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ON INFINITE DIFFERENCE SETS

Preprint

2e boerhaavestraat 49 amsterdam

Printed at the Mathematical Centre, 49, 2e Boerhaavestraat, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O).

AMS(MOS) subject classification scheme (1970): 10A99

On infinite difference sets *)

by

C.L. Stewart & R. Tijdeman

ABSTRACT

Let A_i be an infinite, strictly increasing sequence of non-negative integers with $\bar{d}(A_i) > 0$ for $i = 1,\dots,h$. Let the infinite difference set D_i of A_i be the set of non-negative integers which occur infinitely often as the difference of two terms of A_i. This paper gives several results on infinite difference sets, thereby answering some questions posed by Erdös. It follows from Theorems 1 and 2 that $D_1 \cap \dots \cap D_h$ has positive lower density and does not contain gaps of arbitrary length. There exists even a sequence A with $\underline{d}(A) > 0$ whose infinite difference set equals $D_1 \cap \dots \cap D_h$. Theorem 4 says that the collection of infinite difference sets associated with sequences of positive upper density is a filter on the set of all subsets of the non-negative integers. It follows from Theorem 6 that an infinite difference set need not contain an infinite arithmetical progression. Theorems 7 and 8 are related to a problem of Motzkin. He asked how dense a sequence A can be if its difference set does not contain any elements from a given set K. It is a consequence of Theorem 8 that if $k_1, k_2, \dots$ is a sequence of positive integers such that $\sum_j k_j/k_{j+h} < \infty$ for some positive integer h, then there exists a sequence A with $\underline{d}(A) > 0$ such that k_j is not contained in the difference set of A for $j = 1,2,\dots$. All proofs in the paper are elementary and self-contained. Further most results are quantitative; for example, in the cases above where it is stated that $\underline{d}(A) > 0$ we in fact give explicit lower bounds for $\underline{d}(A)$.

KEY WORDS & PHRASES: *infinite difference sets, Motzkin, uniform distribution*

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*) This report will be submitted for publication elsewhere

1. INTRODUCTION

Let A be a sequence; throughout this paper sequences are understood to be infinite, strictly increasing and composed of non-negative integers. We define D, the infinite difference set of A, to be the set of those non-negative integers which occur infinitely often as the difference of two terms of A. Plainly D has no positive terms if and only if $a_{i+1} - a_i \to \infty$ as $i \to \infty$. Note that D contains zero. We shall be interested in the case when $\bar{d}(A) > 0$. Then D certainly contains more than one term. In fact, see Corollary 1, §2, $\underline{d}(D) \geq \bar{d}(A)$ in this case. Here $\bar{d}$ and $\underline{d}$ denote the (natural asymptotic) upper and lower density respectively.

Let h be a positive integer and let $A_1,\ldots,A_h$ be sequences with positive upper densities $\varepsilon_1,\ldots,\varepsilon_h$ respectively. Erdös asked whether $D_1 \cap \ldots \cap D_h$, the intersection of the associated infinite difference sets, necessarily contains positive terms. We shall show that in fact the intersection has positive lower density. We put

$$C_1 = \varepsilon_1 \quad \text{and} \quad C_h = \prod_{i=1}^{h} (\varepsilon_i/5 \log(h+1)) \qquad \text{for } h \geq 2,$$

and we prove

<u>THEOREM 1</u>. *If* $\bar{d}(A_i) \geq \varepsilon_i$ for $i = 1,\ldots,h$ *then there exists a sequence* A *with* $\underline{d}(A) \geq C_h$ *such that*

$$D \subseteq D_1 \cap \ldots \cap D_h.$$

In fact it follows from Theorem 3 that Theorem 1 remains true even with the stronger conclusion $D = D_1 \cap \ldots \cap D_h$.

By Corollary 1 we have $\underline{d}(D) \geq \bar{d}(A)$ and thus we see from the above theorem that

$$\underline{d}(D_1 \cap \ldots \cap D_h) \geq C_h.$$

Apart from the factor 5 log(h+1), which appears in the definition of C_h, Theorem 1 is best possible. For let $n_1,n_2,\ldots,n_h$ be positive integers and put $A_1 = \{a \mid a \geq 0 \text{ and } a \equiv 0 \pmod{n_1}\}$ and $A_i = \{a \mid a \geq 0$ and $a \equiv 0,1,\ldots,n_1\ldots n_{i-1}-1 \pmod{n_1\ldots n_i}\}$ for $i = 2,\ldots,h$. We then have $d(A_i) = 1/n_i$ for $i = 1,\ldots,h$. Furthermore $D_1 = \{a \mid a \geq 0 \text{ and } a \equiv 0 \pmod{n_1}\}$ while $D_i = \{a \mid a \geq 0 \text{ and } a \equiv 0, \pm 1, \pm 2,\ldots, \pm (n_1\ldots n_{i-1}-1) \bmod (n_1\ldots n_i)\}$ for $i = 2,\ldots,h$. An easy induction shows that $D_1 \cap\ldots\cap D_h =$ $= \{a \mid a \geq 0 \text{ and } a \equiv 0 \pmod{n_1\ldots n_h}\}$. Therefore $d(D_1\cap\ldots\cap D_h) =$ $= (\Pi_{i=1}^h n_i)^{-1} = \Pi_{i=1}^h d(A_i) = \Pi_{i=1}^h \varepsilon_i$.

One might ask whether $D_1\cap\ldots\cap D_h$ can contain gaps of arbitrary length. It will follow as a consequence of our next theorem that this is not possible. Independently Prikry [7] has obtained this result by means of a theorem of Hindman [5]. Further his proof remains valid if D_i is replaced by $\{x \mid \bar{d}(A_i\cap A_i+x) > 0\}$ for $i = 1,\ldots,h$; here $A + k$ is the set $\{a + k \mid a\varepsilon A\}$. We denote the non-negative integers by $\mathbb{N}_0$ and we prove

THEOREM 2. *There exist* r *integers* $k_1,\ldots,k_r$ *such that*

$$\bigcup_{j=1}^{r} \{(D_1\cap\ldots\cap D_h) + k_j\} \supseteq \mathbb{N}_0$$

with

$$r \leq C_h^{-\log 3/\log 2}.$$

It follows from Theorem 2 that $D_1\cap\ldots\cap D_h$ cannot contain gaps of size larger than twice the maximum in absolute value of the k_j's. For if there was a larger gap the integers closest to the middle of the gap would not be in the union of the sets $(D_1\cap\ldots\cap D_h) + k_j$ contradicting Theorem 2. We observe that it is vain to hope for an estimate for $\max_j |k_j|$ in terms of the ε_i's. For let A_1 denote the set of integers of the form $3nt + i$ for $i = 1,\ldots,t$ and $n = 0,1,2,\ldots$. Then D_1 consists of the non-negative integers of the form $3nt \pm i$ for $i = 0,\ldots,t$ and $n = 0,1,2,\ldots$ and so contains infinitely many gaps of length t. On the other hand $d(A_1) = 1/3$.

Theorems 1 and 2 show that infinite difference sets possess a certain regularity. This might suggest that every infinite difference set associated with a sequence of positive upper density has a density. However this is certainly not the case since we have

THEOREM 3. *Let* D *be the infinite difference set of a sequence* A. *Let* E *be a set of non-negative integers with* $D \subseteq E$. *Then there exists a sequence* B *with* $\bar{d}(B) = \bar{d}(A)$ *and* $\underline{d}(B) = \underline{d}(A)$ *whose infinite difference set is* E.

An immediate consequence of this result is that there exist sequences A with $\bar{d}(A) = \underline{d}(A) > 0$ for which $\bar{d}(D) > \underline{d}(D)$. Further Theorem 3 is a step in the proof of the following theorem concerning $\mathbb{D}$, the collection of infinite difference sets associated with sequences of positive upper density. Let $P(\mathbb{N}_0)$ denote the set of all subsets of $\mathbb{N}_0$. We have

THEOREM 4. $\mathbb{D}$ *is a filter on* $P(\mathbb{N}_0)$. *Furthermore all cofinite subsets of* $\mathbb{N}_0$ *which contain zero are in* $\mathbb{D}$.

$\mathbb{D}$ is not an ultrafilter. For there exist disjoint sets B_1 and B_2 satisfying $B_1 \cup B_2 = \mathbb{N}_0$ and $\underline{d}(B_1) = \underline{d}(B_2) = 0$; by Corollary 1 every infinite difference set associated with a sequence of positive upper density has a positive lower density and thus neither B_1 nor B_2 is in $\mathbb{D}$.

We define the difference set of a finite or infinite sequence A to be the set of those non-negative integers which occur as the difference of two elements of A and we denote this set by $\mathcal{D}(A)$. It is interesting to note that the collection of all difference sets associated with sequences of positive upper density does not form a filter. First the collection does not satisfy the superset property. Observe that while $\mathcal{D}(E) = E$, where E denotes the non-negative even integers, there exists no sequence A with $\mathcal{D}(A) = E \cup \{1\}$. Secondly the collection does not satisfy the intersection property as the following example shows. Put $A = \{a \mid a \geq 0$ and $a \equiv 0 (\mathrm{mod}\ 10)\} \cup \{7\}$ and $B = \{b \mid b \geq 0$ and $b \equiv 7 (\mathrm{mod}\ 10)\} \cup \{0\}$; it is readily checked that $\mathcal{D}(A) \cap \mathcal{D}(B) = A$ and that there is no sequence C of positive upper density with $\mathcal{D}(C) = A$. It would be desirable to explicitly describe those sets which are infinite difference sets or difference sets of sequences of positive upper density. A first attempt for the case of difference sets has been made by Ruzsa [9].

Obviously one always has $D \subseteq \mathcal{D}(A)$. On the other hand we have

THEOREM 5. *Given a sequence* A *with positive upper density there exists a sequence* A' *with* $\bar{d}(A) \leq \underline{d}(A')$ *such that* $\mathcal{D}(A') \subseteq D$.

It follows from the above theorem that we may replace D by $\mathcal{D}(A)$ in the statement of Theorem 1; hence plainly the analogous statement of Theorem 1 holds with difference sets in place of infinite difference sets.

An infinite difference set need not contain an infinite arithmetical progression. In fact we shall show that for every α with $0 < \alpha < 1$ there exist sequences A with density α for which the intersection of $\mathcal{D}(A)$ with any infinite arithmetical progression of difference v is a set of density at most $2\alpha/v$. Let $|X|$ be the cardinality of a set X and denote the set $\{0,1,\ldots,n-1\}$ by $\hat{n}$. We have

<u>THEOREM 6</u>. *Let* θ *be an irrational number and let* α *be a number between* 0 *and* 1. *There exist uncountably many sequences* A *with density* α *for which*

$$\limsup_{n\to\infty} \frac{|\mathcal{D}(A)\cap E\cap\hat{n}|}{|E\cap\hat{n}|} \le 2\alpha$$

for every sequence $E = \{e_1,e_2,\ldots\}$ *such that* $\{\theta e_k\}_{k=1}^{\infty}$ *is uniformly distributed modulo* 1.

It is well known (see e.g. [6] Ch.1, Theorem 4.1) that for any sequence $E = \{e_1,e_2,\ldots\}$ the sequence $\{\eta e_k\}_{k=1}^{\infty}$ is uniformly distributed modulo 1 for almost all real numbers η. Hence, given countably many sequences $E^{(i)} = \{e_k^{(i)}\}$ we can find an irrational number θ for which $\{\theta e_k^{(i)}\}$ is uniformly distributed modulo one for all i. In particular it follows from Theorem 6 that for every α with $0 < \alpha < 1$ there exists a sequence A with density α such that

$$\limsup_{n\to\infty} \frac{|\mathcal{D}(A)\cap E\cap\hat{n}|}{|E\cap\hat{n}|} \le 2\alpha$$

for every arithmetical progression $\{ak+b\}_{k=1}^{\infty}$ with $a,b \in \mathbb{N}_0$, $a > 0$, for every geometrical progression $\{ab^k\}_{k=1}^{\infty}$ with $a,b \in \mathbb{N}_0$, $a > 0$, $b > 1$, and for every sequence $\{P(k)\}_{k=1}^{\infty}$, where $P(x)$ is a non-constant polynomial mapping $\mathbb{N}_0$ into $\mathbb{N}_0$.

Theorem 7 concerns sequences which have a non-empty intersection with every infinite difference set D associated with a sequence A of positive upper density. We prove that there are arbitrarily thin sequences of positive integers with this property.

THEOREM 7. *For every sequence* $f_1, f_2, \ldots$ *there exists a sequence* $E = \{e_1, e_2, \ldots\}$ *with* $e_j \geq f_j$ *for all* j *such that for every sequence* A

$$\liminf_{n\to\infty} \frac{|D\cap E\cap \hat{n}|}{|E\cap \hat{n}|} \geq \bar{d}(A).$$

The sequence E constructed for the proof of Theorem 7 has the property that for all positive integers h, $\liminf_{i\to\infty} e_{i+h}/e_i = 1$. In Theorem 8 we show, by contrast, that for every well spaced sequence K of positive integers there exists a sequence A of positive lower density such that $D \cap K = \emptyset$. More precisely we have, on setting

$$c_1 = \tfrac{1}{4} \quad \text{and} \quad c_h = (20 \log(h+1))^{-h} \qquad \text{for } h = 2,3,\ldots,$$

THEOREM 8. *Let* $k_1, k_2, \ldots$ *be a sequence of positive integers. There exists a sequence* A *with*

$$\text{(1)} \qquad \underline{d}(A) \geq \max_{1\leq h<\infty} \{c_h \prod_{j=1}^{\infty} (1-k_j/k_{j+h})^2\},$$

such that $k_j \notin \mathcal{D}(A)$ *for* $j = 1,2,\ldots$.

Thus, for example, if we take $k_j = j!$ and $h = 2$ in Theorem 8 we find that there exists a sequence with positive lower density at least 1/2000 which does not have a factorial as the difference of two terms.

We find from Theorem 8 that whenever $\sum_{j=1}^{\infty} k_j/k_{j+h} < \infty$ for some positive integer h, there exists a sequence A of positive lower density with $k_j \notin \mathcal{D}(A)$. We believe that this condition is too stringent. In fact we conjecture that if for some positive integer h

$$\liminf_{j\to\infty} k_{j+h}/k_j > 1,$$

then there exists a sequence A of positive upper density such that $k_j \notin \mathcal{D}(A)$ for $j = 1,2,\ldots$. In order to prove this conjecture it suffices, by Theorems 1 and 5, to prove that if a sequence $E = \{e_1, e_2, \ldots\}$ has the property that $E \cap D \neq \emptyset$ for every sequence A of positive upper density then

$$\liminf_{i\to\infty} e_{i+1}/e_i = 1.$$

Theorem 8 and the above conjecture are related to a general problem of Motzkin who asked how dense a sequence A can be if $\mathcal{D}(A)$ does not contain any elements from a given set K. Cantor and Gordon [1] and more recently Haralambis [4], have obtained some results in this connexion, mainly for finite sets K. Sarközy [10], [11] and [12] considered the case of some interesting infinite sets K. He obtained results like: if A is a sequence with positive upper density then two distinct elements of A differ by a square. Furstenberg [3], using the methods of ergodic theory, has also proved this result. Erdös and Hartman [2] asked the question: for which sets K does there exist an infinite sequence A with $\mathcal{D}(A) \cap K = \emptyset$. In response, Rotenburg [8] showed that the condition $k_{i+1} - k_i \to \infty$ as $i \to \infty$ is a sufficient one. In conclusion we should like to thank M. Best and P. Erdös for some helpful comments.

§2. PRELIMINARY LEMMAS

For any subset T of $\hat{n}$ and any integer a we put $T(a) = T + a \cap \hat{n}$ where $T + a$ denotes the set of numbers $t + a$ with $t \in T$. We prove

LEMMA 1. *Let* δ *and* ε *satisfy* $0 < \delta < 1$, $0 < \varepsilon < 1$. *If* T *is a subset of* $\hat{n}$ *with* $|T| \geq \varepsilon n$ *then there exist integers* $k, a_1, \ldots, a_k$ *and a set* E *with* $|E| \leq \delta n$ *such that*

$$T \cup T(a_1) \cup \ldots \cup T(a_k) = \hat{n} - E$$

and such that

$$k \leq 2[(\log \delta) / \log(1-\varepsilon)].$$

PROOF. We first observe that $C(a) = T(a) \cup T(a-n)$ is a cyclic shift of T for $a = 0,\ldots,n-1$ and hence $|C(a)| \geq \varepsilon n$. Further, given any subset G of $\hat{n}$

with θn terms for $0 \le \theta \le 1$ we may find an integer b for which $|C(b) \cap G| \ge \varepsilon\theta n$. To see this note that each integer from $\hat{n}$ is contained in at least εn of the cyclic shifts $C(0),\dots,C(n-1)$. Thus $\sum_{a=0}^{n-1} |C(a) \cap G| \ge \varepsilon\theta n^2$ and as a consequence $|C(b) \cap G| \ge \varepsilon\theta n$ for some integer b as required.

Now set $G_1 = \hat{n}\backslash T$. We have $|G_1| = \theta_1 n$ where $\theta_1 \le 1 - \varepsilon$ since $|T| \ge \varepsilon n$. By the above paragraph we may find an integer b_1 such that $|C(b_1) \cap G| \ge \varepsilon\theta_1 n$ and thus $G_2 = \hat{n}\backslash\{T \cup C(b_1)\}$ satisfies $|G_2| = \theta_2 n$ for $\theta_2 \le \theta_1 - \varepsilon\theta_1 \le (1-\varepsilon)^2$. Iterating this argument $\ell-1$ times yields integers $b_1,\dots,b_{\ell-1}$ and a set $G_\ell = \hat{n}\backslash\{T \cup C(b_1)\cup\dots\cup C(b_{\ell-1})\}$ satisfying $|G_\ell| \le (1-\varepsilon)^\ell n$. On recalling that $C(b_i) = T(b_i) \cup T(b_i-n)$ we see that if $\ell -1 = [\log \delta \,/\, \log(1-\varepsilon)]$ then $T \cup T(b_1) \cup T(b_1-n)\cup\dots\cup T(b_{\ell-1}) \cup T(b_{\ell-1}-n) =$ $= \hat{n}\backslash G_\ell$ where $|G_\ell| \le \delta n$. Putting $2(\ell-1) = k$, $b_i = a_{2i-1}$ and $b_i - n = a_{2i}$ for $i = 1,\dots,\ell-1$ and $G_\ell = E$ the lemma follows.

<u>LEMMA 2</u>. *Let* A *be a sequence with* $\bar{d}(A) = \varepsilon > 0$. *For any positive integer* b *there are at least* $[\varepsilon r]$ *of the integers* $b,2b,\dots,rb$ *in* D.

<u>PROOF</u>. Split A into b subsequences $A_j = A \cap \{ib+j\}_{i=0}^{\infty}$ for $j = 0,1,\dots,b-1$. At least one of the sequences A_j satisfies $\bar{d}(A_j) \ge \varepsilon/b$. We define the sequence B by $i \in B$ if and only if $ib + j \in A_j$ for this particular value of j. Let D_0 be the infinite difference set of B. It is clear that if $d \in D_0$ then $bd \in D_j \subset D$. Hence, it suffices to prove that at least $[\varepsilon r]$ of the integers $1,2,\dots,r$ belong to D_0.

Since $\bar{d}(B) \ge \varepsilon$, there are infinitely many integers m_i such that $|B \cap [m_i,m_i+r]| > \varepsilon r$. By the box principle there is a set of $[\varepsilon r] + 1$ integers $b_0,\dots,b_{[\varepsilon r]}$ with $0 \le b_0 < b_1 <\dots< b_{[\varepsilon r]} \le r$ such that for infinitely many integers m_i one has $m_i + b_k \in B$ for $k = 0,1,\dots,[\varepsilon r]$. It follows that $b_k - b_0$ $(k = 1,\dots,[\varepsilon r])$ are $[\varepsilon r]$ differences which occur in D_0. This proves our assertion whence the lemma follows.

<u>COROLLARY 1</u>. *For any sequence* A *we have* $\underline{d}(D) \ge \bar{d}(A)$.

<u>PROOF</u>. If $\bar{d}(A) = 0$ the corollary plainly holds. If $\bar{d}(A) > 0$ the result follows on taking $b = 1$ in Lemma 2.

§3. PROOF OF THEOREM 1

Let $A_1,\ldots,A_h$ be sequences with $\bar{d}(A_i) \geq \varepsilon_i > 0$ for $i = 1,\ldots,h$. We first observe that if $\varepsilon_i > \frac{1}{2}$ for some integer i then $D_i = \mathbb{N}_0$. For if there is a positive integer k which is not in D_i then the sequence $A_i' = A_i \cup A_i + k$ satisfies $\bar{d}(A_i') \geq 2\bar{d}(A_i)$ which is plainly impossible for $\varepsilon_i > \frac{1}{2}$. To see that $\bar{d}(A_i') \geq 2\bar{d}(A_i)$ note that $|A_i \cap A_i + k| = \ell < \infty$ by assumption and thus for all $n > 0$,

$$|A_i' \cap \hat{n}| \geq |A_i \cap \hat{n}| + |A_i + k \cap \hat{n}| - \ell \geq 2|A_i \cap \hat{n}| - k - \ell$$

from which the conclusion follows. Accordingly we may assume that the ε_i's are all at most $\frac{1}{2}$.

We shall construct, for each positive integer n, a set $W_n = W$ with

(2) $$W \subseteq \hat{n},\ |W| \geq C_h n \text{ and } \mathcal{D}(W) \subseteq D_1 \cap \ldots \cap D_h.$$

Since $\bar{d}(A_i) \geq \varepsilon_i$, for each positive integer n there are infinitely many integers k for which $(A_i - k) \cap \hat{n}$ contains at least $\varepsilon_i n$ terms. By the pigeon hole principle there exists an infinite subsequence $k_1, k_2, \ldots$ of the k's for which $(A_i - k_1) \cap \hat{n} = (A_i - k_2) \cap \hat{n} = \ldots$. Set $T_i = (A_i - k_1) \cap \hat{n}$. Plainly we have $T_i \subseteq \hat{n}$, $|T_i| \geq \varepsilon_i n$ and

(3) $$\mathcal{D}(T_i) \subseteq D_i.$$

Thus if $h = 1$ we may take $W = T_1$ and (2) holds.

Assume that $h \geq 2$. On setting $\delta = (h+1)^{-1}$, $\varepsilon = \varepsilon_i$ and $T = T_i$ in Lemma 1 we conclude that there exist integers $k(i)$, $a_{i,1},\ldots,a_{i,k(i)}$ and a set E_i with $|E_i| \leq n/(h+1)$ such that

(4) $$T_i \cup T_i(a_{i,1}) \cup \ldots \cup T_i(a_{i,k(i)}) = \hat{n} \backslash E_i$$

and such that

(5) $$k(i) \leq 2[-\log(h+1)/\log(1-\varepsilon_i)],$$

for $i = 1,\ldots,h$. Put

$$F = \bigcap_{i=1}^{h} (\hat{n}\backslash E_i).$$

By construction $|F| \geq n/(h+1)$. Setting $a_{i,0} = 0$, so that $T_i = T_i(a_{i,0})$, we find from (4) that

$$F \subseteq \bigcup \bigcap_{i=1}^{h} T_i(a_{i,j(i)}),$$

where the union is taken over the $(k(1)+1)\ldots(k(h)+1)$ h-tuples $(j(1),\ldots,j(h))$ with $0 \leq j(i) \leq k(i)$. Thus for at least one h-tuple the set

$$W = T_1(a_{1,j(1)})\cap\ldots\ldots\cap T_h(a_{h,j(h)})$$

contains at least

(6) $$w = n/(h+1)(k(1)+1)\ldots(k(h)+1)$$

terms. Clearly $\mathcal{D}(W) \subseteq \cap_{i=1}^{h} \mathcal{D}(T_i(a_{i,j(i)}))$ and therefore, since $\mathcal{D}(T_i(a)) \subseteq \mathcal{D}(T_i)$ for all integers a, $\mathcal{D}(W) \subseteq \cap_{i=1}^{h} \mathcal{D}(T_i)$. Thus from (3),

$$\mathcal{D}(W) \subseteq D_1\cap\ldots\cap D_h.$$

Since W is plainly contained in $\hat{n}$ we need only show that $|W| \geq w \geq C_h n$. We have from (6) and (5) that

$$w \geq \frac{n}{h+1} \prod_{i=1}^{h} \left(\frac{2\log(h+1)}{-\log(1-\varepsilon_i)} + 1\right)^{-1}$$

which, since $0 < \varepsilon_i \leq \frac{1}{2}$ for $i = 1,\ldots,h$, gives

$$w \geq \frac{n}{h+1} \prod_{i=1}^{h} \left(\frac{-\log(1-\varepsilon_i)}{2\log(h+1) + \log 2}\right).$$

We may now use the inequality $-\log(1-x) \geq x$, which holds for $0 \leq x \leq \frac{1}{2}$, and the fact that $h \geq 2$ to deduce that

$$w \geq n \prod_{i=1}^{h} \left(\frac{\varepsilon_i}{\sqrt{3}(2+\log 2/\log 3)\log(h+1)} \right) .$$

It is easily checked that $w \geq C_h n$. Thus (2) is seen to hold for $h \geq 2$ as well as for $h = 1$.

We construct the sequence A from the sets W_n in the following way. For $n = 1,2,3,\ldots$ we put

$$A \cap \left[\binom{n}{2}, \binom{n+1}{2} \right) = W_{n-[\log n]} + \binom{n}{2} .$$

The sequence A is well defined since, for $n \geq 1$, $W_{n-[\log n]} + \binom{n}{2} \subseteq [\binom{n}{2}, \binom{n}{2}+n)$ and $\binom{n}{2} + n = \binom{n+1}{2}$. We now show that $\underline{d}(A) \geq C_h$. Given a positive integer m we define k by the inequalities $\binom{k}{2} \leq m < \binom{k+1}{2}$. From (2) we have $|W_n| \geq C_h n$ for $n = 1,2,\ldots$ and thus

$$|A \cap \hat{m}| \geq \sum_{n=1}^{k-1} |W_{n-[\log n]}| \geq \sum_{n=1}^{k-1} C_h (n-[\log n])$$

$$\geq C_h \left(\binom{k}{2} - k[\log k]\right).$$

Therefore

$$\frac{|A \cap \hat{m}|}{m} \geq \frac{C_h \binom{k}{2} - k[\log k]}{\binom{k+1}{2}} \geq C_h - \frac{2(1+[\log k])}{k} .$$

Letting m and hence k tend to infinity we see that $\underline{d}(A) \geq C_h$.

Finally we show that $D \subseteq D_1 \cap \ldots \cap D_h$. The terms of A in the interval $[\binom{n}{2}, \binom{n+1}{2})$ differ, by construction, by at least $[\log n]$ from the terms of the interval $[\binom{n+1}{2}, \binom{n+2}{2})$. Thus if a difference occurs infinitely often in A it must occur as the difference of two elements from the interval $[\binom{m}{2}, \binom{m+1}{2})$ for some positive integer m and so it must be contained in $\mathcal{D}(W_{m-[\log m]})$. From (2) we see that the difference is contained in $D_1 \cap \ldots \cap D_h$ as required. This completes the proof.

§4. PROOF OF THEOREM 2

We set $D^0 = D_1 \cap \ldots \cap D_h$ and $\ell_0 = 0$. For $i \geq 1$ we define ℓ_i and D^i whenever $D^{i-1} \not\supseteq \mathbb{N}_0$ by the following inductive process: set ℓ_i equal to the smallest positive integer which is not in D^{i-1} and put $D^i = D^{i-1} \cup D^{i-1} + \ell_i \cup D^{i-1} - \ell_i$. We shall prove that $D^s \supseteq \mathbb{N}_0$ for some positive integer s satisfying $s \leq [-\log C_h/\log 2]$ where C_h is defined as before. This will establish the theorem since

$$D^s = \bigcup_{i=1}^{r} (D^0+k_i) = \bigcup_{i=1}^{r} \{(D_1 \cap \ldots \cap D_h) + k_i\}$$

where the k_i are the $r = 3^s$ finite sums of the form $a_1+\ldots+a_s$ with a_i one of 0, ℓ_i or $-\ell_i$ for $i = 1,\ldots,s$.

For any integer n we may construct, as in the proof of Theorem 1, a set $W = W(n) \subseteq \hat{n}$ with $|W| \geq C_h n$ for which $\mathcal{D}(W) \subseteq D_1 \cap \ldots \cap D_h$.

We now set $W = W^0$ and we define W^i to be $W^{i-1} \cup W^{i-1} + \ell_i$, for $i = 1,\ldots,s$. For any non-empty set A and any integer ℓ it is readily checked that $\mathcal{D}(A \cup A+\ell) \subseteq \mathcal{D}(A) \cup \mathcal{D}(A) + \ell \cup \mathcal{D}(A) - \ell$ and therefore that $\mathcal{D}(W^i) = \mathcal{D}(W^{i-1} \cup W^{i-1}+\ell_i) \subseteq \mathcal{D}(W^{i-1}) \cup \mathcal{D}(W^{i-1}) + \ell_i \cup \mathcal{D}(W^{i-1}) - \ell_i$. From the definition of D^i and the fact that $\mathcal{D}(W^0) \subseteq D^0 = D_1 \cap \ldots \cap D_h$ we conclude that $\mathcal{D}(W^i) \subseteq D^i$ for $i = 0,\ldots,s$. Therefore ℓ_{i+1} does not occur as the difference of two terms in W^i since by assumption ℓ_{i+1} is not in D^i. Accordingly $W^i \cap W^i + \ell_{i+1} = \emptyset$ so that $|W^{i+1}| = 2|W^i|$ and thus $|W^s| = 2^s|W^0| \geq 2^s C_h n$. On the other hand $W^s \subseteq [0, n+\ell_1+\ldots+\ell_s]$ and therefore

$$2^s C_h n \leq n + \ell_1 + \ell_2+\ldots+\ell_s + 1.$$

Dividing by n and letting n tend to infinity we see that $2^s C_h \leq 1$ whence

$$s \leq [-\log C_h/\log 2]$$

as required.

§5. PROOF OF THEOREM 3

Let $E = \{e_1, e_2, \ldots\}$. We construct a sequence $F = \{f_1, f_2, \ldots\}$ by setting, for $n = 1,2,\ldots$: $f_n = e_j$ where j is the unique integer satisfying both $n = \binom{m}{2} + j$ and $1 \le j \le m$ for some positive integer m. Note that every element of E occurs infinitely often as a term of F.

We now construct B. The terms of B are the integers $3^{e_n} + e_n$ and $3^{e_n} + e_n + f_n$ and those integers of A which do not lie in the intervals $[3^{e_n}, 3^{e_n} + 3e_n]$ for $n = 1,2,\ldots$. Since E is an increasing sequence of non-negative integers, $e_n \ge n - 1$ and $\lim_{n\to\infty} (3ne_n)/3^{e_n} = 0$, whence B differs from A only on a set of density zero. Thus $\bar{d}(B) = \bar{d}(A)$ and $\underline{d}(B) = \underline{d}(A)$.

The intervals $[3^{e_n}, 3^{e_n} + 3e_n]$ are disjoint for $n = 1,2,\ldots$. Further $f_n \le e_n$ for all $n > 0$ and thus the difference of an element of B from the interval $[3^{e_n}, 3^{e_n} + 3e_n]$ with one not from this interval is $\ge e_n$. Since $e_n \to \infty$ as $n \to \infty$ the infinite difference set of B is equal to the union of those integers which occur infinitely often as the difference of two terms of B neither of which is in $\bigcup_{n=1}^{\infty} [3^{e_n}, 3^{e_n} + 3e_n]$ with those integers which occur as the difference of two terms of B in $[3^{e_n}, 3^{e_n} + 3e_n]$ for infinitely many integers n. The former set is plainly contained in $D \subseteq E$ while the latter set is exactly E since $3^{e_n} + e_n + f_n - (3^{e_n} + e_n) = f_n$ is the only positive integer which occurs as the difference of two terms of B from $[3^{e_n}, 3^{e_n} + 3e_n]$ and since every element of E occurs infinitely often as a term of F. This completes the proof.

§6. PROOF OF THEOREM 4

To prove that $\mathbb{D}$ is a filter on $\mathcal{P}(\mathbb{N}_0)$ we must show that (i) $\mathbb{D} \ne \emptyset$, (ii) $D \ne \emptyset$ for $D \in \mathbb{D}$, (iii) if $D \in \mathbb{D}$ and $D \subseteq E \subseteq \mathbb{N}_0$ then $E \in \mathbb{D}$, (iv) $D_1 \cap D_2 \in \mathbb{D}$ for $D_1, D_2 \in \mathbb{D}$. Properties (i) and (ii) are readily seen to hold. Property (iii) follows from Theorem 3. Property (iv) follows from property (iii) and Theorem 1. Therefore $\mathbb{D}$ is a filter on $\mathcal{P}(\mathbb{N}_0)$.

Further we must show that every cofinite subset of $\mathbb{N}_0$ which contains zero is in $\mathbb{D}$. Given a set of positive integers $n_1 < n_2 < \ldots < n_k$ we consider the set of positive multiples of $n_k + 1$. This has an infinite difference set which does not contain $n_1,\dots,n_k$ and so by the superset property (iii) we can find a D which is exactly $\mathbb{N}_0 \setminus \{n_1,\dots,n_k\}$. This completes the proof.

§7. PROOF OF THEOREM 5

Put $\varepsilon = \bar{d}(A)$ and let n be any positive integer. We prove first that there exist infinitely many integers m such that $|A \cap [m,m+k)| \geq \varepsilon k$ for $k = 1,\dots,n$. Suppose this statement is false. Then for every $m \geq m_0$ there exists a k_m with $1 \leq k_m \leq n$ such that $|A \cap [m,m+k_m)| < \varepsilon k_m$. Put

$$\varepsilon' = \{\max \frac{i}{k} \mid i,k \in \mathbb{N}_0,\ 1 \leq k \leq n,\ \frac{i}{k} < \varepsilon\}.$$

Note that $\varepsilon' < \varepsilon$ and that for every $m \geq m_0$ we have $|A \cap [m,m+k_m)| \leq \varepsilon' k_m$. Define the sequence m_0, $m_1, m_2, \dots$ inductively by putting $m_{j+1} = m_j + k_j$. Let x be at least m_0 and define J by the inequalities $m_J \leq x < m_{J+1}$. Since for every positive integer j we have $|A \cap [m_j,m_{j+1})| \leq \varepsilon'(m_{j+1}-m_j)$ the number of elements of A less than x is at most

$$m_0 + \varepsilon'(m_J-m_0) + x - m_J \leq \varepsilon' x + m_0 + n.$$

Thus $\bar{d}(A) \leq \varepsilon'$ which is a contradiction.

Let $r_1^{(n)}, r_2^{(n)}, \dots$ be a sequence such that $|A \cap [r_j^{(n)}, r_j^{(n)}+k)| \geq \varepsilon k$ for $j = 1,2,\dots$ and $k = 1,\dots,n$. We consider the sets $(A-r_j^{(n)}) \cap \hat{n}$. By the pigeon hole principle there exists on infinite subsequence $\{s_j^{(n)}\}$ of $\{r_j^{(n)}\}$ such that $(A-s_j^{(n)}) \cap \hat{n}$ is the same set $S^{(n)}$ for every j. We obtain in this way a set $S^{(n)}$, for every positive integer n, such that for $k = 1,\dots,n$ the number of elements less than k is at least εk. We now construct the sequence A' by induction. Suppose $A' \cap \hat{n}$ has been constructed in such a way that there are infinitely many integers ν with $S^{(\nu)} \cap \hat{n} = A' \cap \hat{n}$. We put $n \in A'$ if and only if there are infinitely many integers ν' among these integers ν with $n \in S^{(\nu')}$. It follows that there are infinitely many integers ν with $S^{(\nu)} \cap \widehat{(n+1)} = A' \cap \widehat{(n+1)}$. By construction the number of elements of A' less than n is equal to the number of elements of $S^{(\nu)}$ less than n for some $\nu > n$ and hence is at least εn. Thus $\underline{d}(A') \geq \varepsilon$. Let a_1' and a_2' be any two elements of A' with $a_1' < a_2'$. Then a_1' and a_2' are in $S^{(\nu)}$ for some integer ν. Therefore $a_1' + s_j^{(\nu)} \in A$ and $a_2' + s_j^{(\nu)} \in A$ for $j = 1,2,\ldots$ whence $a_2' - a_1' \in D$. This completes the proof.

Note that we have even proved that the Schnirelmann density of $A' + 1$ is at least ε since $|A' \cap [0,n)| \geq \varepsilon n$ for every positive integer n.

§8. PROOF OF THEOREM 6

Suppose $\{\theta e_k\}_{k=1}^{\infty}$ is uniformly distributed modulo 1 and θ is irrational. Define A to be the sequence composed of those integers n in $\mathbb{N}_0$ for which an integer m exists with $n\theta - m \in (\gamma, \gamma+\alpha)$. (Note that there are uncountably many choices for γ and that for two different choices the corresponding sequences are different.) Since $\{n\theta\}_{n=1}^{\infty}$ is uniformly distributed, $d(A) = \alpha$. If $n \in \mathcal{D}(A)$, then $n = n_1 - n_2$ with $n_1, n_2 \in A$, there exist m_1, m_2 such that $\gamma < n_1\theta - m_1 < \gamma + \alpha$, $\gamma < n_2\theta - m_2 < \gamma + \alpha$ and hence $-\alpha < (n_1 - n_2)\theta - (m_1 - m_2) < \alpha$. Hence, $\mathcal{D}(A)$ consists of integers $n \in \mathbb{N}_0$ for which an integer m exists with $n\theta - m \in (-\alpha, \alpha)$. Since $\{\theta e_k\}$ is uniformly distributed, this implies that

$$\limsup_{n\to\infty} \frac{|\mathcal{D}(A) \cap E \cap \hat{n}|}{|E \cap \hat{n}|} \leq 2\alpha.$$

§9. PROOF OF THEOREM 7

Clearly we may assume that $\alpha = \bar{d}(A) > 0$ and that

$$(7) \qquad f_{n+1}/f_n > n, \qquad \text{for all } n.$$

Put $Q(m) = \binom{m}{2}$ for $m = 1,2,3,\ldots$. We define the sequence E by setting

$$e_{Q(m-1)+j} = j\, f_{Q(m)}$$

for $j = 1,2,\ldots,m-1$ and $m = 2,3,\ldots$. It follows that

$$e_{Q(m-1)+j} \geq f_{Q(m)} = f_{Q(m-1)+m} \geq f_{Q(m-1)+j}$$

for these values of j and m. Thus $e_j \geq f_j$ for all j. Further, by (7), for $m \geq 2$,

$$e_{Q(m)} = (m-1)f_{Q(m)} \leq Q(m)f_{Q(m)} < f_{Q(m+1)} = e_{Q(m)+1}.$$

It follows that the sequence E is strictly increasing and further that the elements of E in the interval $[f_{Q(m)}, f_{Q(m+1)})$ are $jf_{Q(m)}$ for $j = 1,2,\ldots,m-1$. Hence, by Lemma 2, the number of elements of $D \cap E$ in the interval $[f_{Q(m)}, f_{Q(m+1)})$ is at least $[(m-1)\alpha]$. Let n be an integer larger than f_1. Take m such that $f_{Q(m)} \leq n < f_{Q(m+1)}$. Then, as the numbers e_j are distinct,

$$\frac{|D\cap E\cap \hat{n}|}{|E\cap \hat{n}|} \geq \frac{[\alpha]+[2\alpha]+\ldots+[(m-2)\alpha]}{1+2+\ldots+(m-1)} \geq \frac{\binom{m-1}{2}\alpha - m}{\binom{m}{2}} .$$

Hence,

$$\liminf_{n\to\infty} \frac{|D\cap E\cap \hat{n}|}{|E\cap \hat{n}|} \geq \lim_{m\to\infty} \frac{\binom{m}{2}\alpha - 2m}{\binom{m}{2}} = \alpha.$$

§10. PROOF OF THEOREM 8

We shall first show that for any sequence $k_1, k_2, \ldots$ of positive integers there exists a sequence B with

$$\underline{d}(B) \geq \tfrac{1}{4} \prod_{j=2}^{\infty} \left(1 - \frac{k_{j-1}}{k_j}\right)^2 ,$$

such that $k_j \notin \mathcal{D}(B)$ for $j = 1,2,\ldots$. We construct the terms of B in order of increasing size by the following inductive process. We put $k_0 = 1$. B contains the even integers in $[0,k_1)$. The terms of B in $[k_j, k_{j+1})$ for $j = 1,2,3,\ldots$ are those integers x such that $x - k_j - k_{j-1} \in B$.

Assume, for $j \geq 0$, that $k_j \in \mathcal{D}(B)$ so that $x - y = k_j$ for x and y in B and assume further that k_j is minimal. It is readily verified that B does not contain two consecutive integers and thus $k_0 \notin \mathcal{D}(B)$ whence we may assume that $j \geq 1$. Let x be minimal and define ℓ by $k_\ell \leq x < k_{\ell+1}$. If

$\ell > j$ then, since by construction $[k_\ell, k_\ell + k_{\ell-1})$ contains no terms of B, both x and y are in the same interval $[k_\ell, k_{\ell+1})$. Therefore both $x - k_\ell - k_{\ell-1}$ and $y - k_\ell - k_{\ell-1}$ are in B and this contradicts the minimality of x. Thus x lies in $[k_j, k_{j+1})$ and so $x - k_j - k_{j-1} \in B$. But $y = x - k_j$ so that $y - (x - k_j - k_{j-1}) = k_{j-1}$ and this contradicts the minimality of k_j. Thus $k_j \notin \mathcal{D}(B)$ for $j = 1,2,\ldots$.

In order to compute a lower bound for $\underline{d}(B)$ we show by induction that the number N_ℓ of elements of B in $[0,k_\ell)$ satisfies the inequality

$$(8) \qquad N_\ell \geq \tfrac{1}{2}(k_\ell - k_{\ell-1}) \prod_{j=2}^{\ell-1} \left(1 - \frac{k_{j-1}}{k_j}\right)^2 \qquad \text{for } \ell = 2,3,\ldots .$$

Note that by construction the number of terms of B in the intervals $[0,k_\ell), [k_\ell + k_{\ell-1}, 2k_\ell + k_{\ell-1}), \ldots, [s(k_\ell + k_{\ell-1}), s(k_\ell + k_{\ell-1}) + k_\ell)$, where s is subject to $s(k_\ell + k_{\ell-1}) + k_\ell \leq k_{\ell+1}$, are equal. Thus $(s+1)(k_\ell + k_{\ell-1}) \leq k_{\ell+1} + k_{\ell-1}$, whence $s \leq [(k_{\ell+1} + k_{\ell-1})/(k_\ell + k_{\ell-1})]$ and we have

$$(9) \qquad N_{\ell+1} \geq \left[\frac{k_{\ell+1} + k_{\ell-1}}{k_\ell + k_{\ell-1}}\right] N_\ell \qquad \text{for } \ell = 1,2,\ldots .$$

In particular

$$N_2 \geq \left[\frac{k_2+1}{k_1+1}\right] N_1 \geq \frac{k_2 - k_1}{k_1+1} \cdot \frac{k_1+1}{2} = \tfrac{1}{2}(k_2 - k_1),$$

which proves the induction hypothesis for $\ell = 2$. Suppose now that (8) is valid for ℓ. Then, by (9),

$$\begin{aligned} N_{\ell+1} &\geq \left[\frac{k_{\ell+1} + k_{\ell-1}}{k_\ell + k_{\ell-1}}\right] N_\ell \\ &\geq \tfrac{1}{2} \frac{k_{\ell+1} - k_\ell}{k_\ell + k_{\ell-1}} (k_\ell - k_{\ell-1}) \prod_{j=2}^{\ell-1} \left(1 - \frac{k_{j-1}}{k_j}\right)^2 \\ &\geq \tfrac{1}{2}(k_{\ell+1} - k_\ell) \prod_{j=2}^{\ell} \left(1 - \frac{k_{j-1}}{k_j}\right)^2 , \end{aligned}$$

which proves (8) with ℓ replaced by $\ell + 1$.

Let n be any positive integer larger than k_2. Define ℓ by $k_\ell \leq n < k_{\ell+1}$. If $n \geq 2k_\ell$, we have by the above argument that the number of elements of B not exceeding n is at least

$$\tfrac{1}{2}(n-k_\ell) \prod_{j=2}^{\ell} \left(1 - \frac{k_{j-1}}{k_j}\right)^2 \geq \tfrac{1}{4}n \prod_{j=2}^{\infty} \left(1 - \frac{k_{j-1}}{k_j}\right)^2 .$$

If $n < 2k_\ell$, then the number of elements of B not exceeding n is at least

$$N_\ell \geq \tfrac{1}{2}k_\ell \prod_{j=2}^{\infty} \left(1 - \frac{k_{j-1}}{k_j}\right)^2 > \tfrac{1}{4}n \prod_{j=2}^{\infty} \left(1 - \frac{k_{j-1}}{k_j}\right)^2 .$$

Thus

$$\underline{d}(B) \geq \tfrac{1}{4} \prod_{j=2}^{\infty} \left(1 - \frac{k_{j-1}}{k_j}\right)^2$$

as required.

We observe that the expression in brackets in (1) is bounded above by 1 and that $c_h \to 0$ as $h \to \infty$. Therefore it assumes a maximum for some integer. Let that integer be h. We split our given sequence $\{k_j\}_{j=1}^{\infty}$ into h subsequences $\{k_{jh+i}\}_{j=0}^{\infty}$ where $i = 1,2,\ldots,h$. On applying the above result to each of the h subsequences we obtain h sequences B_i $(i = 1,\ldots,h)$ with

$$\underline{d}(B_i) \geq \tfrac{1}{4} \prod_{j=1}^{\infty} (1 - k_{(j-1)h+i}/k_{jh+i})^2$$

such that $k_{(j-1)h+i} \notin \mathcal{D}(B_i)$ for $j = 1,2,\ldots$. By Theorem 1 there exists a sequence A' with

$$\underline{d}(A') \geq 4^h c_h \prod_{i=1}^{h} \underline{d}(B_i) \geq c_h \prod_{j=1}^{\infty} (1-k_j/k_{j+h})^2,$$

such that $k_j \notin D'$. Finally, according to Theorem 5 there exists a sequence A with $\underline{d}(A) \geq \underline{d}(A')$ for which $\mathcal{D}(A) \subseteq D'$. This completes the proof.

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ADDENDUM

We can now prove the conjecture stated in the introduction. This result is a consequence of Theorem 9. Set $c_1 = 1/3$ and $c_h = (15 \log(h+1))^{-h}$ for $h = 2,3,\ldots$. We have

THEOREM 9. *If* $k_1, k_2, \ldots$ *is a sequence of positive integers satisfying* $k_{j+h}/k_j \geq 4$ *for some positive integer* h *and for* $j = 1,2,\ldots$ *then there exists a sequence* A *with* $\underline{d}(A) \geq c_h$ *for which* $k_j \notin \mathcal{D}(A)$ *for* $j = 1,2,\ldots$.

Observe that if $k_{j+h}/k_j \geq \alpha > 1$ for $j = 1,2,\ldots$, and if g is an integer with $g \geq \log 4/\log \alpha$, then $k_{j+gh}/k_j \geq 4$ for $j = 1,2,\ldots$ since

$$\frac{k_{j+hg}}{k_j} = \frac{k_{j+gh}}{k_{j+(g-1)h}} \cdots \frac{k_{j+h}}{k_j} \geq \alpha^g \geq 4.$$

Further, if $\liminf_{j\to\infty} k_{j+\ell}/k_j > 1$ for some positive integer ℓ then there exists a real number α with $\alpha > 1$ such that $k_{j+\ell}/k_j \geq \alpha$ for $j = 1,2,\ldots$. Thus we may apply Theorem 9 with $h = g\ell$ to conclude that there exists a sequence A of positive lower density with $k_j \notin \mathcal{D}(A)$ for $j = 1,2,\ldots$ as was asserted previously.

To illustrate Theorem 9 we show that there eixsts a sequence A with $\underline{d}(A) \geq 1/9$ which does not have a factorial as the difference of two terms. First we apply Theorem 9 with $h = 1$ to the numbers $j!/3$ for $j = 3,4,\ldots$ and then we multiply the terms of the obtained sequence by 3 to give the required sequence A.

PROOF OF THEOREM 9

We shall first prove the theorem for the case $h = 1$. To this end we shall construct a number θ with $0 < \theta < 1$ such that $\|k_j\theta\| \geq 1/3$ for $j = 1,2,\ldots$, where $\|x\|$ denotes the distance from x to the nearest integer. This will suffice since we may then take A to be composed of those integers n for which $0 \leq n\theta - [n\theta] < 1/3$. If θ is irrational it follows from the uniform distribution of θ modulo 1 that $d(A) = 1/3$. If θ is rational it is easily seen that $d(A) \geq 1/3$. Further $\mathcal{D}(A)$ is contained in the set of those integers n with $\|n\theta\| < 1/3$ and thus $k_j \notin \mathcal{D}(A)$ for $j = 1,2,\ldots$.

To obtain θ we first construct a sequence of real numbers $\gamma_1,\gamma_2,\ldots$ by the following inductive process. Put $\gamma_1 = (3k_1)^{-1}$ and let γ_{n+1} for $n = 1,2,\ldots$, be the smallest number x from the interval $[\gamma_n,\gamma_n+(4k_n)^{-1}]$ for which $xk_{n+1} = m + 1/3$ for some integer m. Such a number exists since $k_{n+1} \geq 4k_n$. Set $I_n = [\gamma_n,\gamma_n+(3k_n)^{-1}]$ for $n = 1,2,\ldots$. Observe that by construction $\|k_n x\| \geq 1/3$ for every x in I_n and in fact, since $I_{n+1} \subseteq$ $\subseteq [\gamma_n,\gamma_n+(4k_n)^{-1}+(3k_{n+1})^{-1}] \subseteq I_n$ for $n = 1,2,\ldots$, $\|k_j x\| \geq 1/3$ for $j = 1,\ldots,n$ for every x in I_n. The intervals $[0,1] \supseteq I_1 \supseteq I_2 \supseteq \ldots$ form a nest and so we may take

$$\theta = \bigcap_{i=1}^{\infty} I_i .$$

We then have $\|k_j\theta\| \geq 1/3$ for $j = 1,2,\ldots$. This proves the theorem for the case $h = 1$.

In the general case we split the sequence $\{k_j\}_{j=1}^{\infty}$ into h subsequences $\{k_{jh+i}\}_{j=0}^{\infty}$ where $i = 1,\ldots,h$. On applying the above argument to each of the h subsequences we obtain sequences A_i, $i = 1,\ldots,h$, with $d(A_i) \geq 1/3$ such that $k_{jh+i} \notin \mathcal{D}(A_i)$ for $j = 0,1,2,\ldots$. By Theorem 1 there exists a sequence A' with $\underline{d}(A') \geq c_h$ such that $D' \subseteq D_1 \cap \ldots \cap D_h$ and hence such that $k_j \notin D'$ for $j = 1,2,\ldots$. Finally, by Theorem 5 there exists a sequence A with $\underline{d}(A) \geq \bar{d}(A')$ for which $\mathcal{D}(A) \subseteq D'$. The result now follows.

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