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A COMPARISON OF BOUNDS OF DELSARTE AND LOVÁSZ

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A comparison of bounds of Delsarte and Lovász*)

by

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ABSTRACT

We compare two upper bound functions: Delsarte's linear programming bound (an upper bound for the cardinality of cliques in association schemes) and Lovász's θ-function (an upper bound for the Shannon capacity of a graph). We show that the two bounds can be treated in a unified fashion. Delsarte's linear programming bound can be generalized to a bound $\theta'(G)$ for the independence number $\alpha(G)$ of arbitrary graphs G, such that $\theta'(G) \le \theta(G)$. On the other hand, if the edge set of G is the union of some classes of a symmetric association scheme, $\theta(G)$ may be calculated by means of linear programming. We show that for such graphs G the product $\theta(G)\cdot\theta(\bar{G})$ is equal to the number of vertices of G.

KEY WORDS & PHRASES: *linear programming bound, association scheme, Shannon capacity, positive semi-definite, codes, constant weight codes.*

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1. INTRODUCTION

The purpose of this note is to compare two upper bound functions, both being bounds for numbers motivated by more or less information-theoretical problems: Delsarte's linear programming bound, an upper bound for the cardinality of cliques in association schemes, and Lovász's θ-function, yielding an upper bound for the Shannon capacity of a graph. The first bound may be conceived of as a bound for the independence number $\alpha(G)$ of certain graphs G, whereas Lovász's bound limits $\alpha(G^k)$, the independence number of the normal product of k copies of G.

We first give, in brief, these two bounds and their theoretical background. (A *graph* is an undirected graph, without loops or multiple edges.)

Association schemes and Delsarte's linear programming bound (Delsarte [2], cf. MacWilliams & Sloane [5]). A pair $(X,\mathcal{R})$, where $\mathcal{R} = (R_0,\dots,R_n)$ is a partition of $X\times X$, is called a *(symmetric) association scheme*, with *intersection numbers* p_{ij}^k $(i,j,k=0,\dots,n)$, if

(1) $$R_0 = \{(x,x) \mid x\in X\};$$

(2) $$R_k^{-1} = \{(y,x) \mid (x,y)\in R_k\} = R_k, \text{ for } k = 0,\dots,n;$$

(3) for all $i,j,k = 0,\dots,n$, and $(x,y)\in R_k$:
$$|\{z \mid (x,z)\in R_i \text{ and } (z,y)\in R_j\}| = p_{ij}^k.$$

So $p_{ij}^k = p_{ji}^k$. We may consider the pairs (X,R_i) as graphs $(i=1,\dots,n)$. (X,R_i) is regular of valency $v_i = p_{ii}^0$ $(v_0 = 1)$. Therefore $p_{ij}^0 = \delta_{ij}v_i$. Let D_i be the adjacency matrix of (X,R_i); D_0 is the identity matrix. Since, by (3), the symmetric matrices $D_0,\dots,D_n$ commute there exists a matrix $P = (P_k^u)_{k,u=0}^n$ such that $P_k^0,\dots,P_k^n$ are the eigenvalues of D_k $(k=0,\dots,n)$, and the eigenvalues $P_0^u,\dots,P_n^u$ of $D_0,\dots,D_n$, respectively, have a common eigenvector (u=0,...,n). We may assume that $P_k^0 = v_k$ for all k. Set

(4) $$Q_k^u = \frac{\mu_u}{v_k} P_k^u,$$

where μ_u is the dimension of the common eigenspace of $D_0,\dots,D_n$ belonging to $P_0^u,\dots,P_n^u$, respectively (u=0,...,n). It can be shown that

(5) $$\sum_{u=0}^{n} P_k^u Q_\ell^u = m\cdot\delta_{k\ell} \quad\text{and}\quad \sum_{k=0}^{n} P_k^u Q_k^v = m\cdot\delta_{uv},$$

where $m = |X|$. So P and $m^{-1}\cdot Q^T$ represent inverse matrices.

Coding theorists are interested in two families of association schemes: the families of Hamming schemes and Johnson schemes, respectively.

Let n and q be natural numbers and let X be the set of vectors of length n, with entries in $\{0,\dots,q-1\}$. Moreover let, for k = 0,...,n:

(6) $$R_k = \{(x,y)\in X\times X \mid d_H(x,y) = k\},$$

where $d_H(x,y)$ denotes the *Hamming distance* between the vectors x and y, i.e. the number of coordinate places in which x and y differ. Let $\mathcal{R} = (R_0,\dots,R_n)$. As can be checked easily $(X,\mathcal{R})$ is a symmetric association scheme; schemes obtained in this way are called *Hamming schemes*.

For Hamming schemes the values of v_k, μ_u and P_k^u are given by:

(7) $$v_k = \binom{n}{k}\cdot(q-1)^k, \quad \mu_u = \binom{n}{u}\cdot(q-1)^u,$$

$$P_k^u = K_k(u) = \sum_{j=0}^{k} (-1)^j (q-1)^{k-j}\binom{u}{j}\binom{n-u}{k-j} = \sum_{j=0}^{k} (-q)^j (q-1)^{k-j}\binom{n-j}{k-j}\binom{u}{j},$$

for k,u = 0,...,n ($K_k(u)$ is the *Krawtchouk polynomial* of degree k in the variable u).

The second family is obtained as follows. Let v and n be natural numbers and let X be the set of 0,1-vectors of length v with exactly n ones ($n \le \frac{1}{2}v$). Moreover, let, for k = 0,...,n,

$$(8) \qquad R_k = \{(x,y) \in X \times X \mid d_J(x,y) = k\},$$

where $d_J(x,y) = \frac{1}{2}d_H(x,y)$ is the *Johnson distance* between x and y. Let $\mathcal{R} = (R_0,\ldots,R_n)$. Then $(X,\mathcal{R})$ is a symmetric association scheme; schemes constructed in this manner are called *Johnson schemes*. Their parameters are:

$$(9) \qquad v_k = \binom{n}{k}\binom{v-n}{n-k}, \quad \mu_u = \binom{v}{u} - \binom{v}{u-1} = \frac{v-2u+1}{v-u+1}\cdot\binom{v}{u},$$

$$P_k^u = E_k(u) = \sum_{j=0}^{k} (-1)^{k-j}\binom{n-j}{k-j}\binom{n-u}{j}\binom{v-n+j-u}{j} = \sum_{j=0}^{k} (-1)^{j}\binom{u}{j}\binom{n-u}{k-j}\binom{v-n-u}{k-j},$$

for k,u = 0,...,n ($E_k(u)$ is the *Eberlein polynomial* of degree 2k in the variable u).

(A third family of symmetric association schemes is formed by *strongly regular graphs*. These are exactly those graphs (X,R_1) such that $(X,\mathcal{R})$ is a symmetric association scheme, where $\mathcal{R} = (R_0,R_1,R_2)$, $R_2 = (X\times X)\setminus(R_0 \cup R_1)$. It follows that the complementary graph of a strongly regular graph is strongly regular.)

The main problem in combinatorial coding theory is to estimate the maximum size of any subset C (a "code") of (the set X in) Hamming and Johnson schemes such that no two elements in C have (Hamming or Johnson) distance less than a given value d. A generalized translation of this problem into the language of association schemes needs the notion of an M-clique; given $0 \in M \subset \{0,\ldots,n\}$ a subset Y of X is an M-*clique* if $(x,y) \in \bigcup_{k\in M} R_k$ for all $x,y \in Y$.

So the coding problem is to determine the maximum cardinality of $\{0,d,d+1,\dots,n\}$-cliques in Hamming and Johnson schemes.

To obtain an upper bound for the size of cliques in a symmetric association scheme $(X,\mathcal{R})$ define, for $Y\subset X$, the *inner distribution* $(a_0,\dots,a_n)$ of Y by

$$(10)\qquad a_k = \frac{|R_k\cap(Y\times Y)|}{|Y|},$$

for $k = 0,\dots,n$; so $a_0 = 1$ and $\sum_{k=0}^n a_k = |Y|$. Moreover, if Y is an M-clique then $a_k = 0$ if $k\notin M$. Delsarte showed that, for the inner distribution of any sybset Y of X, one has

$$(11)\qquad \sum_{k=0}^{n} a_k Q_k^u \geq 0,$$

for $u = 0,\dots,n$. Therefore, for M-cliques Y one has

$$(12)\qquad |Y| \leq \max\{\textstyle\sum_{k=0}^n a_k \mid a_0,\dots,a_n\geq 0;\ a_0=1;\ a_k=0 \text{ for } k\notin M;\ \sum_{k=0}^n a_kQ_k^u \geq 0\} = \min\{\textstyle\sum_{u=0}^n b_u \mid b_0,\dots,b_n\geq 0;\ b_0=1;\ \sum_{u=0}^n b_uP_k^u\leq 0 \text{ for } k\in M\setminus\{0\}\}.$$

The equality in (12) follows from the duality theorem of linear programming. This bound on the size of cliques is called *Delsarte's linear programming bound*. One may apply linear programming techniques to calculate its value - see [1] for applications in coding theory.

The following result of Delsarte shows that the linear programming bound is a sharpening of the Hamming bound in coding theory. Let $(X,\mathcal{R})$ be a symmetric association scheme, with $\mathcal{R} = (R_0,\dots,R_n)$, and let $0\in M\subset\{0,\dots,n\}$ and $\overline{M} = \{0\}\cup(\{0,\dots,n\}\setminus M)$. Then

(13) the product of the linear programming bound for M-cliques and the linear programming bound for $\overline{M}$-cliques is at most $|X|$.

Hence $|Y|\cdot|Z|\le|X|$ for M-cliques Y and $\overline{M}$-cliques Z. Taking $M = \{0,d,d+1,\dots,n\}$ in a Hamming scheme the Hamming bound follows.

The Shannon capacity and Lovász's bound. Lovász [4] introduced, for any graph G, the number $\theta(G)$, which is an upper bound for the "Shannon capacity" $\Theta(G)$. Let $\alpha(G)$ be the maximum number of independent (i.e. pairwise non-adjacent) points in a graph G, and let $G\cdot H$ denote the (normal) product of graphs G and H, i.e. the point set of $G\cdot H$ is the cartesian product of the point sets of G and H, whereas two distinct points of $G\cdot H$ are adjacent iff in both coordinate places the elements are adjacent or equal. G^k denotes the product of k copies of G.

Shannon [9] introduced the following number for graphs G:

(14) $$\Theta(G) = \sup_k \sqrt[k]{\alpha(G^k)} = \lim_{k\to\infty} \sqrt[k]{\alpha(G^k)} ,$$

which is called the *Shannon capacity* of G.

If one considers the points of G as letters in an alphabet, two points being adjacent iff they are "confoundable", then $\alpha(G^k)$ may be interpreted as the maximum number of k-letter messages such that any two of them are inconfoundable in at least one coordinate place.

Since $\alpha(G)^k\le\alpha(G^k)$, it follows that $\alpha(G)\le\Theta(G)$. Equality does not hold in general; e.g. $\alpha(C_5) = 2$, whereas $\alpha(C_5^2) = 5 \le \Theta(C_5)^2$. Lovász showed that, in fact, $\Theta(G) = \sqrt{5}$. Actually, he gave a general upper bound for $\Theta(G)$ as follows.

Let $G = (V,E)$ be a graph, with vertex set $V = \{1,\dots,n\}$, and define

(15) $$\theta(G) = \min\{\ell ev A \mid A=(a_{ij}) \text{ is a symmetric } n\times n\text{-matrix such that } a_{ij} = 1 \text{ if } \{i,j\}\notin E\},$$

where $\ell ev A$ denotes the largest eigenvalue of A. Now, if $\alpha(G) = k$, each matrix A satisfying the conditions mentioned in (15) has a $k\times k$ all-one principal submatrix (with largest eigenvalue k), hence $\ell ev A\geq k$. Therefore $\alpha(G) \leq \theta(G)$. Since, as Lovász proved, $\theta(G\cdot H) = \theta(G)\cdot\theta(H)$ for all graphs G and H, one has $\alpha(G^k)\leq\theta(G)^k$, which yields the stronger inequality $\Theta(G)\leq\theta(G)$ (Haemers [3] showed the existence of graphs G with $\Theta(G)<\theta(G)$). Moreover Lovász showed

(16) $$\theta(G) = \max\{\textstyle\sum_{i,j} b_{ij} \mid B=(b_{ij}) \text{ is an } n\times n \text{ positive semi-definite matrix, with } \mathrm{Tr}B=1, \text{ and } b_{ij}=0 \text{ whenever } \{i,j\}\in E\}.$$

So $\theta(G)$ may be considered as both a maximum and a minimum, which makes the function θ easier to handle. Lovász found, inter alia, for graphs G (with n points):

(17) $\theta(G)\cdot\theta(\overline{G}) \geq n$ (where $\overline{G}$ is the complementary graph), with equality if G is vertex-transitive;

and

(18) $\theta(G) \leq \dfrac{-n\lambda_n}{\lambda_1-\lambda_n}$ if G is regular (λ_1 and λ_n being the largest and smallest eigenvalues of the adjacency matrix of G), with equality if G is edge-transitive.

A consequence of (18) is: let $v\geq 2n$ and let $K(v,n)$ be the graph whose vertices are the n-subsets of some fixed v-set, two vertices being adjacent iff they are disjoint; such graphs are called *Kneser-graphs*. Then

(19) $\theta(K(v,n)) = \binom{v-1}{n-1}$

(by (18) it is sufficient to calculate the eigenvalues of $K(v,n)$), generalizing the Erdős-Ko-Rado theorem, which says that $\alpha(K(v,n)) = \binom{v-1}{n-1}$.

The theories of Delsarte and Lovász appear to have certain common characteristics, such as bounding cliques or independent sets in graphs, using eigenvalue-techniques on matrices determined by graphs, yielding relations between a graph and its complement, and being applicable to allied structures such as "constant weight codes" and Kneser-graphs. The purpose of this note is to go further into this relationship.

Clearly, Delsarte's linear programming bound may be conceived of as an upper bound for $\alpha(G)$ for graphs G whose edge set is the union of some classes R_i of a symmetric association scheme $(X,\mathcal{R})$. We show that Delsarte's bound can be extended to a bound $\theta'(G)$ for $\alpha(G)$ for arbitrary graphs G; the description of $\theta'(G)$ has many features in common with Lovász's $\theta(G)$. It will follow that $\theta'(G) \leq \theta(G)$ (in general $\theta'(G) \neq \theta(G)$). On the other hand, if the edge set of G is the union of some classes of a symmetric association scheme $(X,\mathcal{R})$ the number $\theta(G)$ may be calculated by means of a linear program obtained from (12) by dropping the nonnegativity constraints for $a_0,\ldots,a_n$. It follows that also for such graphs G one has $\theta(G)\cdot\theta(\overline{G}) = |X|$ (cf. (13) and (17)).

2. A COMPARISON OF THE BOUNDS OF DELSARTE AND LOVÁSZ

First recall the following strong form of the duality theorem of linear programming. Let C and D be closed convex cones in $\mathbb{R}^k$ and $\mathbb{R}^m$, respectively, with dual cones C^* and D^*, respectively (that is, C^* consists of all vectors in $\mathbb{R}^k$ having a nonnegative inner product with each element of C). Let M be a real-valued m×k-matrix, and let $c\in \mathbb{R}^k$ and $d\in \mathbb{R}^m$. Then

$$\text{(20)} \qquad \max\{cx \mid x\in C;\ d-Mx\in D\} = \min\{yd \mid y\in D^*;\ yM-c\in C^*\},$$

provided that the object sets are nonempty and closed. Furthermore, notice that the closed convex cone of all real-valued symmetric positive semi-definite n×n-matrices, conceived of as n^2-vectors, has as dual cone the set of real-valued n×n-matrices U such that $y^TUy\geq 0$ for all real n-vectors y. (So symmetric matrices in the dual cone are positive semi-definite.) For convenience, we use the following inner product notation for n×n-matrices $A = (a_{ij})$ and $B = (b_{ij})$:

$$\text{(21)} \qquad A*B = \sum_{i,j=1}^{n} a_{ij}\cdot b_{ij},$$

that is, $A*B = \mathrm{Tr}(A^TB)$. So $A*I = \mathrm{Tr}A$ and $A*J = \sum_{i,j=1}^{n} a_{ij}$.

Let G be a graph, with point set $\{1,\ldots,n\}$. Lovász defined

$$\text{(22)} \qquad \theta(G) = \max\{\textstyle\sum_{i,j} b_{ij} \mid B=(b_{ij}) \text{ is a symmetric positive semi-definite } n\times n\text{-matrix with } \mathrm{Tr}B=1, \text{ and } b_{ij}=0 \text{ if } \{i,j\}\in E\} = \min\{\ell ev A \mid A=(a_{ij}) \text{ is a symmetric } n\times n\text{-matrix with } a_{ij}=1 \text{ if } \{i,j\}\notin E\}.$$

Now define $\vartheta'(G)$ as follows.

(23) $$\vartheta'(G) = \max\{\textstyle\sum_{i,j} b_{ij} \mid B=(b_{ij}) \text{ is a nonnegative symmetric positive semi-definite } n\times n\text{-matrix with } \mathrm{Tr}B=1, \text{ and } b_{ij}=0 \text{ if } \{i,j\}\in E\},$$

so the difference with (22) is the restriction of the range for the maximum to nonnegative matrices B.

THEOREM 1. $\alpha(G) \le \vartheta'(G) \le \vartheta(G)$.

PROOF. Clearly, $\vartheta'(G) \le \vartheta(G)$. Suppose $Y\subset\{1,...,n\}$ is an independent set with $\alpha(G)=k$ elements. Define $b_{ij} = 1/k$ if $i,j\in Y$, and $b_{ij} = 0$, otherwise. Then $B=(b_{ij})$ is nonnegative and positive semi-definite with trace 1, and $b_{ij}=0$ if $\{i,j\}\in E$. Furthermore, $\sum_{i,j} b_{ij} = k$. Hence $\alpha(G) = k \le \vartheta'(G)$. □

THEOREM 2. $\vartheta'(G) = \min\{\ell ev A \mid A=(a_{ij}) \text{ is a symmetric } n\times n\text{-matrix with } a_{ij}\ge 1 \text{ if } \{i,j\}\notin E\}$.

PROOF. By definition

(24) $$\vartheta'(G) = \max\{B*J \mid B=(b_{ij}) \text{ is a symmetric positive semi-definite } n\times n\text{-matrix such that: } B*I=1,\ B*F_{ij}=0 \text{ for } \{i,j\}\in E, \text{ and } B*F_{ij}\ge 0 \text{ for } \{i,j\}\notin E\},$$

where F_{ij} is the $n\times n$-(0,1)-matrix with only ones in the positions (i,j) and (j,i). From the above-mentioned form of the linear programming duality theorem it follows that this maximum equals

(25) $\min\{\lambda \in \mathbb{R} \mid M=(m_{ij})$ is a symmetric $n\times n$-matrix; $m_{ij}\leq 0$ if $\{i,j\}\notin E$; $\lambda I+M-J$ is positive semi-definite$\}$.

Putting $A = J-M$, one has, since, for symmetric A, the largest eigenvalue of A is equal to the minimum value of λ such that $\lambda I-A$ is positive semi-definite,

(26) $\theta'(G) = \min\{\ell ev A \mid A=(a_{ij})$ is a symmetric $n\times n$-matrix such that $a_{ij}\geq 1$ if $\{i,j\}\notin E\}$. □

Since the largest eigenvalue of a matrix is not increased by decreasing diagonal elements we may suppose that the minimum is attained by some A with ones on the diagonal.

We secondly prove that for graphs derived from symmetric association schemes $\theta'(G)$ coincides with Delsarte's linear programming bound.

Let $(X,\mathcal{R})$ be a symmetric association scheme, with $\mathcal{R} = (R_0,\ldots,R_n)$, and let $0\in M\subset\{0,\ldots,n\}$. Let $G=(X,E)$ be the graph with $E = \bigcup_{i\notin M} R_i$. Clearly, M-cliques in the association scheme coincide with independent sets in G.

THEOREM 3. $\theta'(G)$ *is equal to the linear programming bound for* M-*cliques in* $(X,\mathcal{R})$.

PROOF. The linear programming bound is, by definition (cf. (12))

(27) $\max\{\sum_{k=0}^{n} a_k \mid a_0,\ldots,a_n\geq 0;\ a_0=1;\ a_k=0$ for $k\notin M$; $\sum_{k=0}^{n} a_k Q_k^u \geq 0$ for $u=0,\ldots,n\}$.

Let $a_0,\ldots,a_n$ attain this maximum, and put

$$(28)\qquad (b_{ij}) = B = \sum_{k=0}^{n} \frac{a_k}{m\cdot v_k} D_k,$$

where m, v_k and D_k are as in section 1. Then B satisfies the conditions mentioned in (23); B is positive semi-definite since, by the commutativity of $D_0,\ldots,D_n$, the matrix B has eigenvalues

$$(29)\qquad \sum_{k=0}^{n} \frac{a_k}{m\cdot v_k} P_k^u = \sum_{k=0}^{n} \frac{a_k}{m\cdot \mu_u} Q_k^u,$$

for u=0,...,n. Since $D_k * J = v_k \cdot m$, it follows that $B*J = \sum_{i,j} b_{ij} = \sum_k a_k$. Therefore, the linear programming bound is at most $\theta'(G)$.

To prove the converse, let $b_0,\ldots,b_n$ attain the minimum in (12), and let $\lambda = \sum_u b_u$. Define

$$(30)\qquad A = \lambda I - \sum_{k,u=0}^{n} \frac{b_u}{\mu_u} Q_k^u . D_k + J = \lambda I - \sum_{k=0}^{n} \Big(\sum_{u=0}^{n} \frac{b_u}{\mu_u} Q_k^u - 1\Big)\cdot D_k .$$

Since $\lambda I - A$ has eigenvalues

$$(31)\qquad \sum_{k=0}^{n} \Big(\sum_{u=0}^{n} \frac{b_u}{\mu_u} Q_k^u - 1\Big) . P_k^v = \sum_{u=0}^{n} \frac{b_u}{\mu_u}\cdot m\cdot \delta_{uv} - \delta_{k0} \geq 0$$

(v=0,...,n) the matrix A has largest eigenvalue at most λ. Furthermore, by (4) and (12), $a_{ij} \geq 1$ if $\{i,j\} \notin E$. Therefore, the minimum in (12) is at least the minimum of theorem 2, or the linear programming bound is at least $\theta'(G)$. □

If $(X,\mathcal{R})$ is a Johnson scheme with n classes (cf. Delsarte [2]) and $M=\{0,\ldots,n-1\}$, then $G=K(v,n)$. As Lovász showed that $\theta(K(v,n))=\binom{v-1}{n-1}$, also Delsarte's linear programming bound yields the Erdös-Ko-Rado theorem.

Using techniques similar to those used in the proof of theorem 3 one proves for symmetric association schemes $(X,\mathcal{R})$ and graphs G related as mentioned before theorem 3:

THEOREM 4. $\theta(G) = \max\{\sum_{k=0}^{n} a_k \mid a_0=1;\ a_k=0 \text{ for } k\notin M;\ \sum_{k=0}^{n} a_k Q_k^u \geq 0 \text{ for } u=0,\ldots,n\} =$
$\min\{\sum_{u=0}^{n} b_u \mid b_0,\ldots,b_n \geq 0;\ b_0=1;\ \sum_{u=0}^{n} b_u P_k^u = 0 \text{ for } k\in M\setminus\{0\}\}$.

PROOF. Similar to the proof of theorem 3. □

So for graphs derived from symmetric association schemes there is an easier way to calculate the θ-value. As a generalization of Delsarte's result (13) one has

THEOREM 5. *Let the edge set* E *of the graph* G = (V,E) *be the union of some classes of a symmetric association scheme. Then* $\theta(G)\cdot\theta(\overline{G}) = |X|$.

PROOF. Lovász proved that for all graphs $G: \theta(G)\cdot\theta(\overline{G}) \geq |X|$.
Now suppose E is the union of some classes of an association scheme, as described before theorem 3. Then by theorem 4, $\theta(G) = \sum_k a_k$, for some $a_0,\ldots,a_n$, where $a_0=1$, $a_k=0$ for $k\notin M$ and $\sum_k a_k Q_k^u \geq 0$ for $u=0,\ldots,n$. Set

$$(32)\qquad b_u = \frac{\sum_{k=0}^{n} a_k Q_k^u}{\theta(G)}.$$

Then $b_0,\ldots,b_n \geq 0$ and $b_0=1$; furthermore, for $k\notin M$ (cf. (5)):

$$(33)\qquad \sum_{u=0}^{n} b_u P_k^u = \frac{1}{\theta(G)}\cdot\sum_{u,\ell} a_\ell Q_\ell^u P_k^u = \frac{1}{\theta(G)}\cdot\sum_{\ell} a_\ell\cdot m\cdot\delta_{k\ell} = \frac{a_k\cdot m}{\theta(G)} = 0,$$

so $b_0,\dots,b_n$ satisfy the conditions mentioned in the minimum-side of theorem 4, with $\overline{G}$ instead of G. Also

$$(34)\qquad \sum_{u=0}^{n} b_u = \frac{1}{\theta(G)}\cdot\sum_{k,u} a_k Q_k^u = \frac{1}{\theta(G)}\cdot\sum_k a_k\cdot\sum_u Q_k^u = \frac{1}{\theta(G)}\cdot\sum_k a_k\cdot m\cdot\delta_{k0} = \frac{|X|}{\theta(G)}\,.$$

Since, by theorem 4, $\sum_u b_u \geq \theta(\overline{G})$ we have shown that $\theta(G)\cdot\theta(\overline{G}) \leq |X|$. □

Because there are (many) strongly regular graphs that are not vertex-transitive (cf. Seidel [8]) theorem 5 is not included in (17).

M.R. Best found the following example of a graph G with $\theta'(G) < \theta(G)$. The points of G are all vectors in $\{0,1\}^6$, two vectors being adjacent iff their Hamming distance is at most 3 (so the edge set is the union of some classes of a Hamming scheme). Then $\theta'(G) = 4$ whereas $\theta(G) = \frac{16}{3}$.

After completing this research I learnt that partially similar results have been obtained, independently, by McEliece, Rodemich & Rumsey [7] (cf. [6]). Their functions $\alpha_L(G)$ and $\theta_L(G)$ are equal to $\theta'(G)$ and $\theta(G)$, respectively.

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