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A NOTE ON NONVANISHING FOURIER TRANSFORMS

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A note on nonvanishing Fourier transforms

by

J. van de Lune

ABSTRACT

It is shown that if $\phi \in L^1(\mathbb{R}^+)$, ϕ is monotonically non-increasing on $\mathbb{R}^+$ whereas there exists some interval on which ϕ is strictly decreasing then the Fourier transform $\hat{\phi}$ of ϕ has no real zeros.

Moreover it is shown that the condition "there exists some interval on which ϕ is strictly decreasing" is not a necessary condition in order to have $\hat{\phi}(t) \neq 0$, $\forall t \in \mathbb{R}$.

KEY WORDS & PHRASES: *Fourier transforms, Tauberian theorems.*

1. Recently L.A. RUBEL [1] proposed the following problem: Prove that if $\phi \in L^1 := L^1(\mathbb{R})$, $\phi \geq 0$, $\phi(x) = 0$ for all x outside an interval [a,b] and ϕ is strictly decreasing on [a,b] then the span of all translates of ϕ is dense in L^1.

By Wiener's general Tauberian theorem for L^1 (c.f. [2; p.9]) the span of all translates of some $\phi \in L^1$ is dense in L^1 if and only if the Fourier transform $\hat{\phi}$ of ϕ has no real zeros.

From this it is clear that RUBEL's result is an immediate consequence of the following more general

THEOREM 1. *If $\phi \in L^1$, $\phi(x) = 0$ for all $x < a$ and ϕ is monotonically non-increasing on $[a,\infty)$ whereas there exists some interval $[\alpha,\beta]$ with $\alpha < \beta$, on which ϕ is strictly decreasing, then the Fourier transform $\hat{\phi}$ of ϕ has no real zeros.*

PROOF. Without loss of generality we may assume that a = 0. From the conditions in the theorem it is clear that

(1) $\phi(x) \geq 0$ for all $x \geq 0$,

(2) $\phi(x) < \infty$ for all $x > 0$,

(note that $\phi(+0)$ is possibly not finite)

(3) $\int_0^\infty \phi(x)\,dx > 0$,

(4) $\lim_{x\to\infty} \phi(x) = 0$,

(5) $0 \leq \alpha < \beta$.

We proceed by contradiction. If the theorem is false then there exists a $t_0 \in \mathbb{R}$ such that

(6) $$\hat{\phi}(t_0) = \int_{\mathbb{R}} e^{it_0x}\phi(x)\,dx = \int_0^\infty e^{it_0x}\phi(x)\,dx = 0.$$

From (3) it is clear that $t_0 \neq 0$ and since ϕ is real we have

(7) $$\overline{\hat{\phi}(t_0)} = \hat{\phi}(-t_0)$$

so that we may assume that $t_0 > 0$.
Now choose any $\theta > 0$ and observe that

(8) $$0 = \hat{\phi}(t_0) = \left\{ \int_0^\theta + \int_\theta^\infty \right\} e^{it_0 x} \phi(x)\, dx =$$

$$= \int_0^\theta e^{it_0 x} \phi(x)\, dx + \frac{1}{it_0} \int_\theta^\infty \phi(x)\, d\, e^{it_0 x} =$$

$$= \int_0^\theta e^{it_0 x} \phi(x)\, dx + \frac{1}{it_0} \left\{ \phi(x)\, e^{it_0 x} \Big|_{x=\theta}^{x=\infty} - \int_\theta^\infty e^{it_0 x}\, d\phi(x) \right\} =$$

(since $\lim_{x\to\infty} \phi(x) = 0$)

$$= \int_0^\theta e^{it_0 x} \phi(x)\, dx - \frac{1}{it_0} \phi(\theta)\, e^{i\theta t_0} - \frac{1}{it_0} \int_\theta^\infty e^{it_0 x}\, d\phi(x).$$

It follows that

(9) $$-it_0\, e^{-i\theta t_0} \int_0^\theta e^{it_0 x} \phi(x)\, dx + \phi(\theta) = - \int_\theta^\infty e^{it_0 (x-\theta)}\, d\phi(x).$$

Defining

(10) $$h(\theta) = -it_0\, e^{-i\theta t_0} \int_0^\theta e^{it_0 x} \phi(x)\, dx, \quad (\theta > 0)$$

and

(11) $$h^*(\theta) = \mathrm{Re}\, h(\theta), \quad (\theta > 0)$$

we have

$$(12) \qquad h^*(\theta) + \phi(\theta) = \int_{\theta}^{\infty} \cos t_0(x-\theta)\, d\Psi(x)$$

where $\Psi = -\theta$.

Since $t_0 > 0$ we thus obtain by the substitution $t_0(x-\theta) = u$

$$(13) \qquad h^*(\theta) + \phi(\theta) = \int_{0}^{\infty} \cos u\, d\Psi(\theta + \frac{u}{t_0}) .$$

Now observe that the (non-degenerate) interval

$$(14) \qquad [\alpha t_0, \beta t_0] \subset [0,\infty)$$

contains a subinterval $[\gamma,\delta]$, say, such that for some $\varepsilon > 0$

$$(15) \qquad \cos u \le 1 - \varepsilon \quad \text{for all} \quad u \in [\gamma,\delta] .$$

Since $\Psi = -\phi$ is non-decreasing on $[\theta,\infty)$ it follows that

$$(16) \qquad h^*(\theta) + \phi(\theta) = \{ \int_0^{\gamma} + \int_{\gamma}^{\delta} + \int_{\delta}^{\infty} \} \cos u\, d\, \Psi(\theta + \frac{u}{t_o}) \le$$

$$\le \int_0^{\gamma} d\, \Psi(\theta + \frac{u}{t_o}) + (1 - \varepsilon) \int_{\gamma}^{\delta} d\, \Psi(\theta + \frac{u}{t_0}) + \int_{\delta}^{\infty} d\, \Psi(\theta + \frac{u}{t_0}) =$$

$$= \int_0^{\infty} d\, \Psi(\theta + \frac{u}{t_0}) - \varepsilon \int_{\gamma}^{\delta} d\, \Psi(\theta + \frac{u}{t_0}) =$$

$$= -\Psi(\theta) - \varepsilon \int_{\gamma}^{\delta} d\, \Psi(\theta + \frac{u}{t_0})$$

and consequently

$$(17) \qquad h^*(\theta) \le \varepsilon\{\Psi(\theta + \frac{\gamma}{t_0}) - \Psi(\theta + \frac{\delta}{t_0})\} .$$

Since

$$(18) \qquad |h(\theta)| \le t_0 \int_0^\theta \phi(x)\, dx$$

we have

$$(19) \qquad \lim_{\theta \downarrow 0} h(\theta) = 0$$

so that certainly

$$(20) \qquad \lim_{\theta \downarrow 0} h^*(\theta) = 0.$$

Hence, taking limits for $\theta \downarrow 0$ we obtain from (17) that

$$(21) \qquad 0 \le \varepsilon\{\Psi(\frac{\gamma}{t_0} + 0) - \Psi(\frac{\delta}{t_0} + 0)\}$$

and, since $\varepsilon > 0$, it follows that

$$(22) \qquad \phi(\frac{\gamma}{t_0} + 0) \le \phi(\frac{\delta}{t_0} + 0) \quad .$$

However, observing that

$$(23) \qquad \alpha \le \frac{\gamma}{t_0} < \frac{\delta}{t_0} \le \beta$$

and that ϕ is strictly decreasing on $[\alpha,\beta]$ we must have

$$(24) \qquad \phi(\frac{\gamma}{t_0} + 0) > \phi(\frac{\delta}{t_0} + 0) \quad .$$

This contradicts (22) and completes the proof.

2. In theorem 1, the condition "there exists some (non-degenerate) interval $[\alpha,\beta]$ on which ϕ is strictly decreasing" is not a necessary condition as may be seen from the following

THEOREM 2. *Let* $\{a_n\}_{n=-\infty}^{\infty}$ *be a strictly decreasing sequence such that* $\lim_{n\to\infty} a_n = 0$. *In addition let* $\{w_n\}_{n=-\infty}^{\infty}$ *be a monotonically non-decreasing sequence such that* $\lim_{n\to\infty} w_n = \infty$. *Define* $\phi : \mathbb{R} \to \mathbb{R}$ *by*

$$\phi(x) = \begin{cases} 0 & \text{if } x \le 0 \\ w_n & \text{if } a_{n+1} < x \le a_n \end{cases} \tag{2.1}$$

and let $\phi \in L^1$, i.e.

$$\sum_{n=-\infty}^{\infty} w_n(a_n - a_{n+1}) < \infty \quad . \tag{2.2}$$

Then the Fourier transform $\hat{\phi}$ *of* ϕ *has no real zeros.*

PROOF. Suppose the theorem is false. Then there exists a $t_0 \in \mathbb{R}$ such that $t_0 > 0$ and

$$0 = \hat{\phi}(t_0) = \int_0^{\infty} e^{it_0x}\phi(x)\,dx = \sum_{n=-\infty}^{\infty} w_n \int_{a_{n+1}}^{a_n} e^{it_0x}\,dx = \tag{2.3}$$

$$= \frac{1}{it_0} \sum_{n=-\infty}^{\infty} w_n(e^{it_0a_n} - e^{it_0a_{n+1}})$$

so that

$$\sum_{n=-\infty}^{\infty} w_n\{\cos(t_0a_n) - \cos(t_0a_{n+1})\} = 0 \tag{2.4}$$

or, equivalently

(2.5) $$\lim_{M,N\to\infty} \sum_{n=-M}^{N} w_n\{\cos(t_0a_n) - \cos(t_0a_{n+1})\} = 0.$$

Setting $c_k = \cos(t_0a_k) - 1$ we have

(2.6) $$\sum_{n=-M}^{N} w_n\{\cos(t_0a_n) - \cos(t_0a_{n+1})\} =$$

$$= \sum_{n=-M}^{N} w_n(c_n - c_{n+1})$$

which, by summation by parts,

$$= w_{-M}c_{-M} + \sum_{n=-M+1}^{N} c_n(w_n - w_{n-1}) - w_Nc_{N+1} .$$

Since $\phi \in L^1$ it is clear that $\lim_{N\to\infty} w_{-N} = 0$ and thus, since $|c_{-n}| \le 2$,

(2.7) $$\lim_{N\to\infty} w_{-N}c_{-N} = 0 .$$

We shall now show that

(2.8) $$\lim_{M\to\infty} w_Mc_{M+1} = 0 .$$

We have

(2.9) $$w_Mc_{M+1} = w_M\{\cos(t_0a_{M+1}) - 1\} =$$

$$= \frac{\cos(t_0a_{M+1}) - 1}{(t_0a_{M+1})^2}\, t_0^2\, (w_Ma_{M+1}^2)$$

and since

(2.10) $$\lim_{M\to\infty} t_0 a_{M+1} = 0$$

and

(2.11) $$\lim_{x\to 0} \frac{\cos x - 1}{x^2} = -\frac{1}{2}$$

it suffices to show that

(2.12) $$\lim_{M\to\infty} w_M a_{M+1}^2 = 0 \quad .$$

Since

(2.13) $$\lim_{M\to\infty} a_M = 0 \quad \text{and} \quad 0 < a_{N+1} < a_N$$

it clearly suffices to show that $w_M a_M$ is bounded for $M \to \infty$. Using the monotonicity of w_n we obtain

(2.14) $$w_M a_M = w_M \sum_{n=M}^{\infty} (a_n - a_{n+1}) =$$

$$= \sum_{n=M}^{\infty} w_M (a_n - a_{n+1}) \le \sum_{n=M}^{\infty} w_n (a_n - a_{n+1}) \le \int_0^{\infty} \phi(x)\, dx$$

and (2.8) follows.
Hence

(2.15) $$0 = \sum_{n=-\infty}^{\infty} c_n (w_n - w_{n-1}) = \sum_{n=-\infty}^{\infty} \{\cos(t_0 a_n) - 1\} (w_n - w_{n-1}) \quad .$$

Plainly each term in the above series is ≤ 0 and hence $= 0$. However, $\cos(t_0 a_n) < 1$ for all sufficiently large n and thus, since $\lim_{n\to\infty} w_n = 0$, there exists at least one term in the above series which is non zero. This contradiction completes our proof.

If we replace the condition "$\lim_{n\to\infty} w_n = \infty$" by the condition "$w_n$ is strictly increasing" then we obtain from (2.15) that $\cos(t_0 a_n) - 1 = 0$ for all $n \in \mathbb{N}$ so that

$$(2.16) \qquad (0 <)\ t_0 a_n = k_n \cdot 2\pi \qquad \text{for some } k_n \in \mathbb{Z}$$

contradicting our assumption that $\lim_{n\to\infty} a_n = 0$.

Finally, we may also replace the condition "w_n is strictly increasing" by the condition "w_n is non-decreasing and $w_n < \lim_{m\to\infty} w_m$ for all $n \in \mathbb{Z}$".
We thus obtain the following

THEOREM 3. *If the stepfunction* $\phi : (0,\infty) \to \mathbb{R}$, *(in the sense of theorem 2) belongs to* L^1 *and is monotonically non-increasing and assumes infinitely many different values in every neighborhood of* 0, *then* $\hat{\phi}(t) \neq 0$ *for all* $t \in \mathbb{R}$.

The condition "assumes infinitely many different values in every neighborhood of 0" cannot be deleted in general as may be seen from a simple example such as: $\phi(x) = 1$ if $x \in (0,1)$ and $\phi(x) = 0$ if $x \notin (0,1)$.

From the above considerations it also follows that the Fourier transform of a non-increasing stepfunction (in the sense of theorem 2) can have zeros only when all its jumps take place within a set consisting of the multiples of some positive number.

REFERENCES

[1] RUBEL, L.A., *Advanced Problem 6131*, Amer. Math. Monthly (48)1(1977), p. 62.

[2] WIENER, N.,*Tauberian theorems*, Ann. Math. 33(1932), pp. 1-100.