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INTERPOLATION FOR FRAGMENTS OF THE
PROPOSITIONAL CALCULUS

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Interpolation for fragments of the propositional calculus*

by

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ABSTRACT

We consider the problem of whether the interpolation theorem holds for arbitrary fragments of classical and intuitionistic propositional calculus.

KEY WORDS & PHRASES: *interpolation, classical propositional calculus, intuitionistic propositional calculus, propositional calculus.*

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* This report will be submitted for publication elsewhere.

0. INTRODUCTION

We consider the problem of whether the interpolation theorem holds for arbitrary fragments of the first order propositional calculus. (By "fragment" is meant: fragment of the language, not of the inference rules. An exact definition is given in §1.2.)

We consider, first, classical propositional calculus (§1). There is a positive solution here, due to F. Ville.

Next we turn to intuitionistic propositional calculus (§2). The solution here is negative, i.e. we exhibit a fragment for which interpolation fails.

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Special thanks are due to Prof. G. Kreisel, who encouraged the investigation of the problem considered in §2.

1. CLASSICAL PROPOSITIONAL CALCULUS CPC

1.1. In this section we consider classical (first order) propositional calculus CPC.

Formulas are built up from propositional *parameters* and the constant $\bot$, "false" (or $\top$, "true") by a complete set of connectives, e.g. $\{\wedge, \neg\}$.

<u>NOTATION</u>. "$\vdash$" denotes provability in CPC, and "$\equiv$" denotes provable equivalence between formulas. For a formula A, Par(A) denotes the set of parameters in A.

The *interpolation theorem* says: Suppose $A \vdash B$. Then there is "an interpolant for $A \to B$", i.e. a formula C such that $A \vdash C$ and $C \vdash B$, and $\mathrm{Par}(C) \subseteq \mathrm{Par}(A) \cap \mathrm{Par}(B)$.

<u>REMARK</u>. If $\mathrm{Par}(A) \cap \mathrm{Par}(B) = \emptyset$, then C must be (equivalent to) a propositional constant ($\bot$ or $\top$), and so (accordingly) either $\vdash \neg A$ or $\vdash B$.

This theorem was first (stated and) proved by Craig (for the predicate calculus), using a form of the "Herbrand-Gentzen theorem" ([2], Theorem 5). Another proof (semantic) was given by HENKIN [4]. (An exposition is given in [1], §2.2.20.) Still another proof (proof-theoretical) was given in [8] (actually for the predicate calculus with infinitely long expressions). In this proof one builds up an interpolant for $A \to B$ from a cut-free proof of the sequent $A \mapsto B$, by considering "partitioned sequents". (An exposition is given in [12], §6, where the method is attributed to Maehara.)

1.2. We turn to the problem of interpolation for fragments of CPC, which we now explain.

DEFINITIONS AND NOTATION.

(1) Given a set of propositional connectives $c_1,\ldots,c_k$, the *fragment* $[c_1,\ldots,c_k]$ is the set of all formulas built up (from the parameters) by means of these connectives only.

(2) Any formula $A(p_1,\ldots,p_n)$ of CPC, containing n parameters as shown, defines (in an obvious way) an n-ary *propositional connective* c_A.

(3) Hence any set of formulas $A_1,\ldots,A_k$ determines a *fragment* $\mathcal{F} = [c_{A_1},\ldots,c_{A_k}]$.

(4) We say that *interpolation holds* for a fragment $\mathcal{F}$ if for any formulas A, B in $\mathcal{F}$, if $A \vdash B$ then there is an interpolant for $A \to B$ in $\mathcal{F}$.

PROBLEM 1. Does interpolation hold for arbitrary fragments of IPC?

1.3. EXAMPLES. Consider first some simple fragments. Suppose $A \vdash B$, with A,B in the fragment $\mathcal{F}$ given below. We are looking for an interpolant in $\mathcal{F}$.

(1) $\mathcal{F} = [\wedge]$. In this case, since $A \equiv \bigwedge \mathrm{Par}(A)$ and $B \equiv \bigwedge \mathrm{Par}(B)$, we see that $\mathrm{Par}(B) \subseteq \mathrm{Par}(A)$, and so B itself is an interpolant.

(2) $\mathcal{F} = [\vee]$. Here, similarly, $\mathrm{Par}(A) \subseteq \mathrm{Par}(B)$, so A is an interpolant.

(3) $\mathcal{F} = [\wedge,\vee]$. Put both A and B in disjunctive normal form: say $A = \bigvee_{i<m} A_i$, $B = \bigvee_{j<n} B_j$, with $A_i, B_j \in [\wedge]$. Then it is easy to see that

$$(*) \qquad A \vdash B \iff \forall i < m\ \exists j < n\ \ \mathrm{Par}(B_j) \subseteq \mathrm{Par}(A_i),$$

and so an interpolant for $A \to B$ is found by taking the disjunction of those B_j's for which

$$\exists i < m\ (\mathrm{Par}(B_j) \subseteq \mathrm{Par}(A_i)).$$

(Alternatively, an interpolant is easily found by Maehara's method.)

(4) $\mathcal{F}$ = [→]. Interpolation for this fragment is no longer trivial. A straightforward application of Maehara's method gives an interpolant in [→,∧]. However, a semantic proof of interpolation for [→] was given by VILLARS [13].

We return to the question of a proof-theoretic proof of interpolation for [→] at the end of §2.

1.4. Let us return to the general problem (§1.2) of interpolation for arbitrary fragments of CPC. This was solved positively by F. Ville. Her solution is given in [7] and (better) [8] (chapter 1, Exercises). We repeat it here. It is a corollary of the following theorem.

THEOREM 1. (Ville). *For any formula* A(p) *of* CPC (*perhaps with parameters other than* p):

$$A(p) \vdash A(A(\top)),$$

or equivalently,

$$\exists p A(p) \equiv A(A(\top)).$$

(Note. Here and elsewhere, we introduce second order quantifiers, hence "≡" here means equivalence in second order CPC. However, the formulas denoted schematically by "A", "B", etc., are always first order, and any second order quantifiers are always explicitly displayed.)

PROOF. (We argue informally in classical propositional logic.) Suppose

(1) $\quad A(p).$

Now (classically) $A(\top)$ or $\neg A(\top)$.

CASE 1.

(2) $\quad A(\top).$

So

$$A(\top) \leftrightarrow \top.$$

Hence (substituting $A(\top)$ for $\top$ in (2)): $A(A(\top))$.

CASE 2.

(3) $\neg A(\top)$.

So

(4) $A(\top) \leftrightarrow \bot$.

Also

(5) $p \leftrightarrow \bot$

(Since $p \leftrightarrow \top$ would imply $A(\top)$ by (1), contradicting (3)). Hence (from (4) and (5)):

$$A(\top) \leftrightarrow p.$$

Hence (substituting $A(\top)$ for p in (1)): $A(A(\top))$. □

REMARK. One can also get the "dual" result:

$$A(A(\bot)) \vdash A(p).$$

1.5. Suppose now that

$$A(p,r) \vdash B(r,q).$$

(We consider, for ease of exposition, the simplest non-trivial case, where A and B have one parameter r in common, A has one parameter p not in B, and B has one parameter q not in A.)

Note that for *any* formula F,

(1) $A(F,r) \vdash B(r,q)$

and

(2) $A(p,r) \vdash B(r,F)$.

We see from (1) that we would have an interpolant $A(F,r)$ if we could choose F so that $A(p,r) \vdash A(F,r)$. This is provided by Theorem 1, by taking $F = A(\top,r)$. Hence the following is an interpolant:

$$C' := A(A(\top,r),r).$$

This is in the fragment $[c_A,\top]$. We can now get an interpolant in $[c_A,c_B]$ by substituting $B(r,r)$ for $\top$ in C':

$$C := A(A(B(r,r),r),r),$$

since by (2), $A \vdash B(r,r)$, hence $A \vdash (B(r,r) \leftrightarrow \top)$, and so $A \vdash (C' \leftrightarrow C)$. Further, we still have $C \vdash B$, by (1).

This gives a positive solution to the general problem, since C is in the fragment $[c_A,c_B]$, and hence in any fragment containing A and B.

REMARKS.

(1) An interpolant can also be obtained by the dual procedure (working backwards from the right):

$$B(r,B(r,A(r,r))).$$

(2) In the general case that A and B each contain a number of parameters not in the other, the above procedure is iterated in an obvious way.

2. INTUITIONISTIC PROPOSITIONAL CALCULUS IPC

2.1. We turn to a consideration of analogous problems in the case of (first order) intuitionistic propositional calculus IPC. In this section, the symbols "$\vdash$" and "$\equiv$" are used to denote provability and provable equivalence in IPC, and "$\vdash_c$" and "$\equiv_c$" for the corresponding classical notions.

Once again, the interpolation theorem holds. A (proof-theoretical) proof was given by SCHÜTTE [11]. In fact Maehara's method can be easily modified to apply to the intuitionistic sequent calculus (see [12], §6).

2.2. Again we consider the problem of interpolation for fragments of IPC. As before, we begin with some special cases.

Note first that if A and B are in the fragment [∧,∨] then

$$A \vdash B \iff A \vdash_c B$$

(since (*) in §1.3 also holds with "$\vdash_c$" in place of "$\vdash$"). Hence for the fragments [∧], [∨] and [∧,∨], interpolation holds again, and with the same interpolant as in the classical case.

In the case of the fragment [→] however, Villars' method (§1.3,(4)) clearly does not work for intuitionistic logic; and, again, a straightforward application of Maehara's method provides an interpolant in [→,∧]. However, we *can* get an interpolant in [→], by a refinement of this method (see §2.13).

2.3. Let us again consider the general problem of interpolation for arbitrary fragments (defined as in §1).

PROBLEM 2. (Kreisel). Does interpolation hold for arbitrary fragments of IPC?

2.4. In an attempt to solve Problem 2 in a way analogous to Ville's method for classical logic, we may also consider the following problem.

PROBLEM 3. Does $A(p) \vdash A(A(\top))$ hold in IPC?

The answer here is *negative*, by the following counterexample. Let

$$A(p) = (q \to p) \to q.$$

Then

$$A(\top) \equiv q,$$

so

$$A(A(\top)) \equiv q.$$

But

$$(q \to p) \to q \nvdash q.$$

(Note that $(q \to p) \to q \vdash_c q$. This is just Peirce's law!)

2.5. By contrast, however, we have:

THEOREM 2. (Kreisel). $\exists! p\, A(p) \vdash A(A(\top))$.

PROOF. (We argue informally in second order intuitionistic propositional logic.) Assume

(1) $\qquad A(p)$

and

(2) $\qquad \forall q (A(q) \to (q \leftrightarrow p))$.

We will show that

(3) $\qquad p \leftrightarrow A(\top)$,

and hence (substituting $A(\top)$ for p in (1)) $A(A(\top))$.

To prove the bi-implication (3):

(i) Assume p. Then $p \leftrightarrow \top$. So (substituting $\top$ for p in (1)) $A(\top)$.

(ii) Conversely, assume $A(\top)$. Instantiating (2) with $\top$:

$$A(\top) \to (\top \leftrightarrow p);$$

hence, by modus ponens, $\top \leftrightarrow p$, i.e., p. □

2.6. We can also consider a variant of Problem 3:

PROBLEM 4. (Specker). Given a formula $A(p,\vec{q})$ (where $\vec{q}$ is a list of all parameters in A other than p), is there always some formula $\Phi_A(\vec{q})$ of IPC (in the parameters $\vec{q}$ only) such that

$$A(p,\vec{q}) \vdash A(\Phi_A(\vec{q}),\vec{q}),$$

or, equivalently,

$$\exists p A(p,\vec{q}) \equiv A(\Phi_A(\vec{q}),\vec{q})?$$

EXAMPLE. For $A(p,q) = (q \to p) \to q$ (the counterexample to Problem 3) we can take $\Phi_A(q) = \bot$, or $\neg p$; but no formula $\Phi(q)$ in the *minimal* (i.e. $\bot$-less) fragment will work, since every such formula $\Phi(q)$ is equivalent to either $\top$ or q (by induction on the complexity of Φ), and then (in either case) $A(\Phi(q),q) \equiv q$.

2.7. We now give a *negative solution* to Problem 4. We first define a ternary connective δ.

DEFINITION. $\delta(p,q_1,q_2) := (p \wedge q_1) \vee (\neg p \wedge q_2)$

$$\equiv (p \vee \neg q) \wedge (p \to q_1) \wedge (\neg p \to q_2).$$

This can be read as: p is a *discriminator* for the disjunction $q_1 \vee q_2$.

LEMMA. $\exists p \delta(p,q_1,q_2) \equiv q_1 \vee q_2$.

PROOF. $\Rightarrow$: Trivial.
$\Leftarrow$ (arguing informally in second order intuitionistic propositional logic):

Suppose q_1; then $\delta(\top,q_1,q_2)$, so $\exists p\delta$.
Suppose q_2; then $\delta(\bot,q_1,q_2)$, so $\exists p\delta$.
Hence (by $\vee$-elimination) $q_1 \vee q_2 \vdash \exists p\delta$. □

NOTE. The discriminators in the two cases (in the above proof) are quite different! This gives a clue to the following theorem.

2.8. THEOREM 3. *There is no formula* $\Phi(q_1,q_2)$ *of* IPC *such that*

(1) $\delta(p,\vec{q}) \vdash \delta(\vec{\Phi}(\vec{q}),\vec{q})$.

NOTE. By the lemma, (1) is equivalent to

$$q_1 \vee q_2 \vdash \delta(\Phi(\vec{q}),\vec{q}).$$

In other words, although $q_1 \vee q_2$ implies the existence of a discriminator (by the lemma), such a discriminator is not (first-order) definable from $q_1 \vee q_2$.

PROOF. Suppose, then, that for some formula $\Phi(\vec{q})$ of IPC,

(2) $q_1 \vee q_2 \vdash (\Phi(\vec{q}) \wedge q_1) \vee (\neg\Phi(\vec{q}) \wedge q_2)$.

(i) Substitute $q_1 = \top$ in (2):

$$\vdash \Phi(\top,q_2) \vee (\neg\Phi(\top,q_2) \wedge q_2).$$

Hence, by the *disjunction property* of IPC (see e.g. [12], §6.14(1), with $\Gamma = \emptyset$), either

(3) $\vdash \Phi(\top,q_2)$

or

(4) $\vdash \neg\Phi(\top,q_2) \wedge q_2$.

But (4) implies $\vdash q_2$, which is false; so, by (3):

(5) $\Phi(\top,q_2) \equiv \top$ for all q_2.

(ii) Now substitute $q_2 = \top$ in (2):

$$\vdash (\Phi(q_1,\top) \wedge q_1) \vee \neg\Phi(q_1,\top).$$

Hence, again by the disjunction property:
either

(6) $\vdash \Phi(q_1,\top) \wedge q_1$

or

(7) $\vdash \neg\Phi(q_1,\top)$.

But (6) implies $\vdash q_1$; so, by (7):

(8) $\Phi(q_1,\top) \equiv \bot$ for all q_1.

Now (5) and (8) together yield a contradiction, upon substituting $q_2 = \top$ and $q_1 = \top$. □

Note that in the case of classical logic, a discriminator for $q_1 \vee q_2$ is trivially given by q_1 itself (which is what we get by the method of Theorem 1).

2.9. The following is a modification of Problem 4.

PROBLEM 5. (Scott). Given a formula $A(p,\vec{q})$, are there always formulas $\Phi_1(\vec{q}),\ldots,\Phi_n(\vec{q})$ (for some $n > 0$) such that

(1) $\exists pA(p,\vec{q}) \equiv W_{i=1}^{n} A(\Phi_i(\vec{q}),\vec{q})$?

(For example, with $A = \delta(p,\vec{q})$, we could take $n = 2$, $\Phi_1(\vec{q}) = \top$, $\Phi_2(\vec{q}) = \bot$.)

Here again the answer is no, as shown by the following counterexample (Kreisel). Define

$$A_0(p,q) := q \leftrightarrow (\neg p \vee \neg\neg p).$$

LEMMA. *For each monadic formula* $\Phi(q)$,

(2) $A_0(\Phi(q),q) \equiv q$.

PROOF. First, if $\vdash \neg\Phi(q)$ or $\vdash \neg\neg\Phi(q)$, then (2) is easily seen to hold. Next, consideration of the NISHIMURA lattice [9] shows that the only other possibilities for $\Phi(q)$ are: $\Phi(q) \equiv q$ or $\neg q$ or $\neg\neg q$, and in each of these cases, $A_0(\Phi(q),q) \equiv q \leftrightarrow (\neg q \vee \neg\neg q) \equiv q$. □

So a positive solution to (1) for $A = A_0$ would imply, by (2), $\exists p A_0(p,q) \equiv q$, and hence

(3) $\qquad q \leftrightarrow (\neg p \vee \neg\neg p) \vdash q.$

But if (3) holds, then, by substituting $(\neg p \vee \neg\neg p)$ for q in (3), we get $\vdash \neg p \vee \neg\neg p$, which is false.

REMARK. Define the unary propositional connective $\boxtimes$ by $\boxtimes q := \exists p A_0(p,q)$. Kreisel has proved the following (stronger) results.

(i) $\boxtimes q$ is not equivalent to *any* (first-order) formula of IPC [5].

(ii) There is no positive solution to (1) for $A = A_0$ even when the $\Phi_i(q)$ can be taken in the language extended by $\boxtimes$, i.e. the fragment $[\rightarrow, \wedge, \vee, \bot, \boxtimes]$ (unpublished).

2.10. We now have *two second-order "definitions"* of disjunction $q_1 \vee q_2$:

existential: $\exists p\,\delta(p,q_1,q_2)$

and

universal : $\forall r\,\rho(q_1,q_2,r)$,

where $\rho(q_1,q_2,r) := [(q_1 \rightarrow r) \wedge (q_2 \rightarrow r)] \rightarrow r$ ([10], p.67).

Now consider the valid implication

(1) $\qquad \delta(p,\vec{q}) \rightarrow \rho(\vec{q},r).$

The obvious interpolant for this is $q_1 \vee q_2$, and in fact this is the only interpolant, since any interpolant for (1) must also be one for

$$\exists p\,\delta(p,\vec{q}) \rightarrow \forall r\,\rho(\vec{q},r).$$

However, we can show that $\vee$ is not (first-order) definable from $[\delta,\rho]$; in fact, from a larger fragment:

DEFINITION. F_0 is the fragment $[\delta,\rightarrow,\wedge,\bot]$.

THEOREM 4. *Disjunction is not definable in* F_0.

Two proofs are given below: a proof-theoretical proof (§2.11), and a simple proof, due to A.S. Troelstra, which makes use of finite Heyting lattices (§2.12).

As an immediate corollary, we have (by considering (1) and observing that $\delta,\rho \in F_0$):

COROLLARY. *Interpolation fails for the fragment* F_0.

We conclude §2 with three proofs: the two above-mentioned proofs of Theorem 4, and a proof of interpolation for the fragment $[\rightarrow]$.

2.11. First proof of Theorem 4

The idea, roughly, is to show that (ordinary) disjunction cannot be defined from (the negative fragment and) "exclusive disjunction" (i.e. a disjunction, such as δ, in which each disjunct excludes the other: see (3) below).

S and S' will denote *finite sets* of formulas of the fragment F_0. We interpret such a set *conjunctively*, i.e. read "S" as "$\bigwedge S$".

We will, in fact, show that for any such S, it is impossible that $q_1 \vee q_2 \equiv S$. The proof is by induction on the *length* of S, defined by: $\ell S = \sum_{i=1}^{n} \ell A_i$ (where $S = \{A_1,\ldots,A_n\}$, the A_i's distinct), where ℓA = the length of the formula A as a string at symbols. (What we actually need for the induction to work is: $\ell(A\wedge B) > \ell A + \ell B$, $\ell(A\rightarrow B) > \ell B$, and $\ell(\delta(A,B_1,B_2)) > \max(\ell(A\wedge B_1),\ell(\neg A\wedge B_2))$.)

So assume that $q_1 \vee q_2 \equiv S$, with $S = \{A_1,\ldots,A_n\}$; hence

(1) $\quad q_1 \vee q_2 \vdash S$

and

(2) $\quad S \vdash q_1 \vee q_2$.

We consider various cases, and in each case either get an outright contradiction, or find another set S' with $\ell S' < \ell S$ and $q_1 \vee q_2 \equiv S'$.

CASE 1. One of the A_i's is an *atom*. (We may assume that the atoms of S are among q_1, q_2 and $\bot$, since otherwise we could substitute "$\bot$" for any parameter, and (1) and (2) would still hold.)

CASE 1(a). $A_i = q_1$. Then (from (1)) $q_1 \vee q_2 \vdash q_1$, hence $q_2 \vdash q_1$, which is false.

CASE 1(b). $A_i = q_2$. Similar.

CASE 1(c). $A_i = \bot$. Hence (from (1)) $q_1 \vee q_2 \vdash \bot$, which is false.

CASE 2. One of the A_i's is a *conjunction*, say $A_1 = B_1 \wedge B_2$. Simply replace S by $S' := \{B_1, B_2, A_2, \ldots, A_n\}$. Then $\ell S' < \ell S$ and $q_1 \vee q_2 \equiv S'$.

CASE 3. One of the A_i's is a δ, say $A_1 = \delta(B, C_1, C_2)$. Write $D_1 = B \wedge C_1$ and $D_2 = B \wedge C_2$; so $A = D_1 \wedge D_2$. Note that

(3) $$D_1 \wedge D_2 \vdash \bot.$$

Now from (1),

(4) $$q_1 \vee q_2 \vdash D_1 \vee D_2.$$

We will show that for some j (= 1 or 2):

(5) $$q_1 \vee q_2 \vdash D_j.$$

From (3),

$$q_i \vdash D_1 \vee D_2 \qquad (i = 1,2).$$

Hence (by considering a cut-free proof of the sequent $q_i \mapsto D_1 \vee D_2$):

$$q_i \vdash D_1 \quad \textit{or} \quad q_i \vdash D_2 \qquad (i = 1,2).$$

Now suppose e.g. $q_1 \vdash D_1$ and $q_2 \vdash D_2$. Then by (3), $q_1 \wedge q_2 \vdash \bot$, which is false. Similarly, we cannot have $q_1 \vdash D_2$ and $q_2 \vdash D_1$. Hence, for a *fixed* j (= 1 or 2),

$$q_1 \vdash D_j \quad \textit{and} \quad q_2 \vdash D_j,$$

and hence (5).

(This is the whole point! From the fact that $q_1 \vee q_2$ implies an *exclusive* disjunction ((4), (3)) we can infer that it implies one of the components (5).)

Now let $S' = \{D_j, A_2, \ldots, A_n\}$ (with j as in (4)). Then $q_1 \vee q_2 \vdash S'$, and $S' \vdash S \vdash q_1 \vee q_2$. Hence $q_1 \vee q_2 \equiv S'$. Also $\ell S' < \ell S$.

CASE 4. Assume finally that none of cases 1, 2, 3 applies. Then each A_i has the form $B_i \to C_i$. From (2): consider a cut-free proof of the sequent

(6) $$B_1 \to C_1, \ldots, B_n \to C_n \mapsto q_1 \vee q_2.$$

Let I be the last (non-structural) inference of such a proof. There are two possibilities for I: $\vee$R and $\to$ L.

(i) Assume I = $\vee$R. So (for i = 1 or 2) $S \vdash q_i$.
But then by (1), $q_1 \vee q_2 \vdash S \vdash q_i$, hence $q_1 \vee q_2 \vdash q_i$, which is false.

(ii) Hence I must be $\to$ L. So assume (6) is the conclusion of $\to$ L, with (say) $B_1 \to C_1$ as principal formula. The left upper sequent of I is then:

$$B_1 \to C_1, \ldots, B_n \to C_n \mapsto B_1.$$

Hence $S \vdash B_1$, and hence (by modus ponens, since $S \vdash B_1 \to C_1$)

(7) $$S \vdash C_1.$$

Let $S' = \{C_1, A_2, \ldots, A_n\}$. Then $\ell S' < \ell S$, and (by (7)) $S \vdash S'$. Also (since $C_1 \vdash A_1$) $S' \vdash S$. Hence

$$S' \equiv S \equiv q_1 \vee q_2. \quad \square$$

2.12. Semantic proof of Theorem 4 (Troelstra).

(1) Background. (see e.g. [3], §5.2.)

A *finite Heyting lattice* (or *algebra*) can be defined simply as a finite distributive lattice $L = (X, \wedge, \vee, \leq)$. On such a lattice (with "$\bot$" and "$\top$" denoting the "bottom" and "top" nodes respectively) we can further define the operations

$$x \rightarrow y := \vee \{z \in X \mid x \wedge z \leq y\}$$

and

$$x := \vee \{z \in X \mid x \wedge z = \bot\}.$$

We will actually consider lattices "with assignments" $\mathbb{L} = (L, \phi)$, where ϕ is an assignment of propositional parameters to the nodes of L. Given such a lattice $\mathbb{L}$, we can define a valuation $\nu_{\mathbb{L}}$ of all formulas of IPC to $\mathbb{L}$ in an obvious way.

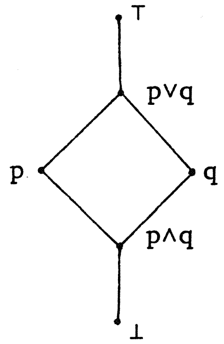
Then IPC is *sound* and *complete* w.r.t. all such lattices, i.e., for any formula A of IPC:

$$\vdash A \quad \text{iff} \quad \nu_{\mathbb{L}}(A) = \top \quad \text{for all finite Heyting lattices } \mathbb{L}\,;$$

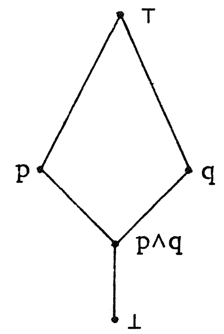
and (hence) for any formulas A and B,

$$A \equiv B \quad \text{iff} \quad \nu_{\mathbb{L}}(A) = \nu_{\mathbb{L}}(B) \qquad \text{for all such } \mathbb{L}.$$

(2) Now consider the lattice

$\mathbb{L}$ =  .

Then the *sublattice* $\mathbb{K}$ generated from {p,q} by {→,∧,⊥,δ} is:

$\mathbb{K}$ =  ,

i.e. the point "p ∨ q" is omitted. This can be seen as follows. First, it is clear that $\mathbb{K}$ is the sublattice generated from {p,q} by {→,∧,⊥}. Secondly, $\mathbb{K}$ is closed in $\mathbb{L}$ under δ. This is because for every x in $\mathbb{K}$, either x = ⊥ or ¬x = ⊥, and so one of the two components of any δ-disjunction must be ⊥. (The point is, once again, that δ is an *exclusive* disjunction!)

REMARK. G. Renardel (unpublished) has used the method of Heyting lattices (and the corresponding Kripke models) to find many more fragments of IPC for which interpolation fails.

2.13. THEOREM. *Interpolation holds for* [→] *in* IPC.

PROOF. We proceed in steps. (1) Suppose D is a formula in [→]. Then $D \equiv \bar{D} \to p$, where p is an atom and $\bar{D}$ is a *finite conjunction* of formulas in [→]. We call p the *leading atom* of D.

(2) Consider a "partitioned sequent" in IPC:

$$S: \quad \Gamma, \Delta \mapsto B$$

where Γ and Δ are finite sets (or sequences) of formulas. An interpolant for S is a formula D such that

$$\Gamma \vdash D \quad \text{and} \quad D,\Delta \vdash B$$

(and, of course, $Par(D) \subseteq Par(\Gamma) \cap Par(\Delta \cup \{B\})$).

(3) Suppose that such a sequent S is provable in IPC, with Γ,Δ and B in $[\to]$. Then it has an interpolant D in $[\to,\wedge]$. (Proof is by induction on the length of a cut-free derivation. This is Maehara's method: see [12], §6.)

D can be written as a conjunction of formulas of $[\to]$:

$$D \equiv \bigwedge_{i=1}^{n} D_i \equiv \bigwedge_{i=1}^{n} (\bar{D}_i \to p_i),$$

with D_i in $[\to]$ and p_i the leading atom of D_i (or possibly $D = \top$).

(4) Furthermore (for D chosen in the ordinary way as above) it can be checked that each p_i (for $i = 1,\ldots,n$) is the leading atom of one of the formulas of Γ. (Again, by induction on the length of a cut-free derivation).

(5) Hence if every formula in Γ has the *same leading atom*, say p_0, then each D_i has leading atom $p_i = p_0$: so $D \equiv \bigwedge_{i=1}^{n} (\bar{D}_i \to p_0)$ (or possibly $D = \top$). But then $D \equiv (D \to p_0) \to p_0$, which is in $[\to]$!

This follows from the equivalence

$$A \to p_0 \equiv ((A \to p_0) \to p_0) \to p_0,$$

by taking $A = \bigvee_{i=1}^{n} \bar{D}_i$.

(6) Suppose finally that $A \vdash B$, with A and B in $[\to]$. Then (taking $\Gamma = A$ and $\Delta = \emptyset$) we get an interpolant for $A \to B$ in $[\to]$ (and, moreover, with the same leading atom as A) (or possibly $= \top$). □

REMARK. This method of proving interpolation for $[\to]$ also works for CPC (see §1.3, example (4)).

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