stichting
mathematisch
centrum

MC

AFDELING ZUIVERE WISKUNDE
(DEPARTMENT OF PURE MATHEMATICS)

ZW 112/78 AUGUSTUS

M.R. BEST

BINARY CODES WITH MINIMUM DISTANCE FOUR

Preprint

2e boerhaavestraat 49 amsterdam

Printed at the Mathematical Centre, 49, 2e Boerhaavestraat, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O).

AMS(MOS) subject classification scheme (1970): 05B40, 52A45, 94A10.

Binary codes with minimum distance four*)

by

M.R. Best

ABSTRACT

A new binary code with length ten and minimum Hamming distance four is presented. It contains fourty words, and is therefore optimal. It gives rise to a new sphere packing in ten dimensions.

Besides, it is proved that a binary code of length eleven and minimum distance four cannot contain eighty words.

Finally all seventeen optimal codes with length twelve, constant weight four and minimum distance four are listed.

KEY WORDS & PHRASES: *code, constant weight code, packing, sphere packing.*

---

*) This report will be submitted for publication elsewhere.

§1. INTRODUCTION

Consider the n-dimensional vector space $\{0,1\}^n$ over the field of two elements. The vectors in this space are called *words*. The *(Hamming-) distance* $d_H(x,y)$ between two words x and y is defined as the number of coordinate places in which they differ. The *(Hamming-) weight* $|x|$ of a word x equals its distance to the origin. The *complement* of a word x is defined as $j - x$, where j is the all-one word. Often a word in $\{0,1\}^n$ will be identified with the subset of $\{1,2,\dots,n\}$ of which it is the characteristic vector.

A subset of $\{0,1\}^n$ is called a *(binary) code* (of *length* n). An $[n,d]$-*code* is a code of length n in which any two words have distance at least d . An $[n,d,w]$-*code* is an $[n,d]$-code in which all words have weight w . The maximum number of codewords of an $[n,d]$-code is denoted by $A[n,d]$. An $[n,d]$-code for which this maximum is achieved is called *optimal*. $A[n,d,w]$ and an *optimal* $[n,d,w]$-code are defined similarly.

Two codes are said to be *equivalent* if one can be obtained from the other by a permutation of the coordinate places, followed by a translation in $\{0,1\}^n$.

The *weight distribution* of a code is the sequence $(W_i)_{i=0}^n$ where W_i equals the number of codewords of weight i . The *weight distribution* of a code C *with respect to* a word x is the weight distribution of $C - x$. The *distance distribution* of C is the sequence $(A_i)_{i=0}^n$ where A_i equals the average number of codewords at distance i from a fixed codeword, i.e.

$$A_i = \frac{1}{|C|} \sum_{x \in C} |\{y \mid y \in C \wedge d_H(x,y) = i\}| \quad .$$

Obviously, the distance distribution is an invariant for equivalent codes, and it equals the average of the weight distributions of the code with respect to all different codewords.

In this report we present a $[10,4]$-code with 40 codewords (which is optimal), and prove the non-existence of an $[11,4]$-code with 80 codewords. Besides, we give all possible $[12,4,4]$-codes and try to describe them, and we give three new $[12,4]$-codes with 144 codewords (which might be optimal).

§2. INEQUALITIES ON WEIGHT AND DISTANCE DISTRIBUTIONS OF BINARY CODES

For each $k, n \in \mathbb{N}$, the *(binary) Kravčuk polynomial* K_k *of degree* k is defined by

$$K_k(x) = \sum_j (-1)^j \binom{x}{j}\binom{n-x}{k-j} \qquad \text{for all } x \in \mathbb{R} ,$$

where

$$\binom{x}{j} = \frac{x(x-1)\cdot\ldots\cdot(x-j+1)}{j!} \qquad \text{for all } x \in \mathbb{R} \text{ and } j \in \mathbb{N} .$$

J. DELSARTE, and independently R.J. McELIECE, H.C. RUMSEY jr. and L.R. WELCH (cf. [3]) proved that the following system of inequalities always holds for binary codes of length n and with the distance distribution $(A_i)_{i=0}^n$:

$$(1) \qquad \sum_{i=0}^{n} A_i K_k(i) \geq 0 \qquad \text{for all } k \in \{0,1,\ldots,n\} .$$

If we combine this system with the obvious inequalities

$$A_i \geq 0 \qquad \text{for all } i \in \{0,1,\ldots,n\} ,$$

$$A_0 = 1 ,$$

and

$$A_i = 0 \qquad \text{for all } i \in \{1,2,\ldots,d-1\} ,$$

for $[n,d]$-codes, we find the constraints of a linear programming problem, in which we want to maximize

$$M = \sum_{i=0}^{n} A_i .$$

The maximum value of M yields an upper bound on $A[n,d]$, the *linear programming bound*.

Since we can assume without loss of generality that in an optimal binary code with even minimum distance only words of even weight occur (otherwise replace one bit by an even parity check), we can assume:

$$A_i = 0 \qquad \text{for } i \in \{0,1,\ldots,n\} \ , \quad i \ \text{ odd.}$$

Since $K_{n-k}(i) = K_k(i)$ for i even, we can confine ourselves in (1) to $k \in \{1,2,\ldots,[\frac{1}{2}n]\}$.

Often the linear programming bound can be improved by adding some extra inequalities (cf. e.g. [1] or [2]). The rest of this section will be devoted to the derivation of some of these.

LEMMA 1. *Let* C *be an* $[n,d]$*-code with weight distribution* $(W_i)_{i=0}^{n}$. *Let* $(p_i)_{i=0}^{n}$ *be a sequence of weakly positive real numbers. Then an* $[n-1,d]$*-code* C' *exists with weight distribution* $(W_i')_{i=0}^{n-1}$ *so that*

$$\sum_{i=0}^{n} (n-i)p_iW_i \le n \sum_{i=0}^{n-1} p_iW_i' \ .$$

PROOF. Let C^j be the code consisting of those words of C which have a zero in their j-th position, and let $(W_i^j)_{i=0}^{n}$ be its weight distribution. Then, by counting the number of zeros in the words of weight i , one finds:

$$(n-i)W_i = \sum_{j=1}^{n} W_i^j \ .$$

Hence

$$\sum_{i=0}^{n} (n-i)p_iW_i = \sum_{i=0}^{n} \sum_{j=1}^{n} p_iW_i^j \le n \max_{j=1}^{n} \sum_{i=0}^{n} p_iW_i^j \ .$$

Let k be the value of j for which this maximum is achieved, and let C' be the code consisting of the words of C^k after deleting the k-th position. Then

$$\sum_{i=0}^{n} (n-i)p_iW_i \le n \sum_{i=0}^{n-1} p_iW_i' \ . \qquad \square$$

Substituting $p_i = \delta_{i,w}$ and $p_i = \delta_{i,n-w}$ respectively, we find:

LEMMA 2.

$$A[n,d,w] \le \lfloor \frac{n}{n-w} A[n-1,d,w] \rfloor$$

and

$$A[n,d,w] \le \lfloor \frac{n}{w} A[n-1,d,w-1] \rfloor \quad .$$

These inequalities are generally known as the Johnson bounds (cf. [5]). If we apply lemma 1 for two weights, we obtain:

LEMMA 3. *Let* C *be an* $[n,d]$*-code with weight distribution* $(W_i)_{i=0}^{n}$. *Let* $i,j \in \{0,1,\ldots,n\}$ *and let* p *and* q *be weakly positive real numbers. Then an* $[n-1,d]$*-code* C' *exists with weight distribution* $(W_i')_{i=0}^{n-1}$ *so that*

$$(n-i)pW_i + (n-j)qW_j \le n(pW_i' + qW_j') \quad .$$

The following inequality is often useful for codes with $n-\frac{1}{2}d$ odd.

LEMMA 4. *Let* C *be an* $[n,d]$*-code with* d *even and with weight distribution* $(W_i)_{i=0}^{n}$. *Then*

$$W_{n-\frac{1}{2}d-1} + (A[n,d,\tfrac{1}{2}d+1] - A[n-\tfrac{1}{2}d+1,d,\tfrac{1}{2}d+1])W_{n-\frac{1}{2}d+1} + $$
$$+ A[n,d,\tfrac{1}{2}d+1] \sum_{i=n-\frac{1}{2}d+2}^{n} W_i \le A[n,d,\tfrac{1}{2}d+1] \quad .$$

PROOF. If $\Sigma_{i=n-\frac{1}{2}d+2}^{n} W_i = 1$, then $W_{n-\frac{1}{2}d-1} = W_{n-\frac{1}{2}d+1} = 0$, in which case the lemma is obvious. Therefore, assume that $\Sigma_{i=n-\frac{1}{2}d+2}^{n} W_i = 0$.

If $W_{n-\frac{1}{2}d+1} = 1$, then the code contains a word x of weight $n-\frac{1}{2}d+1$. Each word of weight $n-\frac{1}{2}d-1$ in the code must have its zero positions disjoint from those of x . Hence there can be at most $A[n-\frac{1}{2}d+1,d,\frac{1}{2}d+1]$ codewords of weight $n-\frac{1}{2}d-1$, i.e. $W_{n-\frac{1}{2}d-1} \le A[n-\frac{1}{2}d+1,d,\frac{1}{2}d+1]$, which proves the lemma in this case.

If $W_{n-\frac{1}{2}d+1} = 0$, then the lemma follows from $W_{n-\frac{1}{2}d-1} \le A[n,d,\frac{1}{2}d+1]$. □

Combination of the lemmas 3 and 4 leads to the following inequality, which is useful for codes with $n-\frac{1}{2}d$ even.

LEMMA 5. *Let* C *be an* $[n,d]$-*code with* d *even and with weight distribution* $(W_i)_{i=0}^{n}$. *Then*

$$(\tfrac{1}{2}d+2)W_{n-\frac{1}{2}d-2} + \tfrac{1}{2}d(A[n-1,d,\tfrac{1}{2}d+1] - A[n-\tfrac{1}{2}d,d,\tfrac{1}{2}d+1])W_{n-\frac{1}{2}d} +$$

$$+ (nA[n-1,d,\tfrac{1}{2}d+1] - (\tfrac{1}{2}d+2)A[n-\tfrac{1}{2}d+2,d,\tfrac{1}{2}d+2])W_{n-\frac{1}{2}d+2} +$$

$$+ nA[n-1,d,\tfrac{1}{2}d+1] \sum_{i=n-\frac{1}{2}d+3}^{n} W_i \le nA[n-1,d,\tfrac{1}{2}d+1] .$$

PROOF. If $\Sigma_{i=n-\frac{1}{2}d+3}^{n} W_i = 1$, then $W_{n-\frac{1}{2}d-2} = W_{n-\frac{1}{2}d} = W_{n-\frac{1}{2}d+2} = 0$, in which case the lemma is obvious. Therefore, assume that $\Sigma_{i=n-\frac{1}{2}d+3}^{n} W_i = 0$.

If $W_{n-\frac{1}{2}d+2} = 1$, then $W_{n-\frac{1}{2}d} = 0$ and, by the same argument as in the proof of lemma 4, $W_{n-\frac{1}{2}d-2} \le A[n-\frac{1}{2}d+2,d,\frac{1}{2}d+2]$, which proves the lemma in this case. Therefore we can also assume that $W_{n-\frac{1}{2}d+2} = 0$.

By lemma 3, an $[n-1,d]$-code C' exists with weight distribution $(W'_i)_{i=0}^{n-1}$ so that (take $i = n-\frac{1}{2}d-2$, $j = n-\frac{1}{2}d$, $p = 1$, $q = A[n-1,d,\frac{1}{2}d+1] - A[n-\frac{1}{2}d,d,\frac{1}{2}d+1]$):

$$(\tfrac{1}{2}d+2)W_{n-\frac{1}{2}d-2} + \tfrac{1}{2}d(A[n-1,d,\tfrac{1}{2}d+1] - A[n-\tfrac{1}{2}d,d,\tfrac{1}{2}d+1])W_{n-\frac{1}{2}d} \le$$

$$\le n(W'_{n-\frac{1}{2}d-2} + (A[n-1,d,\tfrac{1}{2}d+1] - A[n-\tfrac{1}{2}d,d,\tfrac{1}{2}d+1])W'_{n-\frac{1}{2}d}) .$$

By lemma 4, the right hand side does not exceed $nA[n-1,d,\frac{1}{2}d+1]$, which proves the lemma. □

As another application of lemma 1 we take $p_i = 1$ for all i. Then it follows:

LEMMA 6. *Let* C *be an* $[n,d]$-*code with weight distribution* $(W_i)_{i=0}^{n}$. *Then*

$$\sum_{i=0}^{n} (n-i)W_i \le nA[n-1,d] .$$

By iterating this procedure (take $p_i = n-i-1$ in lemma 1 and apply lemma 6 on C', etc.), one finds more generally:

LEMMA 7. *Let* C *be an* $[n,d]$*-code with weight distribution* $(W_i)_{i=0}^{n}$. *Let* $k \in \{0,1,\ldots,n\}$. *Then*

$$\sum_{i=0}^{n} \binom{n-i}{k} W_i \le \binom{n}{k} A[n-k,d] .$$

This generalization can be useful for those k for which the k times shortened code of C is (or might be) optimal.

REMARK. In all above lemmas one is allowed to replace the weight distribution $(W_i)_{i=0}^{n}$ by the distance distribution $(A_i)_{i=0}^{n}$, be averaging over all translates of the code C over the codewords.

§3. THE WEIGHT DISTRIBUTION OF A [12,4]-CODE WITH 160 WORDS

It is known that $A[9,4] = 20$ (cf. BEST et al. [2]). Since obviously $A[n,d] \le 2A[n-1,d]$, it follows that $A[10,4] \le 40$, $A[11,4] \le 80$ and $A[12,4] \le 160$. However, nobody ever succeeded in finding codes that realize these bounds. In this section we shall determine the weight distribution of a binary $[12,4]$-code with 160 codewords - if it existed. In the subsequent sections this will be used to prove the non-existence of such a code.

Let C be a $[12,4]$-code with 160 codewords and with distance distribution $(A_i)_{i=0}^{12}$. As remarked in section 2, we can assume that only even weights occur. Then the constraints in the L.P. system are:

$$\begin{aligned}
&A_0 = 1 , \\
&A_4 \ge 0 , \quad A_6 \ge 0 , \quad A_8 \ge 0 , \quad A_{10} \ge 0 , \quad A_{12} \ge 0 , \\
&12A_0 + 4A_4 - 4A_8 - 8A_{10} - 12A_{12} \ge 0 , \\
&66A_0 + 2A_4 - 6A_6 + 2A_8 + 26A_{10} + 66A_{12} \ge 0 , \\
&220A_0 - 12A_4 + 12A_8 - 40A_{10} - 220A_{12} \ge 0 , \\
&495A_0 - 17A_4 + 15A_6 - 17A_8 + 15A_{10} + 495A_{12} \ge 0 ,
\end{aligned}$$

$$792A_0 + 8A_4 \qquad - 8A_8 + 48A_{10} - 792A_{12} \geq 0 \quad ,$$

$$924A_0 + 28A_4 - 20A_6 + 28A_8 - 84A_{10} + 924A_{12} \geq 0 \quad ,$$

whereas we want to maximize

$$M = A_0 + A_4 + A_6 + A_8 + A_{10} + A_{12} \quad .$$

This problem has a unique solution:

$$A_0 = 1 \quad , \quad A_4 = 55 \quad , \quad A_6 = 58\tfrac{2}{3} \quad , \quad A_8 = 55 \quad , \quad A_{10} = 0 \quad ,$$

$$A_{12} = 1 \quad , \quad M = 170\tfrac{2}{3} \quad .$$

However, we can apply the lemma 5 and 6 with $n = 12$ and $d = 4$. We need:

$$A[10,4,3] = 13 \quad ,$$

$$A[11,4,3] = 17 \quad ,$$

$$A[12,4,4] = 51 \quad .$$

(cf. e.g. BEST et al. [2] or MACWILLIAMS & SLOANE [9]). Lemma 5 yields:

$$A_8 + 2A_{10} \leq 51 \quad .$$

Lemma 6 yields:

$$12A_0 + 8A_4 + 6A_6 + 4A_8 + 2A_{10} \leq 960 \quad .$$

Adding both inequalities to the L.P. system, we find the following *unique* optimal solution:

$$A_0 = 1 \quad , \quad A_4 = 51 \quad , \quad A_6 = 56 \quad , \quad A_8 = 51 \quad , \quad A_{10} = 0 \quad ,$$

$$A_{12} = 1 \quad , \quad M = 160 \quad .$$

(We really need both extra inequalities: neither of them is enough to prove the uniqueness of the solution on its own. However, the last three inequalities of the original L.P. system are superfluous.)

The found distribution is the average of the weight distributions with respect to the various codewords. But A_4 , A_8 , A_{10} , and A_{12} are extremal, so all these weight distributions must be identical, i.e. the code is *regular* with weight distribution $(A_i)_{i=0}^{12}$. $A_{12} = 1$ shows that the code is *self-complementary*, i.e. with each word, also its complement is in the code.

Hence the only way to construct the code is finding an optimal [12,4,4]-code, adding the origin and all complements and praying that still 56 words of weight six fit in. In the next section we shall give all optimal [12,4,4]-codes and try to describe them.

§4. OPTIMAL [12,4,4]-CODES

It follows from the Johnson bound that $A[12,4,4] \leq 51$. On the other hand, several [12,4,4]-codes with 51 codewords are known. KALBFLEISH and STANTON ([7]) proved that in each such code all triples of points are covered by one codeword, except for sixteen non-covered triples, constituting the triangles of two octahedron graphs on disjoint sets of six points. It follows easily that each edge of these graphs is covered by exactly four codewords, each other pair of points by exactly five codewords.

We incorporated this information in a computer program, which executed an exhaustive search for all optimal [12,4,4]-codes by backtracking and isomorphism-rejection. Seventeen non-isomorphic solutions were found. They are listed in the appendix.

If we denote the two sets of six points by A and B , and define an (a,b)-*word* as a word that intersects A in exactly a points and B in exactly b points, then the codewords can be divided into five types: (4,0)-, (3,1)-, (2,2)-, (1,3)-, and (0,4)-words. It is a matter of simple counting that only four types of [12,4,4]-codes exist:

number of:	(4,0)-words	(3,1)-words	(2,2)-words	(1,3)-words	(0,4)-words
Type 0:	3	0	45	0	3
Type 1:	2	4	39	4	2
Type 2:	1	8	33	8	1
Type 3:	0	12	27	12	0 .

KALBFLEISH and STANTON ([7]) as well as MILLS ([10]) gave the following construction for codes of type 0:

1. FIRST CONSTRUCTION. We start from the union of two complete graphs A and B on two disjoint sets of six vertices. Colour the edges of A and B in some arbitrary, but fixed way with five colours. Up to isomorphism, this can be done in only one way. Consider all sets of four vertices $\{a,b,c,d\}$ such that $\{a,b\}$ and $\{c,d\}$ are edges of the same colour in A and B respectively. In this way we find 45 quadruples with minimum mutual distance four.

Besides, any of these quadruples is at distance four from any quadruple of vertices of A. We can choose three of these latter quadruples at mutual distance four. In the same way we can choose three quadruples of vertices of B. Altogether we find 51 quadruples at mutual distance at least four, finishing the construction.

Remains the question how many non-isomorphic codes can be found in this way. The initial 45 quadruples determine uniquely the colourings of A and B. The complements of the three quadruples in A consitute a matching. The same holds for the quadruples in B. Since a matching in A or B is either monochromatic, or consists of three differently coloured edges, we have, by symmetry, the following three possibilities:

1.1. Neither of the matchings is monochromatic.

1.2. Only the matching in B is monochromatic.

1.3. Both matchings are monochromatic.

In case 1.1 the two triples of used colours can:

1.1.1. overlap in one colour (this yields code nr. 2 in the appendix);

1.1.2. overlap in two colours (this yields code nr. 1);

1.1.3. be identical (this yields code nr. 5).

Case 1.2 divides into two possibilities:

1.2.1. One of the edges of the matching in A has the same colour as the matching in B (this yields code nr. 3).

1.2.2. None of the edges of the matching in A has the same colour as the matching in B (this yields code nr. 4).

Case 1.3 also gives rise to two possibilities:

1.3.1. The matchings have the same colour (this yields code nr. 8).

1.3.2. The matchings have different colours (this yields code nr. 9).

2. SECOND CONSTRUCTION. Let A and B be two complete graphs on six labeled vertices. There are fifteen different matchings in B. Each matching can be extended in exactly two different ways to a colouring with five colours (i.e. a partitioning of the edges into five matchings; we do not label the colours). Hence there are exactly six ($=15\cdot 2/5$) different colourings of B. If we call a matching and a colouring *incident* if the matching is one of the colours of the colouring, we find that each matching is incident with two colourings, and that each of the six colourings is incident with five matchings. Hence the colourings and matching form the vertices and edges of another complete graph B^* on six vertices. Fix some isomorphism ϕ from A onto B^*. This isomorphism maps vertices of A to colourings of B, and edges of A to matchings of B.

Consider quadruples consisting of the union of an edge a of A and some edge of B in the matching $\phi(a)$. We find in that way 45 ($=15\cdot 3$) quadruples. We claim that these have mutual distance at least four. Take two quadruples. If their intersections with A are disjoint, our claim is obvious. If the intersections with A are identical, say e, then the intersections with B belong to the same matching $\phi(e)$, and are therefore either identical or disjoint. If the intersections with A are neither disjoint nor identical, they are two edges of A which have exactly one vertex, say x, in common. The corresponding matchings in B therefore belong to the same colouring $\phi(x)$, but are not identical. Hence two edges in these matchings are non-identical, proving our claim.

Here too, we can add three quadruples in A as well as three quadruples in B, both at mutual distance four. The complements of these quadruples form two matchings α and β of A and B respectively. Since $\phi^{-1}(\beta)$

is an edge in A , α and β can be chosen in at least two different ways:

2.1. $\phi^{-1}(\beta) \notin \alpha$. This yields code nr. 6.

2.2. $\phi^{-1}(\beta) \in \alpha$. This yields code nr. 7.

These two codes cannot be isomorphic, since the initial 45 quadruples determine the isomorphism ϕ uniquely. Also the two codes are non-isomorphic with the codes of the first construction, since there the edges of A are combined with only five different matchings of B (the colours), whereas in the second construction all fifteen possible matchings of B are used.

REMARK. The 45 initial quadruples in the second construction can also be constructed in the following way. Consider the well known Steiner system $S(5,6,12)$, and fix a block A . There are 45 blocks which intersect A in exactly four points (cf. e.g. MACWILLIAMS & SLOANE [9], page 71). The symmetric differences of these blocks with A yield the required system of 45 quadruples. However, it is not evident that this system is not isomorphic to that in the first construction.

Thus far, we have explained the nine codes of type 0. Below, we describe a way to construct six of the remaining eight codes.

3. THIRD CONSTRUCTION. Consider a code made by the first construction. Suppose this code contains two quadruples Q and R in A and B respectively which are equicoloured (i.e. there is a bijection ϕ from Q onto R so that corresponding edges have the same colour). Without loss of generality we can assume that the quadruples are coloured as follows with the colours α , β , γ , δ and ε :

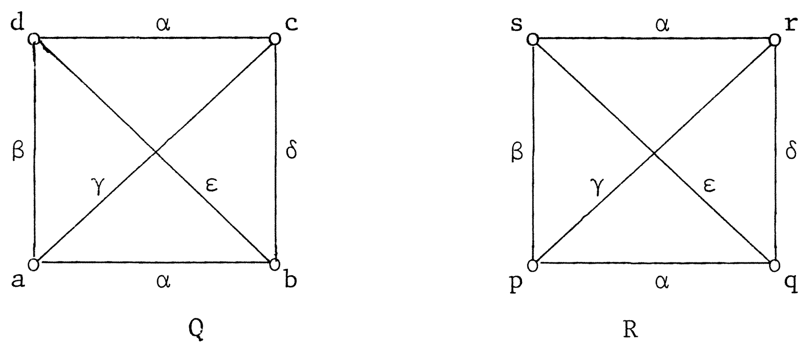

fig. 1

Hence in the code the following eight words occur: $\{a,b,c,d\}$, $\{a,b,r,s\}$, $\{a,c,p,r\}$, $\{a,d,p,s\}$, $\{b,c,q,r\}$, $\{b,d,q,s\}$, $\{c,d,p,q\}$, and $\{p,q,r,s\}$. These words cover exactly those triples which do not contain $\{a,q\}$, $\{b,p\}$, $\{c,s\}$, or $\{d,r\}$. Interchanging in these eight words everywhere the vertices d and r , the same triples will be covered. In this way we construct a new code which still has minimum distance four.

We now trace where the construction can be applied.

The code nrs. 1, 4 and 9 do not contain any equicoloured pair of quadruples.

The code nr. 2, there is one equicoloured pair. The above construction yields code nr. 10.

Code nr. 3 can be transformed in the same way into code nr. 11.

In code nr. 5 even three equicoloured pairs of quadruples occur. They are equivalent under the isomorphism group. The construction invariably yields code nr. 13.

In code nr. 8 also three equicoloured pairs of quadruples exist. They are again equivalent under the automorphism group. But here the construction can be applied three times successively. One obtains respectively code nr. 12, nr. 14, and nr. 15.

We admit the description of the code nrs. 10 - 15 might not be too clarifying. For the code nrs. 16 and 17 the situation is still worse: we have not been able to find any description at all. This is disappointing, since code nr. 16 will turn out to be of special interest.

§5. [12,4]-CODES WITH 160 WORDS

Suppose that a [12,4]-code exists with 160 words. As mentioned already, it must be possible to construct this code from one of the [12,4,4]-codes found in the previous section by adding all complements, the all-zero- and the all-one-word, and then adding as many words of weight six as possible, preserving the minimum distance four.

For codes of type 0 it can be seen that this does not work. There is no possibility to add any (0,6)- or (1,5)-words, since they will always contain a (0,4)-word in the code. Also a (2,4)-word does not fit, since the quadruple in B is coloured with all five colours in any colouring (cf. fig. 1). Hence the hextuple contains a (2,2)-word in the code. Since our [12,4]-code is self-complementary, also (4,2)-, (5,1)-, and (6,0)-words do not fit. Hence the only possible hextuples that can be added are (3,3)-words.

If the constant weight code was made by the first construction, then no (3,3)-words can be added at all: the edges of both triples in A and B are coloured with three colours, which must all be different since otherwise a quadruple consisting of two monochromatic edges would be covered. But only five colours are available.

If the constant weight code was made by the second construction, then several (although not enough) (3,3)-words can be added. Take an arbitrary triple in A . This contains three pairwise adjacent edges, not incident with a same vertex. In B this corresponds to three matchings of which each pair can be extended to a colouring (i.e. it forms a Hamilton-circuit), but which do not belong to a same colouring. It is easily seen that these three matchings together form a $K_{3,3}$. Hence there are exactly two triples which do not cover any edge in one of the matchings. Thus any triple in A can be extended in two ways to a (3,3)-word in the code. This yields in total only $\binom{6}{3}\cdot 2 = 40$ hextuples which can be added. We leave it as an exercise to the reader to show that these words are not disjoint from any quadruple (so that they are not contained in a word of weight eight in the code) and that they have mutual distance at least four.

So far, we have proved that the codes nrs. 1, 2, 3, 4, 5, 8 and 9 are not suited at all to construct a good [12,4]-code in the way described above, while the codes nrs. 6 and 7 only give rise to codes with 144 (= 1 + 51 + 40 + 51 + 1) words. One of the latter was constructed already by JULIN (cf. [6]).

It is not as easy to see how the code nrs. 10 - 17 can be extended. The computer tells us that the only way to construct another [12,4]-code with at least 144 words is to start from code nr. 16. In this case even 44 words of weight six fit, but unfortunately, some of these are at a distance two from each other. These words are situated as follows:

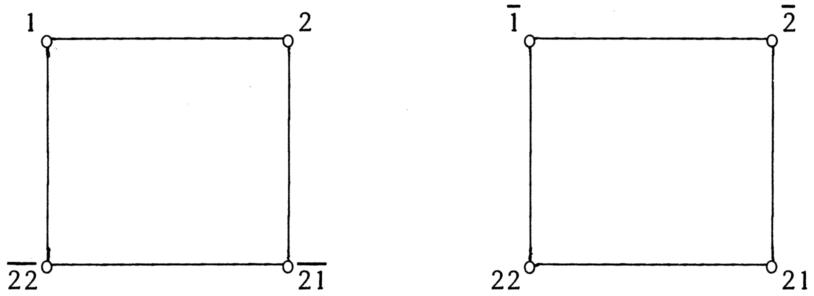

fig. 2

(1 denotes the first word in the list in the appendix, $\overline{1}$ its complement, etc..) Words at distance two are adjacent in the above graph. There are three ways to restore the minimum distance four:

a. omit the words 2 , $\overline{2}$, 22 , and $\overline{22}$;
b. omit the words 1 , $\overline{1}$, 21 , and $\overline{21}$;
c. omit the words $\overline{1}$, 2 , 21 , and $\overline{22}$.

Omission of the words 1 , $\overline{2}$, $\overline{21}$, and 22 yields a code equivalent to the code resulting from c (translation over j).

Recapitulating, we proved that no [12,4]-code exists with 160 words, but we found instead five [12,4]-codes with 144 words. Thus we know: $A[12,4] \leq 159$. In the next section we shall see that even this bound cannot be achieved.

§6. [11,4]-CODES WITH 80 WORDS

We know that $A[11,4] \leq 80$. In this section we shall prove that this bound cannot be achieved.

Let C be an [11,4]-code with 80 words and with distance distribution $(A_i)_{i=0}^{11}$. We can assume that only even weights occur. Then the constraints in the L.P. system are:

$$A_0 = 1 \quad ,$$
$$A_4 \ge 0 \quad , \quad A_6 \ge 0 \quad , \quad A_8 \ge 0 \quad , \quad A_{10} \ge 0 \quad ,$$
$$11A_0 + 3A_4 - A_6 - 5A_8 - 9A_{10} \ge 0 \quad ,$$
$$55A_0 - A_4 - 5A_6 + 7A_8 + 35A_{10} \ge 0 \quad ,$$
$$165A_0 - 11A_4 + 5A_6 + 5A_8 - 75A_{10} \ge 0 \quad ,$$
$$330A_0 - 6A_4 + 10A_6 - 22A_8 + 90A_{10} \ge 0 \quad ,$$
$$462A_0 + 14A_4 - 10A_6 + 14A_8 - 42A_{10} \ge 0 \quad ,$$

whereas we want to maximize

$$M = A_0 + A_4 + A_6 + A_8 + A_{10} \quad .$$

This problem has a unique solution:

$$A_0 = 1 \quad , \quad A_4 = 36\tfrac{2}{3} \quad , \quad A_6 = 29\tfrac{1}{3} \quad , \quad A_8 = 18\tfrac{1}{3} \quad ,$$
$$A_{10} = 0 \quad , \quad M = 85\tfrac{1}{3} \quad .$$

Adding the extra inequalities

$$A_8 + 4A_{10} \le 17 \quad ,$$

and

$$11A_0 + 7A_4 + 5A_6 + 3A_8 + A_{10} \le 440$$

(cf. respectively lemma 4 and lemma 6), we find the following *unique* optimal solution:

$$A_0 = 1 \quad , \quad A_4 = 34 \quad , \quad A_6 = 28 \quad , \quad A_8 = 17 \quad ,$$
$$A_{10} = 0 \quad , \quad M = 80 \quad .$$

The most important conclusion is that there are no words in C at distance 10 from each other. Now consider the code C^* of length twelve consisting of all eighty words of C, each with one extra bit which is always zero, together with all eighty complements. It is easily checked that this is a $[12,4]$-code with 160 words. But in the previous section we proved that such a code does not exist. Hence $A[11,4] \leq 79$. From $A[12,4] \leq 2A[11,4]$ it follows:

THEOREM 1.

$$72 \leq A[11,4] \leq 79$$

and

$$144 \leq A[12,4] \leq 158 \quad .$$

We think that $A[11,4] = 72$ and so $A[12,4] = 144$.

§7. AN OPTIMAL [10,4]-CODE

Thus far, our search for good codes was rather negative, but let us return to the five $[12,4]$-codes with 144 words found in section 5. The first two, resulting from $[12,4,4]$-codes of type 0, where essentially found by JULIN. If they are shortened two times in an appropriate way, $[10,4]$-codes with 38 words are obtained. These were essentially found by GOLAY (cf. [4]).

The other three $[12,4]$-codes with 144 words we found are new. They resulted from the type 3 code nr. 16. Especially the code constructed in a is interesting. There happen to be as many as fourty words in this code which have a zero in the first as well as in the sixth position. Deleting the two bits from these fourty words, we find a $[10,4]$-code with 40 words. Hence:

THEOREM 2.

$$A[10,4] = 40 \quad .$$

The [10,4]-code we find consists of the following 40 words:

0000000000	1001010010	0100011010	1100101110	0011111100
1110000010	1000100011	0011100010	1100010111	0011011011
1101000100	1000011100	0011000101	1010111010	0000111111
1100110000	0111010000	0010101001	1010001111	0111101111
1100001001	0110001100	0010010110	1001101101	1011110111
1011001000	0101101000	0001110001	0110110011	1101111011
1010100100	0101000011	0001001110	0101110110	1110111101
1010010001	0100100101	1111100001	0101011101	1111011110

The automorphism group of the code is the dihedral group of order eight generated by:

$$\alpha = (1,2,4,3)(6,7,9,8)$$

and

$$\beta = (2,3)(5,10)(7,8) \quad .$$

However, if one allows not only permutations, but also translations (in $\{0,1\}^{10}$), then the group acting on the code becomes much larger: it is generated by the affine transformations:

$$\gamma = (1,5,\bar{3},8,6)(2,7,9,\bar{4},10) \quad ,$$

$$\delta = (2,7,\bar{3},8)(5,\bar{6},\overline{10},9) \quad ,$$

and

$$\varepsilon = (5,\overline{10})(6,\bar{6})(7,\bar{7})(8,\bar{8})(9,\bar{9}) \quad .$$

(E.g. δ maps position 2 onto position 7, position 7 onto position 3 with zeros and ones interchanged, position 3 onto position 8 again with zeros and ones interchanged, position 8 onto position 2, etc..) Note that $\alpha = \delta\gamma^3\delta^3\varepsilon\delta\varepsilon$ and $\beta = \delta\varepsilon\delta$.

It is easily checked that the group acts transitively on the positions, and that the transformation $\gamma\delta\gamma$ interchanges the zeros and ones in position 1. It can also be observed that the stabilizer of position 1 is generated by δ and ε and has order 16. Hence the full group has order $2 \cdot 10 \cdot 16 = 320$.

The orbit of the origin under the group has seize $320/8 = 40$, i.e. the group acts transitively on the codewords

If we translate the code over the vector (0011000000) and rearrange the positions according to permutation (2,9,7,8,3,4,6) , the code looks somewhat nicer. It consists of the three dimensional affine subspace

10000 00100
00001 10101
01100 11000
10110 00010
11101 01001
00111 10011
01010 11110
11011 01111

together with four other affine cubes which are obtained by repeatedly executing the "bicyclic" shift (1,2,3,4,5)(6,7,8,9,10) . (Note that each cube, and hence the code, is invariant under the involution $(1,\bar{6})(2,\bar{7})(3,\bar{8})(4,\bar{9})(5,\overline{10})$.)

The code gives rise to a new sphere packing in the ten dimensional Euclidean space. Applying construction A (cf. LEECH & SLOANE [8]) one obtains a packing with contact number 372 $(= 2\cdot 10 + 16\cdot 22)$ and center density $40\cdot 2^{-10} = 5/128 = .0390625$. This last number is a new record.

8. OTHER NEW BOUNDS

Our new [10,4]-code is the basis of an infinite series of codes. By the well known $|u|u+v|$-construction (cf. SLOANE & WHITEHEAD [11] or

MACWILLIAMS & SLOANE [9], chapter 2, section 9), one finds:

THEOREM 3.

$$A[n,4] \geq 5\cdot 2^{n-m-6} \quad \textit{if} \quad n \leq 5\cdot 2^{m} \quad , \quad m \geq 1 \quad .$$

Finally we mention that if one incorporates the extra inequalities derived in section 2 in the L.P. bound, then two other improvements on table 1 in BEST et al. [2] are found:

$$A[21,10] \leq 54$$

and

$$A[22,10] \leq 89 \quad .$$

APPENDIX. THE SEVENTEEN [12,4,4]-CODES WITH 51 WORDS

In this appendix we list the seventeen codes that resulted from the exhaustive search by computer for all optimal [12,4,4]-codes. The words of each code are represented as the rows of an 51×12-matrix.

As remarked in section 4, the non-covered triples form the triangles of two octahedron graphs. The vertices of these graphs have been labeled as follows:

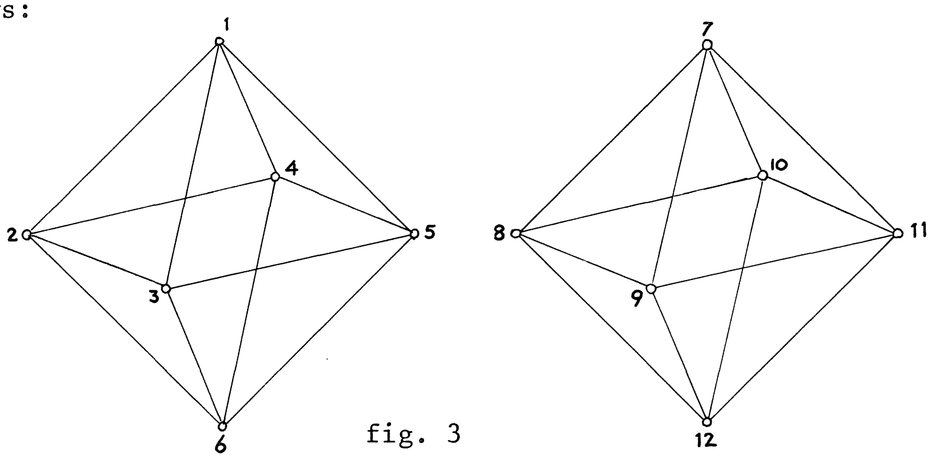

fig. 3

where the labels correspond to the columns of the matrix (numbered from left to right).

If we order the matrices lexicographically with respect to the columns, of each class of isomorphic (= permutation equivalent) matrices the maximal one has been given (modulo the position of the non-covered triples). The seventeen lexicographically maximal matrices have been depicted in decreasing order. The codes nrs. 1 - 9 are of type 0, nrs. 10 - 13 are of type 1, code nr. 14 is of type 2, while the codes nrs. 15 - 17 belong to type 3.

Also during the backtracking, which was done column by column, the maximality was tested regularly. As soon as a new matrix was found, its maximality was tested, and, as a by-product, the automorphism group was printed.

A group is represented by

1. a system of left coset representatives of the stabilizer of the first column of the matrix, and
2. the stabilizer itself, which is represented by

2.1. a system of left coset representatives of the stabilizer of the second column (as a subgroup of the first stabilizer), and

2.2. the stabilizer itself, etc.

This goes on, until the stabilizer is trivial. Identity permutations (as representatives of the stabilizers themselves) have not been printed; a 10 has been printed as an A , an 11 as B , a 12 as a C .

The elements of the groups are the products of coset representatives, from each system one, and in the order they have been given. The order of the group is easily calculated. E.g. the order of the automorphism group of code nr. 7 is $(1+3)(1+3)(1+1) = 32$.

Finally, the hextuples which are at distance four, six, or eight from the quadruples in the code and of which the first coordinate equals zero, have been listed. Obviously, with each hextuple, also its complement has the right distance to all quadruples.

110011000000	Code nr. 1	110011000000	Code nr. 2	110011000000	Code nr. 3
110000110000		110000110000		110000110000	
110000001100	Group:	110000001010	Group:	110000001001	Group:
110000000011		110000000101		110000000110	
101101000000	645231CAB897	101101000000	645231CBA987	101101000000	645231BCA978
101000101000	798BAC132546	101000101000	98C7BA531642	101000101000	------------
101000010010	CBA987654321	101000010100	A7B8C9246135	101000010100	132546CB9A87
101000000101		101000000011	------------	101000000011	
100100100001	Hextuples:	100100100001	132546BC9A78	100100100010	Hextuples:
100100011000		100100011000		100100011000	
100100000110	none	100100000110	Hextuples:	100100000101	none
100010100100		100010100100		100010100100	
100010010001		100010010010	none	100010010001	
100010001010		100010001001		100010001010	
100001100010		100001100010		100001100001	
100001010100		100001010001		100001010010	
100001001001		100001001100		100001001100	
011110000000		011110000000		011110000000	
011000100010		011000100010		011000100001	
011000010100		011000010001		011000010010	
011000001001		011000001100		011000001100	
010100100100		010100100100		010100100100	
010100010001		010100010010		010100010001	
010100001010		010100001001		010100001010	
010010101000		010010101000		010010101000	
010010010010		010010010100		010010010100	
010010000101		010010000011		010010000011	
010001100001		010001100001		010001100010	
010001011000		010001011000		010001011000	
010001000110		010001000110		010001000101	
001100110000		001100110000		001100110000	
001100001100		001100001010		001100001001	
001100000011		001100000101		001100000110	
001010100001		001010100001		001010100010	
001010011000		001010011000		001010011000	
001010000110		001010000110		001010000101	
001001100100		001001100100		001001100100	
001001010001		001001010010		001001010001	
001001001010		001001001001		001001001010	
000110100010		000110100010		000110100001	
000110010100		000110010001		000110010010	
000110001001		000110001100		000110001100	
000101101000		000101101000		000101101000	
000101010010		000101010100		000101010100	
000101000101		000101000011		000101000011	
000011110000		000011110000		000011110000	
000011001100		000011001010		000011001001	
000011000011		000011000101		000011000110	
000000110011		000000110011		000000110011	
000000101101		000000101101		000000101101	
000000011110		000000011110		000000011110	

110011000000	Code nr. 4	110011000000	Code nr. 5	110011000000	Code nr. 6
110000110000		110000110000		110000110000	
110000001001	Group:	110000001100	Group:	110000001100	Group:
110000000110		110000000011		110000000011	
101101000000	231645A7B8C9	101101000000	213465879ACB	101101000000	321654A8C7B9
101000101000	3125648AC79B	101000101000	3516249B7C8A	101000101000	78A9BC124356
101000010100	4561239B7C8A	101000010001	415263A7B8C9	101000010010	A87CB9326154
101000000011	564312C98BA7	101000000110	531642B97CA8	101000000101	------------
100100100010	645231BCA978	100100100100	632541C98BA7	100100100100	153426ABC789
100100011000		100100010010	789ABC123456	100100010001	
100100000101	Hextuples:	100100001001	879ACB213465	100100001010	Hextuples:
100010100001		100010100010	9B7C8A351624	100010100001	
100010010010	none	100010011000	A7B8C9415263	100010011000	000111011001
100010001100		100010000101	B97CA8531642	100010000110	000111100110
100001100100		100001100001	C98BA7632541	100001100010	001011010110
100001010001		100001010100	------------	100001010100	001011101001
100001001010		100001001010	1452367AB89C	100001001001	001101010101
011110000000		011110000000		011110000000	001101101010
011000100100		011000100010	Hextuples:	011000100001	001110011010
011000010001		011000011000		011000010100	001110100101
011000001010		011000000101	none	011000001010	010011011100
010100100001		010100100001		010100100010	010011100011
010100010010		010100010100		010100011000	010101001011
010100001100		010100001010		010100000101	010101110100
010010101000		010010100100		010010100100	010110001110
010010010100		010010010010		010010010010	010110110001
010010000011		010010001001		010010001001	011001001101
010001100010		010001101000		010001101000	011001110010
010001011000		010001010001		010001010001	011010000111
010001000101		010001000110		010001000110	011010111000
001100110000		001100110000		001100110000	011100010011
001100001001		001100001100		001100001001	011100101100
001100000110		001100000011		001100000110	
001010100010		001010100001		001010100010	
001010011000		001010010100		001010010001	
001010000101		001010001010		001010001100	
001001100001		001001100100		001001100100	
001001010010		001001010010		001001011000	
001001001100		001001001001		001001000011	
000110100100		000110101000		000110101000	
000110010001		000110010001		000110010100	
000110001010		000110000110		000110000011	
000101101000		000101100010		000101100001	
000101010100		000101011000		000101010010	
000101000011		000101000101		000101001100	
000011110000		000011110000		000011110000	
000011001001		000011001100		000011001010	
000011000110		000011000011		000011000101	
000000110011		000000110011		000000110011	
000000101101		000000101101		000000101101	
000000011110		000000011110		000000011110	

110011000000	Code nr. 7	110011000000	Code nr. 8	110011000000	Code nr. 9
110000110000		110000110000		110000110000	
110000001010	Group:	110000001001	Group:	110000001001	Group:
110000000101		110000000110		110000000110	
101101000000	623451CBA987	101101000000	214365C98BA7	101101000000	231645897CAB
101000101000	978BCA541632	101000101000	312564978BCA	101000101000	312564978BCA
101000010001	A78BC9236145	101000010100	415263AC8B79	101000010100	456123BCA978
101000000110	------------	101000000011	5134627A8B9C	101000000011	564312CAB897
100100100010	132546C8A9B7	100100100010	624351CB9A87	100100100010	645231ABC789
100100010100	142536B79AC8	100100011000	78A9BC531642	100100011000	798BAC132546
100100001001	153426BCA978	100100000101	879ACB213465	100100000101	879ACB213465
100010100100	------------	100010100100	97B8CA623451	100010100001	987CBA321654
100010011000	12435687A9CB	100010010001	A78BC9153426	100010010010	ACB879654321
100010000011		100010001010	B7A9C8563412	100010001100	BAC798465213
100001100001	Hextuples:	100001100001	C89AB7241635	100001100100	CBA987546132
100001010010		100001010010	------------	100001010001	
100001001100	000111011100	100001001100	135246A87CB9	100001001010	Hextuples:
011110000000	000111100011	011110000000	14253698C7BA	011110000000	
011000100100	001011010011	011000100010	154326C8A9B7	011000100010	none
011000010010	001011101100	011000011000		011000011000	
011000001001	001101010110	011000000101	Hextuples:	011000000101	
010100100001	001101101001	010100100100		010100100001	
010100011000	001110011001	010100010001	none	010100010010	
010100000110	001110100110	010100001010		010100001100	
010010100010	010011011010	010010100001		010010100100	
010010010001	010011100101	010010010010		010010010001	
010010001100	010101001101	010010001100		010010001010	
010001101000	010101110010	010001101000		010001101000	
010001010100	010110001011	010001010100		010001010100	
010001000011	010110110100	010001000011		010001000011	
001100110000	011001001110	001100100001		001100100100	
001100001100	011001110001	001100010010		001100010001	
001100000011	011010000111	001100001100		001100001010	
001010100001	011010111000	001010110000		001010110000	
001010010100	011100010101	001010001001		001010001001	
001010001010	011100101010	001010000110		001010000110	
001001100010		001001100100		001001100001	
001001011000		001001010001		001001010010	
001001000101		001001001010		001001001100	
000110101000		000110101000		000110101000	
000110010010		000110010100		000110010100	
000110000101		000110000011		000110000011	
000101100100		000101110000		000101110000	
000101010001		000101001001		000101001001	
000101001010		000101000110		000101000110	
000011110000		000011100010		000011100010	
000011001001		000011011000		000011011000	
000011000110		000011000101		000011000101	
000000110011		000000110011		000000110011	
000000101101		000000101101		000000101101	
000000011110		000000011110		000000011110	

110011000000	Code nr. 10	110011000000	Code nr. 11	110011000000	Code nr. 12
110000110000		110000110000		110000110000	
110000001010	Group:	110000001001	Group:	110000001010	Group:
110000000101		110000000110		110000000101	
101100100000	351624CBA987	101100100000	351624BCA978	101100100000	32615498C7BA
101001000001	426153BC9A78	101001000010	426153CB9A87	101001001000	421653A87CB9
101000011000	65432187A9CB	101000011000	65432187A9CB	101000010001	624351C8A9B7
101000000110	7A8B9C465213	101000000101		101000000110	789ABC123456
100101000010	897CAB312564	100101000001	Hextuples:	100101000100	98C7BA326154
100100010001	BAC798135246	100100010010		100100011000	A87CB9421653
100100001100	C9B8A7642531	100100001100	000010110111	100100000011	C8A9B7624351
100010100100		100010100100	000010111011	100010100010	------------
100010010010	Hextuples:	100010010001	000101011101	100010010100	1543267BA98C
100010001001		100010001010	001011011010	100010001001	
100001101000	000010110111	100001101000	010000110111	100001100001	Hextuples:
100001010100	000010111011	100001010100	010000111011	100001010010	
100000100011	001011011001	100000100011	010111010001	100000101100	000010101111
011110000000	010000110111	011110000000	011001010110	011110000000	000010111101
011000100100	010000111011	011000100100		011000100100	000101010111
011000010010	011001010101	011000010001		011000011000	001001011011
011000001001		011000001010		011000000011	001110100011
010100101000		010100101000		010100100010	010000101111
010100010100		010100010100		010100010100	010000111101
010100000011		010100000011		010100001001	010111000101
010010100010		010010100001		010010100001	011011001001
010010010001		010010010010		010010010010	011100110001
010010001100		010010001100		010010001100	
010001100001		010001100010		010001101000	
010001011000		010001011000		010001010001	
010001000110		010001000101		010001000110	
001101010000		001101010000		001101000001	
001100001010		001100001001		001100010010	
001100000101		001100000110		001100001100	
001010101000		001010101000		001010110000	
001010010100		001010010100		001010001010	
001010000011		001010000011		001010000101	
001001100010		001001100001		001001100010	
001001001100		001001001100		001001010100	
001000110001		001000110010		001000101001	
000110100001		000110100010		000110101000	
000110011000		000110011000		000110010001	
000110000110		000110000101		000110000110	
000101100100		000101100100		000101110000	
000101001001		000101001010		000101001010	
000100110010		000100110001		000100100101	
000011110000		000011110000		000011100100	
000011001010		000011001001		000011011000	
000011000101		000011000110		000011000011	
000001010011		000001010011		000001001101	
000000101101		000000101101		000000110011	
000000011110		000000011110		000000011110	

110011000000	Code nr. 13	110010100000	Code nr. 14	110010100000	Code nr. 15
110000110000		110001010000		110001010000	
110000001010	Group:	110000001010	Group:	110000001010	Group:
110000000101		110000000101		110000000101	
101100100000	3516249B7C8A	101100100000	624351C8A9B7	101100100000	21436587A9CB
101001001000	426153A8C7B9	101001001000	789ABC123456	101001001000	312564978BCA
101000010100	654321CBA987	101000010001	C8A9B7624351	101000010001	415263A7B8C9
101000000011	789ABC123456	101000000110	------------	101000000110	513462B79AC8
100101000100	9B7C8A351624	100101000100	13524679B8AC	100101000100	624351C8A9B7
100100010010	A8C7B9426153	100100011000	1425367A8B9C	100100011000	789ABC123456
100100001001	CBA987654321	100100000011	1543267BA98C	100100000011	87A9CB214365
100010100001		100011000010		100011000010	978BCA312564
100010011000	Hextuples:	100010010100	Hextuples:	100010010100	A7B8C9415263
100010000110		100010001001		100010001001	B79AC8513462
100001100010	000010101111	100001100001	000010111101	100001100001	C8A9B7624351
100001010001	000010111101	100000110010	000100111011	100000110010	------------
100000101100	000101010111	100000101100	001000110111	100000101100	13524679B8AC
011110000000	001011011001	011110000000	010000101111	011100010000	1425367A8B9C
011000100010	010000101111	011000100100		011010001000	1543267BA98C
011000010001	010000111101	011000011000		011000100100	
011000001100	010111000101	011000000011		011000000011	Hextuples:
010100101000	011001001011	010100100010		010110000100	
010100010100		010100010100		010100100010	000000111111
010100000011		010100001001		010100001001	000001111110
010010100100		010011000001		010011000001	000010111101
010010010010		010010010010		010010010010	000100111011
010010001001		010010001100		010001101000	000111000111
010001100001		010001101000		010001000110	001000110111
010001011000		010001000110		010000110001	001011001011
010001000110		010000110001		010000011100	001100110011
001101000001		001101000001		001110000010	010000101111
001100011000		001100010010		001101000001	010010101101
001100000110		001100001100		001100001100	010101010101
001010110000		001010110000		001010110000	011001011001
001010001010		001010001010		001010000101	011110100001
001010000101		001010000101		001001100010	011111100000
001001100100		001001100010		001001010100	
001001010010		001001010100		001000101001	
001000101001		001000101001		001000011010	
000110100010		000110101000		000110101000	
000110010001		000110010001		000110010001	
000110001100		000110000110		000101110000	
000101110000		000101110000		000101001010	
000101001010		000101001010		000100100101	
000100100101		000100100101		000100010110	
000011101000		000011100100		000011100100	
000011010100		000011011000		000011011000	
000011000011		000010100011		000010100011	
000001001101		000001010011		000010001110	
000000110011		000001001101		000001010011	
000000011110		000000011110		000001001101	

110010100000	Code nr. 16	110010100000	Code nr. 17
110001010000		110001010000	
110000001010	Group:	110000001010	Group:
110000000101		110000000101	
101100100000	624351C8A9B7	101100000001	246135BA7C98
101001001000	78A9BC153426	101001100000	351624C98BA7
101000010001	C89AB7654321	101000011000	4152639CB87A
101000000110	------------	101000000110	564312ABC789
100101000100	13524679B8AC	100101001000	63254187A9CB
100100011000	1425367A8B9C	100100100010	798BAC132546
100100000011	1543267BA98C	100100010100	8A7C9B623451
100011000010		100011000100	9BC78A451623
100010010100	Hextuples:	100010010010	ACB879546132
100010001001		100010001001	B7A9C8264315
100001100001	000000111111	100001000011	C89AB7315264
100000110010	000001111110	100000110001	
100000101100	000011101011	100000101100	Hextuples:
011100000010	000101100111	011100000010	
011010000100	000110011011	011010000001	000000111111
011000110000	000110111100	011000110000	000001101111
011000001001	001001111001	011000001100	000010111011
010110001000	001010011101	010110000100	000100110111
010100100100	001010110110	010100101000	001000111110
010100010001	001100110011	010100010001	010000111101
010011000001	001111000110	010011000010	011111100000
010010010010	010001110101	010010011000	
010001101000	010010101101	010001100100	
010001000110	010100001111	010001001001	
010000100011	010100111010	010000100011	
010000011100	010111010100	010000010110	
001110010000	011000010111	001110001000	
001101000001	011000101110	001101010000	
001100001100	011011001010	001100100100	
001010101000	011101011000	001010100010	
001010000011	011110100001	001010010100	
001001100010	011111100000	001001001010	
001001010100		001001000101	
001000100101		001000101001	
001000011010		001000010011	
000110100010		000110110000	
000110000101		000110000011	
000101110000		000101100001	
000101001010		000101000110	
000100101001		000100011010	
000100010110		000100001101	
000011100100		000011101000	
000011011000		000011010001	
000010110001		000010100101	
000010001110		000010001110	
000001010011		000001110010	
000001001101		000001011100	

REFERENCES

[1] BEST, M.R. & A.E. BROUWER, *The triply shortened binary Hamming code is optimal*, Discrete Math., 17 (1977) 235-245.

[2] BEST, M.R., A.E. BROUWER, F.J. MACWILLIAMS, A.M. ODLYZKO & N.J.A. SLOANE, *Bounds for binary codes of length less than 25*, IEEE Trans. Inform. Theory, IT-24 (1978) 81-93.

[3] DELSARTE, P., *Bounds for unrestricted codes, by linear programming*, Philips Res. Reports, 27 (1972) 272-289.

[4] GOLAY, M.J.E., *Binary coding*, IRE Trans. Inform. Theory, PGIT-4 (1954) 23-28.

[5] JOHNSON, S.M., *A new upper bound for error-correcting codes*, IEEE Trans. Inform. Theory, IT-8 (1962) 203-207.

[6] JULIN, D., *Two improved block codes*, IEEE Trans. Inform. Theory, IT-11 (1965) 459.

[7] KALBFLEISH, J.G. & R.G. STANTON, *Maximal and minimal coverings of* (k-1)-*tuples by* k-*tuples*, Pacific J. Math., 26 (1968) 131-140.

[8] LEECH, J. & N.J.A. SLOANE, *Sphere packings and error correcting codes*, Canad. J. Math., 23 (1971) 718-745.

[9] MACWILLIAMS, F.J. & N.J.A. SLOANE, *The theory of error-correcting codes*, North Holland Publishing Company, Amsterdam 1977.

[10] MILLS, W.H., *On the covering of triples by quadruples*, in: Proc. 5th S-E Conf. Combinatorics, Graph Theory and Computing, Utilitas Math., Winnipeg, 1974, 23-52.

[11] SLOANE, N.J.A. & D.S. WHITEHEAD, *A new family of single-error-correcting codes*, IEEE Trans. Inform. Theory, IT-16 (1970) 717-719.