A lower bound for the permanents of certain (0,1)-matrices

by

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ABSTRACT

We prove that the permanent of a $n \times n$ (0,1)-matrix with exactly three 1's in each row and column is at least $6(\frac{4}{3})^{n-3}$ for $n \geq 3$.

KEY WORDS & PHRASES: (0,1)-*matrices*, *permanent*.

1. Let $n \geq 3$ and let G_n be the class of $n \times n$ (0,1)-matrices with all row and column sums equal to 3. We define

$$(1) \qquad g(n) = \min \{per(A);\ A \in G_n\}$$

VAN LINT [3, pp. 58-62] considers the problem of estimating g(n) and determining

$$g = \lim_{n\to\infty} (g(n))^{1/n}.$$

The existence of the above limit is assured by the observation that $g(m+n) \geq g(m)g(n)$, for all $m,n \geq 3$ (cf. VAN LINT [3] p. 59). By evaluating the permanents of certain circulant matrices, Van Lint proved

$$g \leq \xi \approx 1,465...,$$

where ξ is the real zero of $x^3 - x^2 - 1$. The best lower bound for g(n) obtained thus far was

$$g(n) \geq 3(n-1),$$

proved by HARTFIEL and CROSBY [2], cf. HARTFIEL [1]. In this paper we shall prove

$$g(n) \geq 6(\tfrac{4}{3})^{n-3}$$

which implies that $g \geq 4/3$. The proof is elementary and merely consists of evaluating permanents by expansion.

2. For $n \geq 1$, let U_n be the class of $n \times n$ matrices with entries in $\mathbb{N} \cup \{0\}$ and with all row and column sums equal to 3. Let V_n be the class of $n \times n$ matrices that are obtained by subtracting 1 from any non-zero entry of any matrix $A \in U_n$. We define

$$u(n) = \min \{\mathrm{per}(A);\ A \in \mathcal{U}_n\}$$

$$v(n) = \min \{\mathrm{per}(A);\ A \in \mathcal{V}_n\}.$$

For $x \in \mathbb{R}$, let $\lceil x \rceil$ be the least integer $\geq x$.

<u>LEMMA 1</u>. $u(n) \geq \left\lceil \frac{3}{2} v(n) \right\rceil$.

<u>PROOF</u>. Choose $A \in \mathcal{U}_n$ such that per $A = u(n)$ and let a be the first row vector of A. We may assume that $a = (\alpha_1,\alpha_2,\alpha_3,0,\ldots,0)$. Since $\alpha_1 + \alpha_2 + \alpha_3 = 3$, we find that

$$2a = \alpha_1(\alpha_1-1,\alpha_2,\alpha_3,0,\ldots,0) + \alpha_2(\alpha_1,\alpha_2-1,\alpha_3,0,\ldots,0) + \\ + \alpha_3(\alpha_1,\alpha_2,\alpha_3-1,0,\ldots,0).$$

So 2a is the sum of three vectors d_i $(i = 1,2,3)$ with nonnegative entries and hence 2per A is the sum of the permanents of three matrices in the class $\mathcal{V}_n$. This proves that

$$2u(n) = 2\mathrm{per}\ A \geq 3v(n).$$

Since $u(n) \in \mathbb{N}$, this establishes our lemma. □

<u>LEMMA 2</u>. $v(n) \geq \left\lceil \frac{4}{3} v(n-1) \right\rceil$.

<u>PROOF</u>. Choose $A \in \mathcal{V}_n$ such that per $A = v(n)$. We may assume that the first row vector r is either $(1,1,0,0,\ldots,0)$ or $(2,0,0,\ldots,0)$. We distinguish these two cases.

(i) $r = (1,1,0,0,\ldots,0)$. Let B_1 be the matrix obtained from A by deleting r. Then B_1 has the shape c_1c_2B, where c_1 and c_2 are the first two column vectors of B_1 and B is the remaining matrix. By expanding per A with respect to r we find

$$\mathrm{per}(A) = \mathrm{per}(c_1B) + \mathrm{per}(c_2B) = \mathrm{per}(c_3B),$$

where $c_3 = c_1 + c_2$. The sum of its entries is 3 or 4, since the sum of the entries of the first two columns of A is 5 or 6. If s = 3, then $c_3B \in \mathcal{U}_{n-1}$, so by Lemma 1

(2) $$v(n) = \text{per}(A) \geq u(n-1) \geq \left\lceil \frac{3}{2} v(n-1) \right\rceil.$$

If s = 4, then we write $3c_3$ as sum of the vectors d_i (i = 1,...,4) obtained by subtracting 1 from the non-zero entries of c_3. (Compare the proof of Lemma 1). For i = 1,...,4 one has $d_iB \in \mathcal{V}_{n-1}$, since it has one row sum 2 and one column sum 2 and all other row and column sums equal 3. So

(3) $$v(n) = \text{per}(A) \geq \left\lceil \frac{4}{3} v(n-1) \right\rceil.$$

(ii) r = (2,0,0,...,0). Let c be the first column vector of A. We may assume that either c = (2,1,0,...,0) or c = (2,0,...,0). Let B be the matrix obtained from A by deleting the first row and column. In the first case $B \in \mathcal{V}_{n-1}$ and in the second case $B \in \mathcal{U}_{n-1}$. Thus

(4) $$v(n) = \text{per } A \geq 2 \text{ per } B \geq 2 \min\{u(n-1), v(n-1)\} = 2v(n-1),$$

by Lemma 1.

By considering (2), (3) and (4) we find

$$v(n) \geq \left\lceil \frac{4}{3} v(n-1) \right\rceil,$$

proving the Lemma. □

<u>THEOREM</u>. *Let* g(n) *be defined by* (1). *Then*

$$g(n) \geq 6 \cdot (\tfrac{4}{3})^{n-3}.$$

<u>PROOF</u>. Since $\mathcal{G}_n \subset \mathcal{U}_n$, we have that g(n) ≥ u(n) for n ≥ 3. Now v(1) = 2, so by Lemma 2 v(2) ≥ 3, v(3) ≥ 4 and

$$v(n) \geq 4 \cdot (\tfrac{4}{3})^{n-3}$$

for $n \geq 3$. Hence by Lemma 1

$$u(n) \geq 6 \cdot (\frac{4}{3})^{n-3}$$

for $n \geq 3$, thus proving the theorem. □

3. By estimating less roughly, we find the following lower bounds for u(n) and v(n) from Lemmas 1 and 2

n	low. bnd. v(n)	low. bnd. u(n)
1	2	3
2	3	5
3	4	6
4	6	9
5	8	12
6	11	17
7	15	23
8	20	30

For n = 1 to 6 these bounds represent the exact values of u(n) and v(n). The matrix in $\mathcal{U}_n$ for which the permanent equals u(n) is - up to isomorphism - unique for n = 1 to 6. For n = 5,6 these matrices are

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix} ; \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

respectively. It is not too hard to show that v(7) = 15, v(8) = 21 and that u(7) = g(7) = 24, realized by the incidence matrix of the Fano plane.

The question whether or not g > 4/3 still remains open. It also seems hard to generalize the above ideas to matrices with higher row and column sums and obtain a better multiplicative constant than 4/3. Such a generalization is important in connection to the so-called Van der Waerden conjecture that the permanent of a doubly stochastic n×n-matrix is at least $n!\, n^{-n}$.

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