v)|\) is as small as possible.

We first prove that \(I(u,v) = \{u,v\}\). Suppose \(w \in I(u,v) \setminus \{u,v\}\). Since \(|I(u,w)| < |I(u,v)|\), there exists an edge \(A\) such that \(u \in A\) and \(w \notin A\). It follows that \(v \in A\) (\(u\) and \(v\) cannot be separated). So \(w \in I(u,v) \subseteq A\), for \(A\) is \(I\)-convex, contradicting \(w \notin A\). Therefore \(I(u,v) = \{u,v\}\).

Now let \(B = \{z \in V | \emptyset \notin I(u,z)\}\).

Then \(V \setminus B = \{z \in V | \emptyset \notin I(z,v)\}\), since \(I(u,z) \cap I(z,v) \cap \{u,v\}\) is a singleton. We assert that \(B\) and \(V \setminus B\) are \(I\)-convex, that is \(B \in E_I^1\). Since \(u \in B\) and \(v \notin B\) this contradicts our assumption that \(E_I^1\) does not separate vertices.

We only prove that \(B\) is \(I\)-convex (the \(I\)-convexity of \(V \setminus B\) can be treated similarly).

Note that for each \(z \in B\) we have \(I(u,z) \subseteq B\), since \(v \notin I(u,z)\). Let \(x,y \in B\) and suppose \(I(x,y) \notin B\). Take \(w \in I(x,y) \setminus B\). Since \(I(u,x) \subseteq I(v,x)\) and \(I(u,y) \subseteq I(v,y)\) we have that

\[
\{z\} = I(u,x) \cap I(x,y) \cap I(y,u) = I(v,x) \cap I(x,y) \cap I(y,v)
\]

for some \(z \in B\). Now also

\[
\{z\} \subseteq I(z,w) \cap I(z,v) \subseteq I(x,y) \cap I(x,v) \cap I(y,v) = \{z\},
\]

since \(z,w \in I(x,y)\) and \(z \in I(u,x) \cap I(u,y) \subseteq I(v,x) \cap I(v,y)\). This implies \(z \in I(w,v)\) according to the observation made at the beginning of the proof of proposition 7. So \(I(z,v) \subseteq I(w,v)\). But, since \(w \notin B\), \(u \notin I(w,v)\) and thus \(u \notin I(z,v)\), that is \(z \in V \setminus B\), contradicting the fact that \(z \in B\). \(\square\)

From propositions 8 and 9 we deduce: let \(I\) be a median interval structure on \(V\), then \(I_{E_I}^1 = I\); and let \((V,E)\) be a maximal Helly copair hypergraph, then \(E_{I_{E_I}} = E\).
3. MEDIAN GRAPHS AND HELLY HYPERGRAPHS

In this section the direct correspondence between median graphs and maximal Helly copair hypergraphs with vertex-set \( V \), mentioned in the theorem, is further elaborated.

3.1. MEDIAN SEMILATTICES. Let \((V, \leq)\) be a partially ordered set (poset). \( v \) is said to \textit{cover} \( u \) \((u, v \in V)\), if \( u \leq v \) and there is no \( w \in V \) such that \( u < w < v \). A \textit{semilattice} \((V, \leq)\) is a poset, in which any two elements \( u, v \) have a greatest lower bound \( u \wedge v \). For \( u, v \in V \) set \([u, v] = \{ w \in V | u \leq w \leq v \}\). The semilattice \((V, \leq)\) is called \textit{distributive} if \([u, v], \leq)\) is a distributive lattice for all \( u, v \in V \). The semilattice is said to satisfy the \textit{coronation property} if for any three elements \( u, v, w \in V \), such that the three least upper bounds \( u \vee v, v \vee w, w \vee v \) exist, there exists a least upper bound \( u \vee v \vee w \).

A \textit{median semilattice} is a distributive semilattice, which satisfies the coronation property. This concept was introduced by SHOLANDER [9].

On a median semilattice \((V, \leq)\) the ternary operation \((u, v, w) = (u \wedge v) \vee (u \wedge w) \vee (w \wedge u) \in V\) can be defined, called the \textit{median} of \( u, v \) and \( w \) (SHOLANDER [9] also characterized medians).

We review some results of SHOLANDER [9] reformulating them in our terminology:

(A) \textit{Each median semilattice} \((V, \leq)\) \textit{yields a median interval structure} \( I_{\leq} \) \textit{on} \( V \), where

\[
I_{\leq}(u, v) = \{ w | w \text{ is the median of } u, v, w \} \quad (u, v \in V).
\]

(B) \textit{Let} \( I \) \textit{be a median interval structure on} \( V \) \textit{and} \( u \in V \). \textit{Define an ordering} \( \leq_{I, u} \) \textit{on} \( V \) \textit{by}

\[
v \leq_{I, u} w \iff v \in I(u, w) \quad (v, w \in V).
\]

Then \((V, \leq_{I, u})\) \textit{is a median semilattice}. Furthermore the correspondences given in (A) and (B) commute.
Let \((V, \preceq)\) be a median semilattice. Then \((V, \preceq)\) can be embedded in a Boolean algebra by an order preserving mapping, which also preserves the covering relation in \((V, \preceq)\).

3.2 CUTSET COLOURINGS. A cutset colouring of a connected graph is a colouring of the edges in such a way that the edges of any colour form a matching as well as a cutset (i.e. a minimal disconnecting edge-set). If we want to establish a cutset colouring of a graph we are forced to colour non-adjacent edges in each circuit of length four with the same colour. So the \(n\)-cube admits a cutset colouring with \(n\) colours, which is uniquely determined up to the labelling of the colours. Deleting the edges with a given colour from the \(n\)-cube breaks the graph up into two components, which both are \((n-1)\)-cubes.

Note that not all connected graphs admit a cutset colouring. Necessary conditions for the existence of a cutset colouring of the edges of a connected graph are for instance that the graph is simple, loopless and bipartite and that it does not contain \(K_{2,3}\) as a subgraph.

3.3 MEDIAN GRAPHS AND MAXIMAL HELLY COPAIR HYPERGRAPHS. The diagraph of a poset \((V, \preceq)\) is the graph with vertex-set \(V\), in which two vertices are joined by an edge iff one of the two covers the other in the poset. Clearly, the diagraph of the Boolean algebra on \(2^n\) elements is the \(n\)-cube. As a consequence of (A) and (B) and propositions 6 and 7 we have

**PROPOSITION 10.** Let \(G\) be a graph. Then \(G\) is a median graph iff \(G\) is the diagraph of a median semilattice.

**PROPOSITION 11.** Let \(G\) be a graph. Then \(G\) is a median graph iff \(G\) is a connected induced subgraph of an \(n\)-cube such that with any three vertices of \(G\) their graph median in the \(n\)-cube also is a vertex of \(G\).

**PROOF.** The only if part follows from proposition 10 and (C).

The if part follows as soon as we have proved that the distance in \(G\) between two vertices equals their distance in the \(n\)-cube. Let \(d\) be the distance function of \(G\) and \(e\) that of the \(n\)-cube. Assume that there are vertices \(u, v\) of \(G\) with \(d(u, v) \neq e(u, v)\) and let \(k := d(u, v)\) be as small as possible.
Note that \( k > 2 \).

Let \( w \) be a vertex of \( G \) with \( d(u,w) = 2 \) and \( d(w,v) = k - 2 \). Let \( z \) be the graph median of \( u,v \) and \( w \) in the \( n \)-cube. Thus \( z \) is a vertex of \( G \).

If \( z = w \) then \( e(u,v) = e(u,w) + e(w,v) = 2 + k - 2 \). So \( z \neq w \). But then, since \( e(u,w) = 2 = d(u,w) \), \( z \) is a common neighbour of \( u \) and \( w \). Now \( e(z,v) = e(w,v) - e(w,z) = k - 2 - 1 = k - 3 \). Thus \( d(u,v) \leq d(u,z) + d(z,v) = 1 + e(z,v) = k - 2 < k \), which is a contradiction. \( \square \)

Let \( G \) be a median graph with vertex-set \( V \). Embed \( G \) in an \( n \)-cube \( K \) with \( n \) as small as possible. Since \( G \) is connected \( G \) has at least one edge of each colour from the cutset colouring of \( K \).

The cutset colouring of \( K \) induces an edge colouring of \( G \). According to proposition 10 with any two vertices \( u \) and \( v \) of \( G \) a shortest \( u,v \)-path of \( u \) lies entirely in \( G \). So the induced edge colouring of \( G \) in fact is a cutset colouring. Any cutset from this colouring induces a copair of \( V \): after deleting the cutset from \( G \) the graph breaks up into two components, the vertex-sets of which form the complementary subsets of the copair. In this way the cutset colouring of \( G \) induces a copair hypergraph \((V,E_G)\). Since \( G \) is an induced subgraph of \( K \) it follows that \( E_G \) consists of \( I_G \)-convex subsets of \( V \). Besides it follows that \( E_G \) separates vertices. And thus according to lemma 3 \((V,E_G)\) is a maximal Helly copair hypergraph. Furthermore \( uv \) is an edge in \( G \) iff \( u \neq v \) and \( \cap \{B \in E_G | u,v \in B \} = \{u,v\} \). That is \( G_{E_G} = G \).

Starting with a maximal Helly copair hypergraph \((V,E)\) then \( G_E = G_{E_E} \) is a median graph with vertex-set \( V \). Moreover \( E \) consists of \( I_E \)-convex subsets of \( V \). But also \( E_{G_E} \) is a Helly copair hypergraph consisting of \( I_{G_E} \)-convex subsets of \( V \). Since both \( E \) and \( E_{G_E} \) are maximal, we have that \( E = E_{G_E} \).

The preceding observations imply that a median graph \( G \), with vertex-set \( V \), admits only one cutset colouring which induces a maximal Helly copair hypergraph. Let us call the copairs of \( V \) induced by this cutset colouring of \( G \) the canonical copairs of \( G \). (In fact it can be proved that up to the labelling of the colours a median graph admits exactly one cutset colouring of its edges, cf. [6].)

Recapitulating we have proved:
PROPOSITION 12. The hypergraph $(V, E)$ is a maximal Helly copair hypergraph iff $E$ consists of the canonical copairs of a median graph with vertex-set $V$.

3.4 CONCLUDING REMARKS. Let $G$ be a connected graph with $n$ vertices, which admits a cutset colouring. Since each cutset contains edges of a spanning tree, the number of colours in the cutset colouring is at most $n - 1$.

LEMMA 13. Let $G$ be a connected graph with $n$ vertices admitting a cutset colouring. Then the number of colours in the cutset colouring is $n - 1$ iff $G$ is a tree.

PROOF. The if part of this lemma is trivial. To prove the only if part let $T$ be a spanning tree of $G$. Then $T$ has $n - 1$ edges, so the edges of $T$ all have different colours. Thereby every edge of $T$ determines exactly one cutset of the colouring. Assume that there is an edge joining $u$ and $v$ in $G$, which is not in $T$. The $u,v$-path in $T$ must contain at least two edges, say $f_1, f_2, \ldots$. But then the edge $uv$ is in the cutset determined by $f_1$ and in the cutset determined by $f_2$, which is a contradiction.  

The term maximum will be used in the sense of: with a maximal number of edges.

PROPOSITION 14. The hypergraph $(V, E)$ is a maximum Helly copair hypergraph iff $E$ consists of the canonical copairs of a tree with vertex-set $V$.

COROLLARY 15. Let $(V, E)$ be a Helly copair hypergraph. Then

$$|E| \leq 2(|V| - 1).$$

COROLLARY 16. (E.C. Milner, cf. [4]). Let $(V, E)$ be a Helly hypergraph. Then

$$|E| \leq 2^{\frac{|V| - 1}{2}} + |V| - 1.$$
REFERENCES


