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ON THE PACKING OF QUADRUPLES WITHOUT COMMON TRIPLES

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On the packing of quadruples without common triples *)

by

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ABSTRACT

We observe that the packing number d(3,4,6m) can be readily inferred from results available in the literature.

KEY WORDS & PHRASES: *packing, quadruple system*

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*) This report will be submitted for publication elsewhere

INTRODUCTION

Let $d(t,k,v)$ denote the maximum cardinality of a family of k-subsets of a v-set such that no two of these k-sets have a t-set in common. JOHNSON [7] showed that $d(t,k,v) \le [\frac{v}{k}d(t-1,k-1,v-1)]$ and $d(t,k,v) \le [\frac{v-k}{k}d(t,k,v-1)]$. The first inequality is well known (and usually ascribed to SCHÖNHEIM [10]) but the second one is often reproved for special cases by some ad hoc arguments. Usually $d(t,k,v)$ equals the upper bound obtained by repeatedly applying these inequalities; I conjecture that given t and k there are only finitely many values of v for which $d(t,k,v)$ does not equal the Johnson bound. SCHÖNHEIM [10] determined $d(2,3,v)$ showing that it always equals the Johnson bound:

$$d(2,3,v) = \begin{cases} [\frac{v}{3}[\frac{v-1}{2}]] & \text{if} \quad v \not\equiv 5 \pmod 6 \\ [\frac{v}{3}[\frac{v-1}{2}]]-1 & \text{if} \quad v \equiv 5 \pmod 6 \end{cases}$$

In BROUWER [3] it is shown that $d(2,4,v)$ equals the Johnson bound iff

$$v \neq 8,9,10,11,17,19.$$

Concerning $d(3,4,v)$ the following is known:

(i) for $v \equiv 2$ or $4 \pmod 6$ HANANI [5] constructed a Steiner system $S(3,4,v)$ proving that $d(3,4,v) = v(v-1)(v-2)/24$.

(ii) from this and the results of SCHÖNHEIM [10] it readily follows that for $v \equiv 1$ or $3 \pmod 6$ we have $d(3,4,v) = v(v-1)(v-3)/24$.

(iii) for $v \equiv 0 \pmod 6$ I announced in [2] that $d(3,4,v) = [\frac{v}{4}\cdot d(2,3,v-1)] = v(v^2-3v-6)/24$.
[The case where $v = 6\cdot 2^n$ was treated in KALBFLEISCH & STANTON [8].] The main purpose of this note is to show that this equality is a trivial consequence of results already available in the literature.

(iv) for $v \equiv 5 \pmod 6$ almost nothing is known. Of course $d(3,4,5) = 1$, and BEST ([1], see also [2]) showed that $d(3,4,11) = 35$. Finally there is some information on the structure of a system meeting the Johnson bound.

(Note that in all known cases $d(3,4,v)$ equals the Johnson bound.)

THE CASE v = 6m

Let $X = I_6 \times Y$ where $I_6 = \{0,1,2,3,4,5\}$ and $|Y| = m$.

MILLS [9] defines a G(m,6,4,3) system to be a collection $\mathcal{B}$ of 4-subsets of X such that every triple in X that is not contained in $I_6 \times \{y\}$ for some $y \in Y$ is contained in exactly one 4-set $B \in \mathcal{B}$. He proceeds to show ([9], theorem 1) that such a system exists for all m.

[This can be reformulated by saying that for each m there exists a triplewise balanced design TBD({4,6},1;6m) such that the blocks of size 6 form a parallelclass.]

Now
$$|G(m,6,4,3)| = \frac{\binom{6m}{3} - m\binom{6}{3}}{\binom{4}{3}} = \frac{v(v-1)(v-2)}{24} - 5m.$$

Adding to a system G(m,6,4,3) the 3m quadruples

$$\{0,1,2,3\} \times \{y\},\ \{0,1,4,5\} \times \{y\},\ \{2,3,4,5\} \times \{y\} \qquad (y \in Y)$$

yields a packing D(3,4,v) containing $\frac{v(v-1)(v-2)}{24} - 2m = v(v^2-3v-6)/24$ quadruples. Since this is the Johnson bound for this case it follows that

$$d(3,4,v) = v(v^2-3v-6)/24$$

for $v \equiv 0 \pmod 6$.

REMARK

KALBFLEISCH & STANTON [8] showed conversely that for any system D(3,4,v) with $v(v^2-3v-6)/24$ quadruples (where v = 6m) it is possible to partition the underlying set X into m sets of six elements such that each of the 8m non-covered triples is contained in one of these 6-sets, and each 6-set contains 8 non-covered triples. But this is the same as saying that each optimal D(3,4,v) system (with v = 6m) can be obtained from a G(m,6,4,3) system in the way described above.

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