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A LOWER BOUND FOR THE LENGTH OF PARTIAL TRANSVERSALS IN A LATIN SQUARE

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A lower bound for the length of partial transversals in a Latin square

by

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ABSTRACT

Any Latin square of order n has a partial transversal of order at least $n - \sqrt{n}$.

KEY WORDS & PHRASES: *latin square, transversal.*

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This report will be submitted for publication elsewhere.

0. INTRODUCTION

Let A be a Latin square of order n (i.e. a square matrix such that each of its rows and columns is a permutation of the set $I_n = \{1,2,\ldots,n\}$). A set $T \subset I_n \times I_n$ is called a partial transversal of A if $|T| = \#\{i \mid (i,j) \in T\} = \#\{j \mid (i,j) \in T\} = \#\{a(i,j) \mid (i,j) \in T\}$ (i.e. no two positions in the same row or column; no two entries the same).

In [3] KOKSMA proved that for $n \geq 3$ each latin square of order n has a partial transversal of length at least $(2n+1)/3$. A simple modification of his method yields a lower bound of $(3n-1)/4$. DRAKE [2] proved the existence of a partial transversal of length at least $3n/4$ for $n \geq 8$. DE VRIES & WIERINGA [4] perfected Koksma's method and were able to prove a lower bound of $(4n-3)/5$ for $n \geq 12$. Here we prove a lower bound roughly of order $n - \sqrt{n}$. (It is sharper than the older bounds for $n \geq 7$.) This result is still far from the best possible; in fact RYSER (see [3]) and BRUALDI (see [1], p. 103) conjectured that any Latin square of order n has a partial transversal of order n-1.

For even n this result would be best possible since a circulant Latin square of even order does not possess a transversal, but it is probably true that each latin square of odd order has a transversal.

1. A LOWER BOUND

<u>THEOREM</u>. *Every latin square of order* n *has a partial transversal of order* $n-r$ *for an* r *with* $r(r+1) \leq n$.

<u>PROOF</u>. Let the longest partial transversal of a given Latin square A have length t and let $r = n - t$. By permuting rows, columns and symbols (if necessary) we may assume that $a(i,i) = i$ for $1 \leq i \leq t$. Let $L = \{t+1,\ldots,n\}$, the set of 'large' numbers.

Define sets A_i $(0 \leq i \leq r)$ by induction on i:

$$A_0 = \emptyset,$$

$$A_i = \{j \mid a(j,t+i) \in A_{i-1} \cup L\}.$$

Define a directed graph G with vertex set $\dot{\bigcup}_{i=1}^{r} A_i \times \{t+i\}$ and edge set

$$\{((x,t+i),(y,t+j)) \mid i < j \text{ and } x = a(y,t+j)\}.$$

<u>LEMMA</u>. G *does not contain a directed path starting in a position* (g,t+i) *where* A *has a large entry, and ending in a position* (h,t+j) *with a large row number (i.e.* $h \in L$*).*

<u>PROOF</u>. Suppose $(g_0,t+i_0),(g_1,t+i_1),\ldots,(g_\ell,t+i_\ell)$ is the shortest such path. Then the collection of $t+1$ positions

$$(g_k,t+i_k) \qquad k = 0,\ldots,\ell$$

and

$$(j,j) \qquad \text{for } j \neq g_k \ (0 \le k \le \ell-1),\ j \le t$$

is a partial transversal, contradicting the definition of t.

For:

(i) Since $1 \le i_0 < i_1 < \ldots < i_\ell$ all positions are in different columns.

(ii) All positions are in different rows since $j \neq g_k$, and if $g_h = g_k$ for $h < k$ then *either* $k = \ell$, and $g_h = g_k \in L$, so that

$$(g_0,t+i_0),\ldots,(g_h,t+i_h)$$

is a shorter path down, *or* $k < \ell$ and $g_h = g_k = a(g_{k+1},t+i_{k+1})$ so that

$$(g_0,t+i_0),\ldots,(g_h,t+i_h),(g_{k+1},t+i_{k+1}),\ldots,(g_\ell,t+i_\ell)$$

is a shorter path down. Contradiction in both cases.

(iii) All entries are different, for $a(g_{k+1},t+i_{k+1}) = g_k$ so that the entries are the numbers $1,\ldots,t$ and $a(g_0,t+i_0)$, where the latter is in L. □

From the lemma it follows that all vertices of G are in rows $1,\ldots,t$.

It is also clear that $|A_i| = |A_{i-1}| + r$. Hence $|A_r| = r^2$. Therefore $r^2 \leq t$, i.e. $r^2 + r \leq n$. □

REMARKS.

(1) For this presentation of our proof we are indebted to A. Schrijver.

(2) From an observation by J.H. van Lint it follows that equality can occur only for $n = 2$, i.e. for $n \geq 3$ we have $n \geq r(r+1) + 1$.

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[4] VRIES, A.J. de & R.M.A. WIERINGA, *Een ondergrens voor de lengte van een partiële transversaal in een Latijns vierkant*, THE Memorandum 78-02.

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Added in proof

Essentially the same results were obtained by D.E. WOOLBRIGHT [5].

[5] WOOLBRIGHT, D.E., *An* n×n *Latin square has a transversal with at least* $n-\sqrt{n}$ *distinct symbols*, J. Combinatorial Theory (A) 24 (1978) 235-237.