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SOME CONVEXITY PROPERTIES OF EULER'S GAMMA FUNCTION

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Some convexity properties of Euler's gamma function

by

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ABSTRACT

This report deals with various convexity properties related to Euler's gamma function. Most of these properties are generalizations of monotonic approximation theorems for integrals.

KEY WORDS & PHRASES: *Convexity, gamma-function.*

1. SOME EXTRAPOLATIONS OF A THEOREM OF OZEKI.

OZEKI [3] has shown for example, that if the sequence $\{a_n\}_{n=1}^{\infty}$ is convex, then also the corresponding sequence of Cesàro means

$$\{\frac{1}{n}\sum_{k=1}^{n} a_k\}_{n=1}^{\infty}$$

is convex (also see Mitrinovic [1;p.202]).

Setting $a_n = -\log n$ for $n \in \mathbb{N}$ and observing that

$$\frac{1}{n}\sum_{k=1}^{n} -\log k = \log(n!)^{-\frac{1}{n}}$$

it follows that

$$\{(n!)^{-\frac{1}{n}}\}_{n=1}^{\infty}$$

is log-convex. (For a brief survey of the theory of convex and log-convex functions we refer to E. Artin, Einführung in die Theorie der Gammafunktion, Teubner, (1931)).

We shall prove that more generally we have

<u>THEOREM 1.1</u>. *The function* $f: \mathbb{R}^+ \to \mathbb{R}^+$ *defined by*

$$f(s) = \{\Gamma(s+1)\}^{-\frac{1}{s}}, \qquad (s > 0)$$

is log-convex on $\mathbb{R}^+$.

This theorem in its turn is a simple consequence of the following

<u>THEOREM 1.2</u>. *Let* a *be a constant* > -1. *Then the function* $f_a: \mathbb{R}^+ \to \mathbb{R}^+$ *defined by*

$$f_a(s) = \left\{\frac{\Gamma(s+a+1)}{\Gamma(a+1)}\right\}^{-\frac{1}{s}}, \qquad s > 0,$$

is log-convex on $\mathbb{R}^+$.

Before proving theorem 1.2 we list a number of lemmas which will be useful throughout this note.

LEMMA 1.1. *For the gamma function*

$$\Gamma(s) = \int_0^\infty e^{-x}x^{s-1}dx, \qquad (s > 0)$$

we have the following representation

$$\Gamma(s+1) = s^s e^{-s}\sqrt{2\pi s}\, e^{\mu(s)}, \qquad (s > 0)$$

where $\mu(s)$ *is Binet's function given by*

$$\mu(s) = \int_0^\infty \frac{e^{-st}}{t}\{\frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2}\}dt, \qquad (s > 0).$$

PROOF. See Sansone and Gerretsen [4; p.216].

LEMMA 1.2. *For* $s > 0$ *we have*

$$\mu'''(s) = \frac{1}{s^2} - \frac{1}{s^3} - \int_0^\infty e^{-st}\frac{t^2}{e^t-1}\,dt.$$

PROOF. From the above integral representation of $\mu(s)$ it is clear that for $s > 0$

$$\mu'''(s) = -\int_0^\infty e^{-st}t^2\{\frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2}\}dt =$$

$$= -\int_0^\infty e^{-st}\frac{t^2}{e^t-1}dt + \int_0^\infty e^{-st}tdt - \frac{1}{2}\int_0^\infty e^{-st}t^2dt.$$

Since

$$\int_0^\infty e^{-st}tdt = \frac{1}{s^2}, \qquad (s > 0)$$

and

$$\int_0^\infty e^{-st} t^2 dt = \frac{2}{s^3}, \qquad (s > 0)$$

our proof is complete. □

As an immediate consequence we have

LEMMA 1.3. *For* s > 0 *we have*

$$-\mu'''(s) + \frac{1}{s^2} > 0. \qquad \square$$

PROOF OF THEOREM 1.2. Let $p = a+1$ so that $p > 0$. Define $\phi(s) = \log f_a(s)$ so that for $s > 0$

$$\phi(s) = -\frac{1}{s}\log\frac{\Gamma(s+p)}{\Gamma(p)} = -\frac{1}{s}\log\frac{p}{s+p}\frac{\Gamma(s+p+1)}{\Gamma(p+1)} =$$

$$= -\frac{1}{s}\{\log p - \log(s+p) + \log\frac{(s+p)^{s+p}e^{-s-p}\sqrt{2\pi(s+p)}\,e^{\mu(s+p)}}{p^p e^{-p}\sqrt{2\pi p}\,e^{\mu(p)}}\} =$$

$$= -\frac{1}{s}\{(-p+\tfrac{1}{2})\log p + (s+p-\tfrac{1}{2})\log(s+p) - s + \mu(s+p) - \mu(p)\}.$$

and

$$\phi'(s) = \frac{1}{s^2}\{(-p+\tfrac{1}{2})\log p + (s+p-\tfrac{1}{2})\log(s+p) - s + \mu(s+p) - \mu(p)\} +$$

$$- \frac{1}{s}\{\log(s+p) - \frac{\frac{1}{2}}{s+p} + \mu'(s+p)\}$$

and

$$\phi''(s) = -\frac{2}{s^3}\{(-p+\tfrac{1}{2})\log p + (s+p-\tfrac{1}{2})\log(s+p) - s + \mu(s+p) - \mu(p)\} +$$

$$+ \frac{2}{s^2}\{\log(s+p) - \frac{\frac{1}{2}}{s+p} + \mu'(s+p)\} - \frac{1}{s}\{\frac{1}{s+p} + \frac{\frac{1}{2}}{(s+p)^2} + \mu''(s+p)\}.$$

In order to prove theorem 1.2 it suffices to show that $\phi''(s) > 0$ for $s > 0$, or, equivalently, that $\psi(s) \overset{\text{def}}{=} s^3\phi''(s) > 0$ for $s > 0$.

Since p > 0 and

$$\psi(s) = -2\{(-p+\tfrac{1}{2})\log p + (s+p-\tfrac{1}{2})\log(s+p) - s + \mu(s+p) - \mu(p)\} +$$

$$+ 2s\{\log(s+p) - \frac{\frac{1}{2}}{s+p} + \mu'(s+p)\} - s^2\{\frac{1}{s+p} + \frac{\frac{1}{2}}{(s+p)^2} + \mu''(s+p)\}$$

it is clear that

$$\lim_{s\downarrow 0} \psi(s) = 0$$

so that the proof is complete if we can show that $\psi'(s) > 0$ for $s > 0$. Since, as one may verify,

$$\psi'(s) = s^2\{-\mu'''(s+p) + \frac{1}{(s+p)^2} + \frac{1}{(s+p)^3}\}$$

it follows from lemma 1.3 that indeed

$$\psi'(s) > 0 \qquad \text{for } s > 0. \qquad \square$$

THEOREM 1.3. *If* $a > -1$ *then the function* $g_a : \mathbb{R}^+ \to \mathbb{R}^+$ *defined by*

$$g_a(s) = \{\Gamma(s+a+1)\}^{-\frac{1}{s}}, \qquad s \in \mathbb{R}^+$$

is log-convex if and only if $0 \le a \le 1$.

PROOF. Sufficiency. If $0 \le a \le 1$ then

$$0 < \Gamma(a+1) \le 1.$$

Hence $\frac{-\log\Gamma(a+1)}{s}$ is convex so that $\{\Gamma(a+1)\}^{-\frac{1}{s}}$ is log-convex on $\mathbb{R}^+$. Since the product of log-convex functions is log-convex it follows from theorem 1.2 that $\{\Gamma(s+a+1)\}^{-1/s}$ is log-convex on $\mathbb{R}^+$.

Necessity. Let $p = a+1$ so that $p > 0$. Define $\phi(s) = \log g_a(s)$ so that

$$\phi(s) = -\frac{1}{s}\log\Gamma(s+p) = -\frac{1}{s}\log\frac{\Gamma(s+p+1)}{s+p}.$$

Since $\phi(s)$ is convex by assumption we have $\phi''(s) \geq 0$ for $s > 0$ and hence

$$\lim_{s \downarrow 0} s^3 \phi''(s) \geq 0.$$

On the other hand we have, as one may verify,

$$\lim_{s \downarrow 0} s^3 \phi''(s) = -2 \log \Gamma(p)$$

so that we must have

$$\log \Gamma(p) \leq 0$$

from which it is clear that $1 \leq p \leq 2$ or, equivalently, that $0 \leq a \leq 1$. □

2. SOME EXTRAPOLATIONS OF A THEOREM OF VAN LINT

VAN LINT [2] has shown that if $f:[a,b] \to \mathbb{R}$ is monotonic and either convex or concave on $[a,b]$, then the sequence of canonic upper-Riemann sums, corresponding to $\int_a^b f(x)dx$, is decreasing.

For any positive constant a let $f_a: [0,1] \to \mathbb{R}$ be defined by

$$f_a(x) = \log(1 + \frac{x}{a}), \qquad x \in [0,1].$$

Since f_a is increasing and concave, VAN LINT's theorem yields that the sequence $\{U_n\}_{n=1}^{\infty}$, defined by

$$U_n = \frac{1}{n} \sum_{k=1}^{n} \log(1 + \frac{k}{na}), \qquad n \in \mathbb{N}$$

is decreasing, or, equivalently, that

$$\log\{\frac{\Gamma(na+n+1)}{(na)^n \Gamma(na+1)}\}^{\frac{1}{n}}$$

is decreasing in n.

We shall prove that more generally we have

THEOREM 2.1. *For any positive constant* a, *the function* $f_a : \mathbb{R}^+ \to \mathbb{R}^+$

$$f_a(s) = \left\{\frac{\Gamma(as+s+1)}{s^s\Gamma(as+1)}\right\}^{\frac{1}{s}}, \qquad (s \in \mathbb{R}^+)$$

is log-convex on $\mathbb{R}^+$.

Before proving this theorem we prove some lemmas.

LEMMA 2.1.

$$\lim_{s\downarrow 0} \{\mu(s) + \frac{1}{2}\log 2\pi s\} = 0.$$

PROOF. For $s > 0$ we have

$$\mu(s) + \frac{1}{2}\log 2\pi s = \log \frac{e^s\Gamma(s+1)}{s^s} . \qquad \square$$

LEMMA 2.2.

$$\lim_{s\downarrow 0} s\mu'(s) = -\frac{1}{2} .$$

PROOF. Observe that for $s > 0$

$$s\mu'(s) = -s\int_0^\infty e^{-st}\left\{\frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2}\right\}dt$$

and

$$\lim_{t\to\infty} \left\{\frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2}\right\} = \frac{1}{2}$$

so that the lemma follows from a well known theorem on Laplace transforms.

LEMMA 2.3.

$$\lim_{s\downarrow 0} s^2\mu''(s) = \frac{1}{2} .$$

PROOF. Observe that for $s > 0$

$$s^2\mu''(s) = s^2\int_0^\infty e^{-st}t\{\frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2}\}dt =$$

$$= s^2\int_0^\infty e^{-st}\frac{t}{e^t-1}\,dt - s + \frac{1}{2}$$

and that

$$0 < \int_0^\infty e^{-st}\frac{t}{e^t-1}\,dt < \int_0^\infty e^{-st}dt = \frac{1}{s}\,. \qquad \square$$

LEMMA 2.4. *The function* $s^3\mu'''(s)$ *is increasing on* $\mathbb{R}^+$.

PROOF. Observe that for $s > 0$

$$s^3\mu'''(s) = -s^3\int_0^\infty e^{-st}t^2\{\frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2}\}dt = \text{(by putting } st = u\text{)}$$

$$= -\int_0^\infty e^{-u}u^2\{\frac{1}{e^{\frac{u}{s}}-1} - \frac{1}{\frac{u}{s}} + \frac{1}{2}\}du.$$

The proof will be complete if we can show that for any fixed $u > 0$ the function

$$\frac{1}{e^{\frac{u}{s}}-1} - \frac{1}{\frac{u}{s}} + \frac{1}{2}\,, \qquad (s \in \mathbb{R}^+)$$

is decreasing, or equivalently that the function

$$\phi(x) \overset{\text{def}}{=\!=} \frac{1}{e^x-1} - \frac{1}{x} + \frac{1}{2}\,, \qquad (x \in \mathbb{R}^+)$$

is increasing. Since

$$\phi'(x) = -\frac{e^x}{(e^x-1)^2} + \frac{1}{x^2}$$

it suffices to show that

$$(e^x-1)^2 > x^2e^x, \qquad (x \in \mathbb{R}^+)$$

or (setting x = 2v and taking square roots)

$$e^{2v}-1 > 2ve^v, \qquad (v \quad 0).$$

Writing

$$e^{2v} - 2ve^v - 1 = \sum_{n=0}^{\infty} c_n v^n$$

it is easily seen that $c_0 = c_1 = c_2 = 0$ and $c_n > 0$ for $n \geq 3$. □

PROOF OF THEOREM 2.1. We set c = a+1 and observe that

$$\left\{\frac{\Gamma(as+s+1)}{s^s\Gamma(as+1)}\right\}^{\frac{1}{s}} = \left\{\frac{\Gamma(cs+1)}{s^s\Gamma(as+1)}\right\}^{\frac{1}{s}} =$$

$$= \left\{\frac{(cs)^{cs}e^{-cs}\sqrt{2\pi cs}\; e^{\mu(cs)}}{s^s(as)^{as}e^{-as}\sqrt{2\pi as}\; e^{\mu(as)}}\right\}^{\frac{1}{s}} =$$

$$= \frac{c^c e^{-c}}{a^a e^{-a}}\left\{\sqrt{\frac{c}{a}}\; e^{\mu(cs)-\mu(as)}\right\}^{\frac{1}{s}}.$$

Hence, the proof is complete if we can prove the following

LEMMA 2.5. *If* a *and* c *are constants such that* c > a > 0 *then the function* $\phi\colon \mathbb{R}^+ \to \mathbb{R}$ *defined by*

$$\phi(s) = \frac{\log c - \log a}{2s} + \frac{\mu(cs)-\mu(as)}{s}, \qquad (s \in \mathbb{R}^+)$$

is convex on $\mathbb{R}^+$.

PROOF. For s > 0 we have

$$\phi'(s) = -\frac{\log\frac{c}{a}}{2s^2} + \frac{c\mu'(cs)-a\mu'(as)}{s} - \frac{\mu(cs)-\mu(as)}{s^2}$$

so that

$$\phi''(s) = \frac{\log \frac{c}{a}}{s^3} + \frac{c^2\mu''(cs) - a^2\mu''(as)}{s} - 2\,\frac{c\mu'(cs) - a\mu'(as)}{s^2} +$$

$$+ 2\,\frac{\mu(cs) - \mu(as)}{s^3} .$$

Hence, if suffices to show that $\psi(s) \overset{\text{def}}{=} s^3\phi''(s) > 0$ for $s > 0$.

Since

$$\psi(s) = \log \frac{c}{a} + s^2\{c^2\mu''(cs) - a^2\mu''(as)\} +$$

$$- 2s\{c\mu'(cs) - a\mu'(as)\} + 2\{\mu(cs) - \mu(as)\}$$

we have by lemmas 2.1 through 2.3 that

$$\lim_{s\downarrow 0} \psi(s) = \lim_{s\downarrow 0} 2\{\mu(cs) + \frac{1}{2}\log c - \mu(as) - \frac{1}{2}\log a\} =$$

$$= \lim_{s\downarrow 0} 2\{\mu(cs) + \frac{1}{2}\log 2\pi cs - \mu(as) - \frac{1}{2}\log 2\pi as\} = 0.$$

Hence, in order to show that $\psi(s) > 0$ for $s > 0$ it suffices to show that $\psi'(s) > 0$ for $s > 0$.

One may verify that

$$s\psi'(s) = s^3\{c^3\mu'''(cs) - a^3\mu'''(as)\}$$

so that the proof is complete by lemma 2.4. □

Still more general we have

THEOREM 2.2. *If* a *and* c *are constants such* $c > a > 0$ *then the function* $f_{a,c}: \mathbb{R}^+ \to \mathbb{R}^+$ *defined by*

$$f_{a,c}(s) = \left\{\frac{s^{as}\Gamma(cs+1)}{s^{cs}\Gamma(as+1)}\right\}^{\frac{1}{s}}, \qquad s \in \mathbb{R}^+$$

is log-convex on $\mathbb{R}^+$.

PROOF. Similar as the proof of theorem 2.1.

For any constant $a > 1$ consider the function $f_a\colon [0,1] \to \mathbb{R}$ defined by

$$f_a(x) = -\log(1 - \frac{x}{a}), \qquad x \in [0,1].$$

This function is increasing and convex so that by VAN LINT's theorem

$$U_n(f_a) \overset{\text{def}}{=\!=} -\frac{1}{n} \sum_{k=1}^{n} \log(1 - \frac{k}{na}) = \frac{1}{n} \log \frac{(an)^n \Gamma(an-n)}{\Gamma(an)}$$

is decreasing in n.

We shall now show that more generally we have

THEOREM 2.3. *If* a *and* b *are constants such that* $a > b > 0$ *then the function* $f_{a,b}\colon \mathbb{R}^+ \to \mathbb{R}^+$ *defined by*

$$f_{a,b}(s) = \left\{\frac{s^{as}\Gamma(bs)}{s^{bs}\Gamma(as)}\right\}^{\frac{1}{s}}, \qquad (s \in \mathbb{R}^+)$$

is log-convex on $\mathbb{R}^+$.

We shall derive this theorem from the following

THEOREM 2.4. *If* a *and* b *are constants such that* $a > b > 0$ *then the function* $g_{a,b}\colon \mathbb{R}^+ \to \mathbb{R}^+$ *defined by*

$$g_{a,b}(s) = \left\{\frac{\sqrt{b}\; s^{as}\Gamma(bs)}{\sqrt{a}\; s^{bs}\Gamma(as)}\right\}^{\frac{1}{s}}, \qquad (s \in \mathbb{R}^+)$$

is log-convex on $\mathbb{R}^+$.

Suppose for the moment that theorem 2.4 has been established. Since $a > b > 0$ it is clear that $(\frac{a}{b})^{1/2s}$ is log-convex and since the product of log-convex functions is log-convex it follows that

$$\left(\frac{a}{b}\right)^{\frac{1}{2s}} g_{a,b}(s) = \left\{\frac{s^{as}\Gamma(bs)}{s^{bs}\Gamma(as)}\right\}^{\frac{1}{s}}, \qquad (s \in \mathbb{R}^+)$$

is log-convex on $\mathbb{R}^+$, proving theorem 2.3.

Before establishing theorem 2.4 we prove

LEMMA 2.6. *For every non-negative integer* n *we have.*

$$\lim_{s\downarrow 0} s^n \mu^{(n)}(s) = 0.$$

PROOF. Observe that

$$s^n \mu^{(n)}(s) = s^n(-1)^n \int_0^\infty e^{-st} t^{n-1}\{\frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2}\}dt$$

$$= (-1)^n \int_0^\infty e^{-u} u^{n-1}\{\frac{1}{e^{\frac{u}{s}}-1} - \frac{1}{\frac{u}{s}} + \frac{1}{2}\}du$$

and that

$$\phi(u,s) \overset{\text{def}}{=\!=} \frac{1}{e^{\frac{u}{s}}-1} - \frac{1}{\frac{u}{s}} + \frac{1}{2}$$

is bounded on $\mathbb{R}^+ \times \mathbb{R}^+$ and that for every fixed $u > 0$, $\phi(u,s)$ is decresing in s such that $\lim_{s\to\infty} \phi(u,s) = 0$. □

PROOF OF THEOREM 2.4. Define $\phi(s) = \log g_{a,b}(s)$ so that, similarly as before, it suffices to show that

$$\psi(s) \overset{\text{def}}{=\!=} \frac{\mu(bs) - \mu(as)}{s}, \qquad (s \in \mathbb{R}^+)$$

is convex on $\mathbb{R}^+$.

From the proof of lemma 2.5 we obtain that

$$s^3\psi''(s) = s^2\{b^2\mu''(bs) - a^2\mu''(as)\} - 2s\{b\mu'(bs) - a\mu'(as)\} +$$

$$+ 2\{\mu(bs) - \mu(as)\} =: \xi(s)$$

so that by lemma 2.6

$$\lim_{s\to\infty} \xi(s) = 0.$$

Similarly as before the proof is complete if we can show that

$$\xi'(s) < 0 \qquad \text{for } s > 0.$$

One may verify that

$$\xi'(s) = s^2\{b^3\mu'''(bs) - a^3\mu'''(as)\}$$

so that $\xi'(s) < 0$ by lemma 2.4. □

Finally we consider the function

$$f_a(x) = -\log(1-\frac{x^2}{a^2}), \qquad x \in [0,1]$$

where $a > 1$ is constant.

The corresponding canonical upper Riemann sums are

$$U_n = \frac{1}{n}\log\left\{\frac{a(na)^{2n}\Gamma(na-n)}{(a+1)\Gamma(na+n)}\right\}$$

and since f_a is increasing and convex these U_n form a decreasing sequence.

As a generalization of this result we have that the function $\phi_a\colon \mathbb{R}^+ \to \mathbb{R}^+$, $(a > 1)$, defined by

$$\phi_a(s) = \left\{\frac{a}{a+1}\,\frac{s^{2s}\Gamma(as-s)}{\Gamma(as+s)}\right\}^{\frac{1}{s}}, \qquad s \in \mathbb{R}^+$$

is log-convex on $\mathbb{R}^+$.

In order to see this we observe that

$$\frac{a}{\sqrt{a^2-1}} > 1$$

so that

$$\left\{\frac{a}{\sqrt{a^2-1}}\right\}^{\frac{1}{s}}$$

is log-convex on $\mathbb{R}^+$.

In theorem 2.4 replace a by a+1 and b by a-1. It follows that

$$\left\{\frac{a}{\sqrt{a^2-1}}\right\}^{\frac{1}{s}} \cdot \left\{\frac{\sqrt{a-1}}{\sqrt{a+1}} \frac{s^{2s}\Gamma(as-s)}{\Gamma(as+s)}\right\}^{\frac{1}{s}} =$$

$$= \left\{\frac{a}{a+1} \frac{s^{2s}\Gamma(as-s)}{\Gamma(as+s)}\right\}^{\frac{1}{s}}$$

is log-convex on $\mathbb{R}^+$, proving our claim.

REFERENCES

[1] MITRINOVIĆ, D.S., *Analytic inequalities*, Springer, (1970).

[2] NIEUW ARCHIEF VOOR WISKUNDE, Third Series, Part 23, No. 3 (1975), Problem 399, pp. 254-256.

[3] OZEKI, N., *On some inequalities*, J. Coll. of Arts and Sciences, Chiba Univ. Vol. 4, No. 3 (1965), pp. 211-214.

[4] SANSONE, G. & J. GERRETSEN, *Lectures on the theory of functions of a complex variable*, Noordhoff, Groningen (1960).