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A LOWER BOUND FOR THE LENGTH OF PARTIAL TRANSVERSALS
IN A LATIN SQUARE
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A lower bound for the length of partial transversals in a Latin square
by
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ABSTRACT

Any Latin square of order $n$ has a partial transversal of order at least $n - \sqrt{n}$.

KEY WORDS & PHRASES: latin square, transversal.

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This report will be submitted for publication elsewhere.
0. INTRODUCTION

Let $A$ be a Latin square of order $n$ (i.e., a square matrix such that each of its rows and columns is a permutation of the set $I_n = \{1, 2, \ldots, n\}$). A set $T \subseteq I_n \times I_n$ is called a partial transversal of $A$ if $|T| = \# \{i \mid (i, j) \in T\} = \# \{j \mid (i, j) \in T\} = \# \{a(i, j) \mid (i, j) \in T\}$ (i.e., no two positions in the same row or column; no two entries the same).

In [3] Koksma proved that for $n \geq 3$ each Latin square of order $n$ has a partial transversal of length at least $(2n + 1)/3$. A simple modification of his method yields a lower bound of $(3n - 1)/4$. Drake [2] proved the existence of a partial transversal of length at least $3n/4$ for $n \geq 8$. De Vries & Wieringa [4] perfected Koksma's method and were able to prove a lower bound of $(4n - 3)/5$ for $n \geq 12$. Here we prove a lower bound roughly of order $n - \sqrt{n}$. (It is sharper than the older bounds for $n \geq 7$.) This result is still far from the best possible; in fact Rysier (see [3]) and Brualdi (see [1], p. 103) conjectured that any Latin square of order $n$ has a partial transversal of order $n - 1$.

For even $n$ this result would be best possible since a circulant Latin square of even order does not possess a transversal, but it is probably true that each Latin square of odd order has a transversal.

1. A LOWER BOUND

**THEOREM.** Every Latin square of order $n$ has a partial transversal of order $n - r$ for an $r$ with $r(r + 1) \leq n$.

**PROOF.** Let the longest partial transversal of a given Latin square $A$ have length $t$ and let $r = n - t$. By permuting rows, columns and symbols (if necessary) we may assume that $a(i, i) = i$ for $1 \leq i \leq t$. Let $L = \{t + 1, \ldots, n\}$, the set of 'large' numbers.

Define sets $A_i$ ($0 \leq i \leq r$) by induction on $i$:

$A_0 = \emptyset$,

$A_i = \{j \mid a(j, t + i) \in A_{i-1} \cup L\}$. 


Define a directed graph $G$ with vertex set $\bigcup_{i=1}^{r} A_i \times \{t+i\}$ and edge set
\[\{((x,t+i),(y,t+j)) \mid i < j \text{ and } x = a(y,t+j)\}.\]

**LEMMA.** $G$ does not contain a directed path starting in a position $(g,t+i)$ where $A$ has a large entry, and ending in a position $(h,t+j)$ with a large row number (i.e. $h \in L$).

**PROOF.** Suppose $(g_0,t+i_0),(g_1,t+i_1),\ldots,(g_\ell,t+i_\ell)$ is the shortest such path. Then the collection of $t+1$ positions
\[(g_k,t+i_k) \quad k = 0, \ldots, \ell\]
and
\[(j,j) \quad \text{for } j \neq g_k \ (0 \leq k \leq \ell-1), \ j \leq t\]
is a partial transversal, contradicting the definition of $t$. For:

(i) Since $1 \leq i_0 < i_1 < \ldots < i_\ell$ all positions are in different columns.

(ii) All positions are in different rows since $j \neq g_k$, and if $g_h = g_k$ for $h < k$ then either $k = \ell$, and $g_h = g_k \in L$, so that
\[(g_0,t+i_0),\ldots,(g_h,t+i_h)\]
is a shorter path down, or $k < \ell$ and $g_h = g_k = a(g_{k+1},t+i_{k+1})$ so that
\[(g_0,t+i_0),\ldots,(g_h,t+i_h),(g_{k+1},t+i_{k+1}),\ldots,(g_\ell,t+i_\ell)\]
is a shorter path down. Contradiction in both cases.

(iii) All entries are different, for $a(g_{k+1},t+i_{k+1}) = g_k$ so that the entries are the numbers $1,\ldots,t$ and $a(g_0,t+i_0)$, where the latter is in $L$. $\square$

From the lemma it follows that all vertices of $G$ are in rows $1,\ldots,t$. 
It is also clear that $|A_i| = |A_{i-1}| + r$. Hence $|A_r| = r^2$. Therefore $r^2 \leq t$, i.e. $r^2 + r \leq n$. 

REMARKS.
(1) For this presentation of our proof we are indebted to A. Schrijver.
(2) From an observation by J.H. van Lint it follows that equality can occur only for $n = 2$, i.e. for $n \geq 3$ we have $n \geq r(r+1) + 1$.

REFERENCES


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Added in proof
Essentially the same results were obtained by D.E. WOOLBRIGHT [5].
