

**stichting
mathematisch
centrum**



AFDELING ZUIVERE WISKUNDE
(DEPARTMENT OF PURE MATHEMATICS)

ZW 108/78

FEBRUARI

A. SCHRIJVER

VERTEX-CRITICAL SUBGRAPHS OF KNESER-GRAPHS

Preprint

2e boerhaavestraat 49 amsterdam

Printed at the Mathematical Centre, 49, 2e Boerhaavestraat, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O).

Vertex-critical subgraphs of Kneser-graphs ^{*)}

by

A. Schrijver

ABSTRACT

We show that if the stable (independent) n -subsets of a circuit with $2n+k$ vertices are split into $k+1$ classes, one of the classes contains two disjoint n -subsets; this yields a $(k+2)$ -vertex-critical subgraph of Lovász' Kneser-graph $KG_{n,k}$.

KEY WORDS AND PHRASES: *Kneser-graph, Kneser's conjecture, vertex-critical subgraph, stable sets, clique.*

^{*)} This report will be submitted for publication elsewhere.

1. INTRODUCTION

Let n and k be natural numbers and let X be a set with $2n+k$ elements. Call a collection of subsets of X a *clique* if it does not contain two disjoint sets. The following question arises naturally: *What is the minimal number of cliques into which the collection of all n -subsets of X can be split?*

[An n -subset is a subset with n elements.]

In 1955 KNESER [5] raised the conjecture that the following splitting has the minimal number of cliques, where we lose no generality by assuming that $X = \{1, \dots, 2n+k\}$. For $i = 1, \dots, 2n+k$, let K_i contain all n -subsets of X whose smallest element is i . Then

$$K_1, \dots, K_{k+1}, K_{k+2} \cup \dots \cup K_{k+n+1}$$

divides the n -subsets of X into $k+2$ cliques. So Kneser conjectured that no splitting of the n -subsets into $k+1$ cliques is possible.

In 1977 LOVASZ [6] was able to prove this conjecture. His interesting proof uses homotopy theory and the following theorem of BORSUK [2] (cf. DUGUNDJI [3]) from 1933, where S^k ($\subset \mathbb{R}^{k+1}$) denotes the k -dimensional sphere:

BORSUK'S THEOREM (closed form): *If $S^k = F_1 \cup \dots \cup F_{k+1}$, where F_1, \dots, F_{k+1} are closed subsets of S^k , then one of the sets F_i contains two antipodal points.*

In 1977 as well, BÁRÁNY [1] demonstrated that the truth of Kneser's conjecture immediately follows from the following form of Borsuk's theorem:

BORSUK'S THEOREM (open form): *If $S^k = U_1 \cup \dots \cup U_{k+1}$, where U_1, \dots, U_{k+1} are open subsets of S^k , then one of the sets U_i contains two antipodal points*

(by using simple topological arguments the two forms of Borsuk's theorem can be deduced from each other), together with a theorem of GALE [4] from

1956:

GALE'S THEOREM: *One can select $2n+k$ points on S^k such that each open hemisphere of S^k contains at least n of these points.*

(In the present paper we give explicitly a possible choice of these $2n+k$ points.) B ar any's method runs as follows. Suppose we could divide all n -subsets of X into $k+1$ cliques, say C_1, \dots, C_{k+1} . In this case we may suppose, without loss of generality, that the $2n+k$ elements of X are situated on S^k in a way as formulated in Gale's theorem. For $i = 1, \dots, k+1$, let U_i be the (open) set of the centers of those hemispheres which enclose an element of C_i (this element being a subset of X and hence of S^k). Since, by Gale's theorem, each open hemisphere includes at least one n -subset of X , we know that $S^k = U_1 \cup \dots \cup U_{k+1}$. Hence Borsuk's theorem assures the existence of two antipodal points in, say, U_i . But antipodal points are the centers of disjoint open hemispheres, and these hemispheres include necessarily disjoint n -subsets in C_i , contradicting the fact that C_i is a clique.

One may translate Kneser's conjecture in the language of graphs, by defining the *Kneser-graph* $KG_{n,k}$ as follows. The vertices of $KG_{n,k}$ are the n -subsets of X , two vertices being adjacent iff they are, as n -subsets, disjoint.

Now dividing n -subsets of X into ℓ cliques coincides with colouring the vertices of $KG_{n,k}$ with ℓ colours such that adjacent vertices have different colours; the vertices coloured with some fixed colour together form a clique. So Kneser's conjecture, i.e. Lovasz' result, can be formulated as: the colouring number of $KG_{n,k}$ equals $k+2$.

Do we always need the graph $KG_{n,k}$ completely to conclude that this graph is not $(k+1)$ -colourable? Evidently not, since, if $k = 1$, the existence of an odd circuit in $KG_{n,1}$ is already enough for knowing that $KG_{n,1}$ is not 2-colourable. Therefore one may ask for minimal not- $(k+1)$ -colourable induced subgraphs of $KG_{n,k}$, i.e. for induced subgraphs of $KG_{n,k}$ which are not $(k+1)$ -colourable, but if we delete any vertex of these subgraphs they will be $(k+1)$ -colourable. Otherwise stated, find collections, consisting of n -subsets of X , which cannot be split into $k+1$ cliques, but whose proper

subcollections all are partitionable into $k+1$ (or less) cliques.

A graph whose proper induced subgraphs all have a lower colouring number than the colouring number c of the graph itself is called *c-vertex-critical*. So we are looking for $(k+2)$ -vertex-critical subgraphs of $KG_{n,k}$; in this note we present such subgraphs.

To this end, define an n -subset X' of $X = \{1, \dots, 2n+k\}$ to be *stable* if for no $i = 1, \dots, 2n+k-1$ both $i \in X'$ and $i+1 \in X'$, nor both $2n+k \in X'$ and $1 \in X'$; i.e. a subset is stable if it does not contain two neighbours in the cyclic ordering of $\{1, \dots, 2n+k\}$.

By giving an explicit embedding of $2n+k$ points on S^k satisfying the claim of Gale's theorem we prove the non- $(k+1)$ -colourability of the subgraph of $KG_{n,k}$ induced by those vertices of $KG_{n,k}$ representing stable n -subsets of $X = \{1, \dots, 2n+k\}$. That is, the collection of stable n -subsets of X cannot be divided into $k+1$ cliques. We also show that this last indeed is possible for each of its proper subcollections.

Note that in case $k = 1$ the stable n -subsets of $\{1, \dots, 2n+1\}$ induce an odd circuit in $KG_{n,1}$.

2. VERTEX-CRITICAL SUBGRAPHS

We first give an explicit embedding of $2n+k$ points on the k -dimensional sphere S^k such that each open hemisphere contains at least n of these points. For this we need the following observation about values of polynomials.

OBSERVATION. *Let $p(x)$ be a non-zero polynomial, with real coefficients, of degree at most k . Then there is a stable n -subset X' of $\{1, \dots, 2n+k\}$ such that $(-1)^i p(i) > 0$ whenever $i \in X'$.*

PROOF. Let $p(x)$ be such a polynomial. Define inductively the sequence $i_0, i_1, i_2, i_3 \dots$ of integers by

- (1) i_0 is the largest nonpositive integer such that $(-1)^{i_0} p(i_0) > 0$;
- (2) i_1 is the smallest positive integer such that $(-1)^{i_1} p(i_1) > 0$;

- (3) i_ℓ is the smallest integer such that $i_\ell \geq i_{\ell-1} + 2$ and
 $(-1)^{i_\ell} p(i_\ell) > 0$, for $\ell = 2, 3, 4, \dots$

Clearly, this sequence is infinite. Now take $X' = \{i_1, \dots, i_n\}$. We are ready once we have proved that $i_n \leq 2n+k$ and $i_n - i_1 \leq 2n+k-2$.

To this end let, for real numbers r and s , $Z(r,s)$ be the number of zeros of $p(x)$ contained in the open interval (r,s) , counting f -fold zeros f times ($f \in \mathbb{N}$). We prove that

- (4) $i_\ell - i_{\ell-1} \leq 2 + Z(i_{\ell-1}, i_\ell)$, for $\ell = 1, 2, 3, \dots$

Appropriately adding some of these inequalities yields the inequalities we need, since $Z(-\infty, i_n) \leq k$.

First remark that if all integers in the open interval (c,d) are zeros of $p(x)$, where c and d are integers, then $Z(c,d) \geq d-c-1$. If furthermore $(-1)^c p(c) (-1)^d p(d) > 0$ then $Z(c,d) \equiv d-c \pmod{2}$, whence $Z(c,d) \geq d-c$.

Now to show (4), for $\ell = 1, 2, 3, \dots$, look at the behaviour of $p(x)$ between $i_{\ell-1}$ and i_ℓ . Let

$$i_{\ell-1} = j_0 < j_1 < \dots < j_m = i_\ell$$

be those integers in the closed interval $[i_{\ell-1}, i_\ell]$ which are not a zero of $p(x)$, and consider the sequence of numbers

$$(-1)^{j_0} p(j_0), (-1)^{j_1} p(j_1), \dots, (-1)^{j_m} p(j_m).$$

By (3) above these numbers are negative except the first and last one and possibly the second one. Hence at most two of the products of two consecutive terms in this sequence are negative, the remaining products being positive.

So, by our remark, we have $Z(j_{s-1}, j_s) \geq j_s - j_{s-1}$, for $s = 1, \dots, m$ with at most two exceptions; but in all cases $Z(j_{s-1}, j_s) \geq j_s - j_{s-1} - 1$.

Therefore, by adding inequalities we get

$$Z(i_{\ell-1}, i_{\ell}) = Z(j_0, j_m) \geq j_m - j_0 - 2 = i_{\ell} - i_{\ell-1} - 2,$$

thus proving (4). \square

Next define, for each natural number i , the vector $v_i \in \mathbb{R}^{k+1}$ by

$$v_i = (-1)^i (1, i^1, i^2, \dots, i^k).$$

By projecting the vectors v_1, \dots, v_{2n+k} onto the sphere S^k we obtain $2n+k$ points

$$w_1 = \frac{v_1}{|v_1|}, \dots, w_{2n+k} = \frac{v_{2n+k}}{|v_{2n+k}|},$$

situated on S^k and satisfying a stronger form of Gale's claim, as stated in the following theorem.

THEOREM 1. *Each open hemisphere of S^k encloses an n -subset of $\{w_i \mid 1 \leq i \leq 2n+k\}$ whose indices form a stable subset of $\{1, \dots, 2n+k\}$.*

PROOF. Choose a hemisphere with center, say, $a = (a_0, \dots, a_k)$. This hemisphere contains the point w_i if and only if the inner product of a and w_i is positive, that is, if and only if

$$(-1)^i (a_0 + a_1 i^1 + a_2 i^2 + \dots + a_k i^k) > 0.$$

Since $p(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_k x^k$ is a non-zero polynomial of degree at most k , the observation gives us that a stable n -subset X' of $\{1, \dots, 2n+k\}$ exists such that $(-1)^i p(i) > 0$ whenever $i \in X'$. Hence the vectors w_i with index i in X' have the required properties. \square

Now we are able to prove, in a manner analogous to Bárány's manner to prove Kneser's conjecture, the following sharpening of Kneser's conjecture.

THEOREM 2. *It is not possible to divide the stable n -subsets of $\{1, \dots, 2n+k\}$ into $k+1$ cliques.*

PROOF. Suppose C_1, \dots, C_{k+1} are cliques such that each stable n -subset of $\{1, \dots, 2n+k\}$ is in at least one of them. Let U_i consist of all centers of those open hemispheres which enclose any n -subset $\{w_i \mid i \in X'\}$ with $X' \in C_i$ ($i=1, \dots, k+1$). By theorem 1, $S^k = U_1 \cup \dots \cup U_{k+1}$. Since each U_i is open, the open form of Borsuk's theorem involves the existence of two antipodal points in some U_i . Since antipodal points are the centers of disjoint open hemispheres, there are disjoint n -subsets in C_i . This contradicts the fact that C_i is a clique. \square

Let, by definition, the *reduced Kneser-graph* $KG'_{n,k}$ have as vertices all stable n -subsets of $\{1, \dots, 2n+k\}$, two of them being adjacent if they are disjoint. So $KG'_{n,k}$ is an induced subgraph of $KG_{n,k}$. Theorem 2 in fact asserts that the vertices of $KG'_{n,k}$ cannot be coloured with $k+1$ colours without unicoloured adjacent vertices, so its colouring number equals $k+2$. We conclude with showing that the collection of stable n -subsets is minimal (under inclusion) with these properties, in other words

THEOREM 3. *The reduced Kneser-graph $KG'_{n,k}$ is $(k+2)$ -vertex-critical.*

PROOF. By theorem 2 it is enough to show that if S is a stable n -subset of $\{1, \dots, 2n+k\}$ then the collection

$$\{X' \mid X' \text{ is a stable } n\text{-subset of } \{1, \dots, 2n+k\}, \text{ and } X' \neq S\}$$

can be split into $k+1$ cliques. So choose S .

Consider the circuit with vertices $1, \dots, 2n+k$, two vertices i and j being adjacent iff $i \equiv j + 1$ or $i \equiv j - 1 \pmod{2n+k}$. Let the set S' consist of all elements of S together with all points adjacent in C to any element of S . We may split the set S' uniquely into disjoint classes T_1, \dots, T_m , such that each of them induces on C a path with both end points in $S' \setminus S$, and such that no class contains two adjacent points of $S' \setminus S$ (except in the trivial case $k = 1$). So S' has $2n+m$ points, and there remain $k-m$ points which are in $\{1, \dots, 2n+k\} \setminus S'$. Each of these remaining points determines

a clique consisting of all stable n -subsets containing the point; this provides us with the first $k-m$ clique H_1, \dots, H_{k-m} .

Let a and b be the two end points of the path determined by T_1 ; cliques H_{k-m+1} and H_{k-m+2} , respectively, have as elements all stable n -subsets containing a and b , respectively.

The cliques H_1, \dots, H_{k-m+2} together contain all stable n -subsets of $\{1, \dots, 2n+k\}$ except those completely contained in $S' \setminus \{a, b\}$. Now observe that

- (1) each stable n -subset contained in $S' \setminus \{a, b\}$ either encloses $T_1 \cap S$ or encloses, for some $j = 2, \dots, m$, $T_j \setminus S$,

and

- (2) each stable n -subset contained in $S' \setminus \{a, b\}$ and different from S meets some $T_j \setminus S$ ($j=2, \dots, m$).

(1) and (2) imply that the collections

$$H_{k-m+j+1} = \{X' \mid X' \text{ is a stable } n\text{-subset of } S' \setminus \{a, b\} \text{ such that } T_j \setminus S \subset X' \text{ or both } T_1 \cap S \subset X' \text{ and } X' \cap (T_j \setminus S) \neq \emptyset\}$$

($j=2, \dots, m$) together contain all stable n -subsets of $S' \setminus \{a, b\}$ different from S . Since $T_1 \cap S$ and $T_j \setminus S$ are nonempty, the collections

$H_{k-m+3}, \dots, H_{k+1}$ are cliques. Hence H_1, \dots, H_{k+1} partition all stable n -subsets of $\{1, \dots, 2n+k\}$ different from S into $k+1$ cliques. \square

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