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DISJOINTNESS AND QUASIFACTORS IN TOPOLOGICAL DYNAMICS

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Disjointness and quasifactors in topological dynamics *)

by

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ABSTRACT

We present some results about the disjointnessrelations between factors, extensions and quasifactors of minimal topological transformationgroups (ttg's) with arbitrary phase group and compact Hausdorff phase space. Also we characterize some families of minimal ttg's whose members are disjoint from a certain collection of minimal ttg's. For instance we prove, that if the phase group is strongly amenable then the minimal ttg's, which are disjoint from every PI-ttg, are just the weakly mixing ones.

KEY WORDS & PHRASES: *topological transformation group, disjointness, quasifactor, highly proximal extension.*

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*) This report will be submitted for publication elsewhere.

§0. INTRODUCTION

Several authors ([4],[5],[7],[9],[10] and [16]) used quasifactors in the structure theory of minimal topological transformation groups (from now on the term topological transformation group(s) will be abbreviated by ttg('s)). In this paper we shall make an extensive use of the properties of quasifactors in the determination of disjointness classes.

The first section will be devoted to some basic statements about quasifactors and extensions in relation to disjointness. It contains a result important for the rest of this paper(1.5). It states, that given a distal extension $\phi: X \longrightarrow Y$, every quasifactor of X which is disjoint from Y is distal. In section 2 we characterize in terms of quasifactors the collection $\mathcal{K}^{\perp}$ of all minimal ttg's that are disjoint from every member of a certain family $\mathcal{K}$ of minimal ttg's. For nice families $\mathcal{K}$ this gives us relatively smooth descriptions for $\mathcal{K}^{\perp}$ (2.9, 2.10, 2.12, 2.13, 5.4). Section 3 provides a generalization of [11], 4.5, where KEYNES states, that for abelian phase groups two minimal ttg's are disjoint whenever $\mathfrak{G} = uM$ (3.4). In section 4 we characterize (without countability assumptions) the incontractible weakly mixing minimal ttg's as the incontractible minimal ttg's without distal factor; this result is extended in the fifth section to the minimal ttg's disjoint from every PI-ttg. The last section deals with common factor problems, and results in the observation, that for arbitrary phase groups and minimal ttg's X and Y, such that one of them is regular and one of them is in $\mathcal{D}^{\perp\perp}$ ($\mathcal{D}$ is the family of minimal distal ttg's), the disjointness of X and Y is equivalent to having no nontrivial common factor.

We will now introduce some basic notions and notations. For a more comprehensive treatment we refer to [4], [16] and with more notational resemblance [9]. Under a *ttg* we shall understand here an action of T on X, with T an arbitrary topological group and X a compact Hausdorff space. The action of T on X is a continuous mapping $(t,x) \longmapsto tx : T \times X \longrightarrow X$ such that $(ts)x = t(sx)$ and $ex = x$ for all $s,t \in T$ and $x \in X$ (e is the unity of T). Mostly we shall consider T to be understood and denote the ttg by its phase space only. A *homomorphism* of ttg's $\phi: X \longrightarrow Y$ is a continuous map which commutes with the action of T on X and Y i.e. $\phi(tx) = t\phi(x)$ for all $t \in T$

and $x \in X$. If ϕ is a homomorfism onto we shall call ϕ an *extension* and also we shall call X an extension of Y if there is an extension $\phi: X \longrightarrow Y$.

Let (T,M) be the universal minimal ttg for T ([3]), i.e. M is minimal and an extension of every minimal ttg with phase group T. Then M is isomorphic to a maximal left ideal in its enveloping semigroup E(M), it is a semigroup itself and accordingly it acts on every minimal ttg (T,X). Denote the collection of idempotents in M by J, then $M = \cup \{vM \mid v \in J\}$ and every vM is a subgroup in M, while $\{vM \mid v \in J\}$ is a partition of M. For $u \in J$ we can define a nice compact T_1-topology on uM, the so-called τ-topology. In uM we consider the stabilizer for (T,X) with respect to the point x_0, chosen such that $ux_0 = x_0$: $\mathcal{G}(X,x_0) = \{\alpha \in uM \mid \alpha x_0 = x_0\}$. It is called the *Ellisgroup* of (T,X,x_0) and it plays an important role in the structure theory of minimal ttg's ([7],[9]) The Ellisgroups are τ-closed subgroups of uM and moreover every τ-closed subgroup of uM can be obtained as an Ellisgroup.

Every ttg (T,X) induces a *hypertransformation group* $(T,2^X)$ where $2^X = \{A \subseteq X \mid A = \bar{A} \neq \emptyset\}$ (for an explicit treatment of possible topologies on 2^X see [13]. We will use the Vietoris topology on 2^X). Since X is compact Hausdorff, 2^X is compact Hausdorff and the action of T on 2^X defined by $(t,A) \longmapsto tA = \{ta \mid a \in A\}$ is continuous ([12]). Every homomorphism $\phi: X \longrightarrow Y$ induces a homomorphism $\phi^*: 2^X \longrightarrow 2^Y$ defined by $\phi^*(A) = \phi[A]$. If ϕ is open there are also homomorphisms $\overleftarrow{\phi}^{*}: Y \longrightarrow 2^X$ defined by $\overleftarrow{\phi}^{*}(y) = \overleftarrow{\phi}(y)$ and $\overleftarrow{\phi}^{**}: 2^Y \longrightarrow 2^X$ by $\overleftarrow{\phi}^{**}(B) = \overleftarrow{\phi}[B]$, clearly $\overleftarrow{\phi}^{*}$ and $\overleftarrow{\phi}^{**}$ are embeddings. There exist two "actions" of E(M) on 2^X, namely one defined by $pA := \{pa \mid a \in A\}$ and one defined by $p \circ A := \lim t_i A$ ($\{t_i\}_i$ a net in E(M) converging to p) for $p \in E(M)$ and $A \in 2^X$; see [9]. Always $pA \subseteq p \circ A$ but in general the inclusion is strict. If X is a minimal ttg, then a *quasifactor* of X is a minimal subttg of 2^X. It can be shown that every quasifactor has the form $QF(A,X) = \{p \circ A \mid p \in M\}$ for some $A \in 2^X$, QF(A,X) is non-trivial iff $A \neq X$ iff $X \notin QF(A,X)$; obviously X is a non-trivial quasifactor of itself iff X is non-trivial ($X \cong QF(\{x\},X)$).

Finally recall that two minimal ttg's X and Y are called *disjoint* ($X \perp Y$) if $X \times Y$ is minimal. For a family $\mathcal{K}$ of minimal ttg's we denote by $\mathcal{K}^{\perp}$ the collection of minimal ttg's, which are disjoint from every member of $\mathcal{K}$, and $\mathcal{K}^{\perp\perp}$ means $(\mathcal{K}^{\perp})^{\perp}$.

§1. QUASIFACTORS AND EXTENSIONS

For a homomorfism ϕ: X $\longrightarrow$ Y of minimal ttg's (hence ϕ is an extension) we define $2^{\perp\phi}$ by $2^{\perp\phi} = \{A \in 2^X \mid \phi[A] = Y\}$. Then $2^{\perp\phi}$ is a closed and invariant subset of 2^X. The easy proof of the following lemma will be omitted.

1.1. LEMMA. *Let* ϕ: X $\longrightarrow$ Y *be a homomorphism of minimal ttg's. Then*
a. *if* ϕ *is an open map then every quasifactor of* Y *is a quasifactor of* X*;*
b. $\phi^*[QF(A,X)] = QF(\phi[A],Y)$;
c. *for every quasifactor* $\mathcal{X}$ *of* X *holds* $\phi^*[\mathcal{X}]$ *is trivial iff* $\mathcal{X} \subseteq 2^{\perp\phi}$ *iff* $\mathcal{X} \cap 2^{\perp\phi} \neq \emptyset$.

1.2. REMARK. If a ttg X contains a nonempty proper subset Z, which is invariant under M (i.e. for every point in Z, Z contains a minimal subset of its orbitclosure) then X is not minimal. As a consequence we have

1.3. THEOREM. *Let* ϕ: X $\longrightarrow$ Y *be a homomorphism of minimal ttg's. Then*
a. $\mathcal{Y} \not\perp$ X *for every nontrivial quasifactor* $\mathcal{Y}$ *of* Y*;*
b. $\mathcal{X} \not\perp$ Y *for every nontrivial quasifactor* $\mathcal{X}$ *of* X *with* $\mathcal{X} \cap 2^{\perp\phi} = \emptyset$.

PROOF.
a. Since $\mathcal{Y}$ is a nontrivial quasifactor of Y, there exist $B \in \mathcal{Y}$ and $x_0 \in X$ with $\phi(x_0) \notin B$. So the non-empty subset $\mathcal{A} = \{(x,A) \in X \times \mathcal{Y} \mid \phi(x) \in A\}$ of $X \times \mathcal{Y}$ is a proper subset. Let $(x,A) \in \mathcal{A}$ then $\phi(px) = p\phi(x) \in pA \subseteq p \circ A$ for all $p \in M$ so $p(x,A) = (px, p\circ A)$ is in $\mathcal{A}$ and $\mathcal{A}$ is invariant under M. By 1.2 $X \times \mathcal{Y}$ is not minimal, thus $X \not\perp \mathcal{Y}$.
b. Define a subset $\mathcal{A}$ of $\mathcal{X} \times Y$ by $\mathcal{A} = \{(A,y) \in \mathcal{X} \times Y \mid y \in \phi[A]\}$. Then $\mathcal{A} \neq \emptyset$ and because there exist $B \in \mathcal{X}$ and $y_0 \in Y$ with $y_0 \in \phi[B]$, $\mathcal{A}$ is a proper subset of $\mathcal{X} \times Y$. Also $\mathcal{A}$ is invariant under M; indeed, if $x \in \phi[A]$ and $p \in M$ then $px \in p\phi[A] \subseteq p \circ \phi[A] = \phi[p\circ A]$. ($\phi^*$ is a homomorphism!) □

The following are easy consequences of 1.3: a ttg is never disjoint from its non-trivial quasifactors and if ϕ is a highly proximal extension, then $\mathcal{X} \not\perp$ Y for every non-trivial quasifactor $\mathcal{X}$ of X (See the beginning of §2 for the definition of highly proximal extensions and 2.1).

For our next result we need a definition: Let ϕ: X $\longrightarrow$ Y be a homomorfism of minimal ttg's. Call a quasifactor $\mathcal{X}$ of X ϕ-*sectional* whenever

$\phi[A] = Y$ and $\phi[A^c] = Y$ for every $A \in \mathcal{X}$. In particular $\mathcal{X} \subseteq 2^{\perp\phi}$.

1.4. THEOREM. *Let* $\phi: X \longrightarrow Y$ *be an open homomorphism of minimal ttg's. Then* $\mathcal{X} \not\perp Y$ *for every non-trivial quasifactor* $\mathcal{X}$ *of* X *that is not* ϕ*-sectional.*

PROOF. Let $\mathcal{X}$ be a quasifactor of X. For the case that $\mathcal{X} \cap 2^{\perp\phi} = \emptyset$ see 1.3.b. Let $\mathcal{X} \subseteq 2^{\perp\phi}$ be non-trivial and not ϕ-sectional. Define the subset $\mathcal{A}$ of $\mathcal{X} \times Y$ by $\mathcal{A} = \{(A,y) \in \mathcal{X} \times Y \mid \overleftarrow{\phi}(y) \subseteq A\}$, then $\mathcal{A}$ is a nonempty proper subset of $\mathcal{X} \times Y$. Indeed, there are $B \in \mathcal{X}$ and $y_0 \in Y$ with $\phi[B] = Y$ and $y_0 \notin \phi[B^c]$ hence $\overleftarrow{\phi}(y_0) \subseteq B$, moreover for $x \in A \in \mathcal{X}$ we have $\overleftarrow{\phi}(\phi(x)) \nsubseteq A$. Finally $\mathcal{A}$ is invariant under M, because if $(A,y) \in \mathcal{A}$ then $\overleftarrow{\phi}(py) = p \circ \overleftarrow{\phi}(y)$ ($\overleftarrow{\phi}^{*}$ is a homomorphism, since ϕ is open) so $\overleftarrow{\phi}(py) \subseteq p \circ A$ and $p(A,y) = (p\circ A, py) \in \mathcal{A}$ for all $p \in M$. Application of 1.2 on $\mathcal{X} \times Y$ and $\mathcal{A}$ concludes the proof. □

For the main theorem of this section we have to recall the following fact. Let $\phi: X \longrightarrow Y$ be a homomorphism of minimal ttg's and fix $u \in J$, choose $x_0 \in X$ with $ux_0 = x_0$ and put $y_0 = \phi(x_0)$. Let F be the Ellisgroup of Y relative y_0, then ϕ is distal iff $\overleftarrow{\phi}(py_0) = pFx_0$ for every $p \in M$. In particular $pFx_0 = p \circ Fx_0$. (e.g.[9], I. 4.1.)

1.5. THEOREM. *Let* $\phi: X \longrightarrow Y$ *be a distal homomorphism of minimal ttg's. Then every non-trivial quasifactor of* X *that is disjoint from* Y *is distal.*

PROOF. Every distal homomorphism is open, so by 1.4 we may restrict our attention to ϕ-sectional quasifactors $\mathcal{X}$ of X. So let $\mathcal{X}$ be a quasifactor of X and $\mathcal{X} \perp Y$, and fix $x_0 \in X$, $y_0 \in Y$ and $F \subseteq uM$ as above. Since $\mathcal{X}$ is ϕ-sectional we have for every $B \in \mathcal{X}$ and $p \in M$

1.6. $\qquad B \cap pFx_0 \neq \emptyset$ and $B^c \cap pFx_0 \neq \emptyset$.

Now $\mathcal{X} = QF(A,X)$ for an $A \in \mathcal{X}$ with $x_0 \in A = u \circ A$, for $\mathcal{X} = QF(B,X)$ for some $B \in 2^X$ so by 1.6 there is an $f \in F$ with $fx_0 \in u \circ B$ and $A = f^{-1} \circ B \in \mathcal{X}$ satisfies the requirement. Since $\mathcal{X} \times Y$ is minimal we may write $\mathcal{X} \times Y = \{(p\circ A, py_0) \mid p \in M\}$ and (A,y_0) is a u-invariant element of $\mathcal{X} \times Y$.

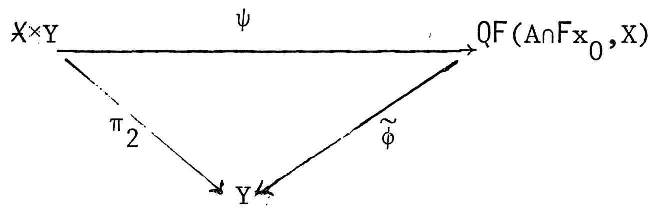
Define $\psi: \mathcal{X} \times Y \longrightarrow 2^X$ by $(p\circ A, py_0) \longmapsto (p\circ A) \cap \overleftarrow{\phi}(py_0)$. Clearly ψ is well defined and equivariant (commutes with the actions on $\mathcal{X} \times Y$ and 2^X).

We claim that ψ is continuous too, and so ψ is a homomorphism of minimal ttg's. First observe that $(p\circ A) \cap \phi^{\leftarrow}(py_0) = (p\circ A) \cap pFx_0 = p(A\cap Fx_0) = p \circ (A\cap Fx_0)$. For let $x \in (p\circ A) \cap \phi^{\leftarrow}(py_0) = (p\circ A) \cap pFx_0$ say $x = pfx_0 \in p \circ A$ for some $f \in F$. Since $fx_0 \in up^{-1}(p\circ A) \subseteq up^{-1} \circ (p\circ A) = u \circ A = A$ it follows that $x \in p(A\cap Fx_0)$. As it is clear that $p(A\cap Fx_0) \subseteq p \circ (A\cap Fx_0) \subseteq (p\circ A) \cap (p\circ Fx_0) = (p\circ A) \cap pFx_0$ this proves our observation. It follows that $\psi[\mathcal{X} \times Y] = \{p \circ (A\cap Fx_0) \mid p \in M\} = QF(A\cap Fx_0,X)$. Let $\alpha\colon M \longrightarrow \mathcal{X} \times Y$ be defined by $\alpha(p) = (p\circ A, py_0)$ and $\gamma\colon M \longrightarrow QF(A\cap Fx_0,X)$ by $\gamma(p) = p \circ (A\cap Fx_0)$. Then α and γ are quotient maps and $\psi \circ \alpha = \gamma$, so ψ is continuous, which proves our claim.

Now define $\tilde{\phi}\colon QF(A\cap Fx_0,X) \longrightarrow Y$ by $\tilde{\phi}((p\circ A)\cap pFx_0) = py_0$. Since $(p\circ A) \cap pFx_0 = (q\circ A) \cap qFx_0$ implies $pFx_0 \cap qFx_0 \neq \emptyset$ and so $py_0 = qy_0$, it follows that $\tilde{\phi}$ is well defined. In addition $\tilde{\phi}$ is equivariant and by a similar argument as for the continuity of ψ, $\tilde{\phi}$ is continuous and so $\tilde{\phi}$ is a homomorphism of minimal ttg's. Next we show that $\tilde{\phi}$ is not injective. To this end, use 1.6 in order to choose $p \in M$ and $f \in F$ with $pfx_0 \notin p \circ A$. There exists a $q \in M$ with $q \circ A \neq p \circ A$ and $pfx_0 \in q \circ A$ so $(p\circ A) \cap pFx_0 \neq (q\circ A) \cap pFx_0$. As $(q\circ A, py_0) \in \mathcal{X} \times Y$ we can find an $r \in M$ with $(q\circ A, py_0) = r \circ (A, y_0) = (r\circ A, ry_0)$ and consequently $(q\circ A) \cap pFx_0 = (r\circ A) \cap rFx_0$. It follows that $pFx_0 \cap rFx_0 \neq \emptyset$ and so $py_0 = ry_0$, whereas $(p\circ A) \cap pFx_0 \neq (r\circ A) \cap rFx_0 = (q\circ A) \cap pFx_0$.

Finally we show that $\tilde{\phi}$ is distal, or equivalently, $\tilde{\phi}^{\twoheadleftarrow}(py_0) = pF(A\cap Fx_0)$ for every $p \in M$. If $f \in F$ then $\tilde{\phi}(pf(A\cap Fx_0)) = \tilde{\phi}((pf\circ A)\cap pFx_0) = py_0$, so $pF(A\cap Fx_0) \subseteq \tilde{\phi}^{\twoheadleftarrow}(py_0)$. Conversely consider $q \in M$ with $qy_0 = \tilde{\phi}(q(A\cap Fx_0)) = py_0$. Then $pFx_0 = qFx_0$ and $up^{-1}q \in F$. Now choose $v \in J$ with $vp = p$ then $vq = pf$ for $f = up^{-1}q \in F$, and $vq(A\cap Fx_0) = pf(A\cap Fx_0)$. Since $q(A\cap Fx_0) \subseteq qFx_0 = pFx_0 = vpFx_0 \subseteq vMx_0$ we know that $vq(A\cap Fx_0) = q(A\cap Fx_0)$, so $q(A\cap Fx_0) = pf(A\cap Fx_0) \subseteq pF(A\cap Fx_0)$, therefore $\tilde{\phi}^{\twoheadleftarrow}(py_0) \subseteq pF(A\cap Fx_0)$ and ϕ is distal.

If we consider the following commutative diagram



with $\pi_2(p\circ A, py_0) = py_0$ for $p \in M$, we may conclude from lemma II.3 of [1], that $\mathcal{X}$ has a non-trivial distal factor, say W. In fact W is obtained as a quasifactor of $QF(A\cap Fx_0, X)$ as follows:
Define $\psi^{\#}: \mathcal{X} \longrightarrow 2^{QF(A\cap Fx_0,X)}$ by $\psi^{\#}(p\circ A) = \{(p\circ A) \cap rFx_0 \mid r \in M \text{ and } r \circ A = p \circ A\}$. Then $\psi^{\#}$ is a homomorphism and W is defined as $\psi^{\#}(\mathcal{X})$. But $\psi^{\#}$ is injective; for let $p \circ A \neq q \circ A$, $x \in (p\circ A) \setminus (q\circ A)$ and $r \in M$ with $x \in rFx_0$, then $(p\circ A) \cap rFx_0 \neq (q\circ A) \cap rFx_0$. Since $\{mFx_0 \mid m \in M\}$ is a partition of X we may conclude that $(p\circ A) \cap rFx_0 \notin \psi^{\#}(q\circ A)$ and so $\psi^{\#}(p\circ A) \neq \psi^{\#}(q\circ A)$.
The compactness of $\mathcal{X}$ and W now gives $\mathcal{X} \cong W$ and consequently, $\mathcal{X}$ is distal. □

We do not have to hope for an analogue of 1.3a where we compare quasifactors $\mathcal{X}$ of X with Y. The following example shows that if $\phi: X \longrightarrow Y$ is distal, then a ϕ-sectional quasifactor $\mathcal{X}$ of X can be disjoint from Y. Let S be the unit circle and define the transformation group $(\mathbb{R}, S_p)$ by $(t, e^{i\psi}) \longmapsto e^{i(\psi+pt)}$. Choose α irrational, then $(\mathbb{R}, \mathbb{T}) = (\mathbb{R}, S_1 \times S_\alpha)$ is a minimal torus action, and it is equicontinuous. Let $\phi: \mathbb{T} \longrightarrow S$, be the projection in the first coordinate; ϕ is a homomorphism of minimal ttg's and clearly ϕ is distal. Define $A = \bar{A} = \{(e^{i\psi}, 1) \mid 0 \le \psi < 2\pi\}$ it is easy to see that $QF(A, \mathbb{T})$ is a ϕ-sectional quasifactor of $\mathbb{T}$ and that it is isomorphic with $(\mathbb{R}, S_\alpha)$. Since $(\mathbb{R}, \mathbb{T})$ is minimal it follows that $(\mathbb{R}, S_1) \perp (\mathbb{R}, S_\alpha)$ and so $(\mathbb{R}, S_1) \perp (\mathbb{R}, QF(A, \mathbb{T}))$. Observe that the obvious fact that $QF(A, \mathbb{T})$ is distal is in accordance with 1.5.

§2. QUASIFACTORS AND DISJOINTNESS CLASSES

In [2] the authors gave a fruitful generalization of almost one-to-one extensions, the so called highly proximal extensions (h.p. extension for short). We shall first summarize a few aspects of it, that are useful for our purpose: the characterization of disjointness classes in terms of quasifactors.

Let $\phi: X \longrightarrow Y$ be a homomorphism of minimal ttg's; ϕ will be called *highly proximal* (h.p.) if for some $y \in Y$ there is a net $\{t_n\}$ in T, such that the net $\{t_n \overleftarrow{\phi}(y)\}$ tends to a singleton in the hyperspace topology. For the proofs of the following lemmas we refer to [2].

2.1. LEMMA. *For a homomorphism* $\phi\colon X \longrightarrow Y$ *of minimal ttg's the following are equivalent:*

a. ϕ *is an* h.p. *extension.*

b. *Every non empty open subset of* X *contains a fiber* $\overleftarrow{\phi}(y)$ *for some* $y \in Y$.

c. $2^{\perp\phi} = \{X\}$.

d. *If* $y \in Y$, $x \in \overleftarrow{\phi}(y)$ *and* $p \in M$ *then* $p \circ \overleftarrow{\phi}(y) = \{px\}$.

The collection of all minimal ttg's can be partitioned in h.p. equivalence classes; two minimal ttg's are called h.p. equivalent if they have a common extension via h.p. extensions. Every equivalence class contains a unique maximal element: the maximal h.p. extension of each of the members of the equivalence class. Such a minimal ttg will be called *maximal highly proximal*. If $\gamma\colon M \longrightarrow X$ is an extension, then for every $x_0 \in X$, $X^* = QF(\overleftarrow{\gamma}(x_0),M)$ is the maximal highly proximal extension of all members of the equivalence class of X (X^* is independent of the choice of $x_0 \in X$). Similar to [5], prop.8.3 we have:

2.2. LEMMA. *The following are equivalent for a minimal ttg* X.

a. X *is maximal highly proximal (i.e.* $X = X^*$*).*

b. X *is an open image of* M.

c. *Every homomorphism* $\phi\colon Y \longrightarrow X$ *of minimal ttg's is open.*

The relation between h.p. extensions and disjointness is given by

2.3. LEMMA. *Let* X_1,X_2 *and* Y_1,Y_2 *be two h.p. equivalent pairs of minimal ttg's then* $X_1 \perp Y_1$ *iff* $X_2 \perp Y_2$. *In particular* $X \perp Y$ *iff* $X^* \perp Y$.

2.4. THEOREM. *Let* X *and* Y *be minimal ttg's.*

a. *If* Y *is a factor of* X, *then* Y^* *is a factor of* X^*.

b. $X \not\perp Y$ *iff* Y *has a non-trivial quasifactor, which is a factor of* X^*.

PROOF. a is Theorem I.1(iii) of [2], and the "only if" in b is lemma II.4 of [2]. The "if-part" of b is a simple corollary of 1.3 and 2.3. □

Finally we recall corollary II.1 of [2]: Let X,X_1 and Y be minimal ttg's.

2.5. If X_1 is a proximal extension of X (an extension via a proximal homomorphism) and X_1 has a distal factor Y, then Y is a factor of X.

Let K be a family of minimal ttg's; we denote by $[K]$ the smallest collection L of minimal ttg's with:

(i) $K \subseteq L$;

(ii) $X \in L$ and $\phi: Y \longrightarrow X$ an h.p. extension then $Y \in L$;

(iii) $X \in L$ and $\phi: X \longrightarrow Y$ a homomorphism then $Y \in L$.

The following lemma characterizes $[K]$.

2.6. LEMMA.

a. $[K] = \{Z \mid Z$ *is a factor of* Y^* *for some* $Y \in K\}$

b. $[K^{\perp}] = K^{\perp}$ *and* $[K] \subseteq K^{\perp\perp}$.

PROOF.

a. Clearly $\{Z \mid Z$ is a factor of Y^* for some $Y \in K\}$ is closed under factors and contains K. Let $\phi: Y^* \longrightarrow Z$ be a homomorphism for some $Y \in K$ and let $\psi: Z' \longrightarrow Z$ be an h.p. extension for a minimal Z'. From 2.4.a we know that Z^* is a factor of Y^*. Since Z' and Z are h.p. equivalent, Z' is a factor of Z^*, hence of Y^*. Now $\{Z \mid Z$ is a factor of Y^* for some $Y \in K\}$ satisfies (i), (ii) and (iii), and clearly it is minimal under these conditions.

b. Follows from 2.3 and the obvious fact that if $X \perp Y$ then every factor of X is disjoint from Y.

EXAMPLES.

(i) Let $\mathcal{D}$ be the collection of minimal distal ttg's and D the universal minimal distal ttg (the phase group is fixed and understood) then $[\mathcal{D}] = [\{D\}] = \{Z \mid Z$ is a factor of $D^*\}$.

(ii) Let P be the collection of minimal proximal ttg's and P the universal minimal proximal ttg then $P = [P] = [\{P\}]$

(iii) Let F be a τ-closed subgroup of $G = uM$. Then $QF(u \circ F, M)$ is the universal minimal proximal extension of minimal ttg's with Ellisgroup F ([9], IX. 3.3.(2)). Now $[\{QF(u \circ F, M)\}] = M(F) = \{Y \mid Y$ is minimal and $\mathcal{G}(Y, y_0) \supseteq F$ for some $y_0 = uy_0 \in Y\}$.

To prove this we need the following definition and fact ([9],X.1.1). Let ϕ: X $\longrightarrow$ Y be a homomorphism of minimal ttg's, that respects the base points $x_0 = ux_0 \in X$ and $y_0 = uy_0 \in Y$ and let $F = \mathcal{J}(Y,y_0)$. ϕ is called a *RIC-extension* if for every $p \in M$: $\phi^{\leftarrow}(py_0) = p \circ Fx_0$. Every RIC-extension is an open map. Since QF(u∘F,M) is an image of M under a RIC-extension it follows from 2.2 that QF(u∘F,M) is maximal highly proximal. Since for every τ-closed subgroup $F' \supseteq F$, QF(u∘F',M) is an image of QF(u∘F,M) (under a RIC-extension defined by $p \circ F \longmapsto p \circ F'$) and every minimal Y with Ellisgroup F' is an image of QF(u∘F'M) (under a proximal extension) it follows that $[\{QF(u\circ F,M)\}] = \mathcal{M}(F)$. Remark that QF(u∘G,M) = P and $\mathcal{M}(G) = \mathcal{P}$.

Also observe that if QF(u∘G,M) $\neq \{*\}$ then for every τ-closed subgroup K of G holds QF(u∘K,M) $\notin \mathcal{M}(F)$, for QF(u∘K,M) and QF(u∘F,M) have QF(u∘G,M) as a non-trivial common factor.

2.7. THEOREM. *Let $\mathcal{K}$ be a family of minimal ttg's. For a minimal ttg* X *the following are equivalent:*

a. $X \in \mathcal{K}^{\perp}$;

b. $X \in [\mathcal{K}]^{\perp}$;

c. $\mathcal{X} \notin [\mathcal{K}]$ *for every non-trivial quasifactor* $\mathcal{X}$ *of* X.

PROOF. c $\Rightarrow$ b Assume the existence of a $Z \in [\mathcal{K}]$ with $X \not\perp Z$ then (2.4.b) X has a non-trivial quasifactor $\mathcal{X}$ which is a factor of Z^* and consequently $\mathcal{X} \in [\mathcal{K}]$. b $\Rightarrow$ a Since $\mathcal{K} \subseteq [\mathcal{K}]$ we know $[\mathcal{K}]^{\perp} \subseteq \mathcal{K}^{\perp}$. a $\Rightarrow$ c Suppose that $\mathcal{X} \in [\mathcal{K}]$ for some non-trivial quasifactor $\mathcal{X}$ of X. Then there is a $Y \in \mathcal{K}$ such that $\mathcal{X}$ is a factor of Y^*, so $X \not\perp Y$ (2.4.b) and $X \notin \mathcal{K}^{\perp}$. □

Denote the family of almost periodic minimal ttg's with $\mathcal{AP}$.

2.8. LEMMA.

a. $X \in \mathcal{D}$ *iff every non-trivial quasifactor of* X *is distal.*

b. $X \in \mathcal{AP}$ *iff every non-trivial quasifactor of* X *is almost periodic.*

PROOF.

a. Theorem 1.5 with Y trivial.

b. By [12], $X \in \mathcal{AP}$ iff 2^X is almost periodic (and X minimal). In addition, if 2^X is almost periodic then every non-trivial quasifactor of X is almost periodic, and this in turn implies $X \in \mathcal{AP}$. □

2.9. THEOREM. *Let* K *be* $\mathcal{D}$ *or* AP *and let* K *be the universal minimal* K*-ttg. Then for minimal* X *the following are equivalent:*

a. $X \in K^{\perp}$;

b. $X \perp K$;

c. X *admits no non-trivial factors in* K *;*

d. X *admits no non-trivial quasifactors in* $[K]$.

PROOF. The equivalence of a, b and d is just 2.7, and a $\Rightarrow$ c is trivial. c $\Rightarrow$ a Suppose $X \notin K^{\perp}$ then $X \not\perp Y$ for some $Y \in K$. According to 2.4.b there exists a non-trivial quasifactor $\mathcal{Y}$ of Y, which is a factor of X^*. Since $\mathcal{Y} \in K \subseteq D$ (2.8) and using (2.5) $\mathcal{Y}$ turns out to be a non-trivial K-factor of X. □

2.10. COROLLARY. *Let* K *be* $\mathcal{D}$ *or* AP*, and let* X *be a minimal ttg. Then* $X \in K^{\perp\perp}$ *iff every non-trivial quasifactor of* X *has a non-trivial* K*-factor.*

2.11. COROLLARY. $\mathcal{D}^{\perp\perp} = AP^{\perp\perp}$ *and consequently* $\mathcal{D}^{\perp} = AP^{\perp}$.

PROOF. Since $AP \subseteq \mathcal{D}$ it is obvious that $AP^{\perp\perp} \subseteq \mathcal{D}^{\perp\perp}$. Let $X \in \mathcal{D}^{\perp\perp}$, then every non-trivial quasifactor $\mathcal{X}$ of X has a non-trivial distal factor. Since in [6] ELLIS proved (without countability assumptions), that every non-trivial minimal point-distal ttg (and, consequently, every non-trivial element of $\mathcal{D}$) admits a non-trivial AP-factor. From this and 2.10 it is clear that $\mathcal{D}^{\perp\perp} \subseteq AP^{\perp\perp}$ so $\mathcal{D}^{\perp\perp} = AP^{\perp\perp}$ and $\mathcal{D}^{\perp} = \mathcal{D}^{\perp\perp\perp} = AP^{\perp\perp\perp} = AP^{\perp}$.

Remark that every non-trivial minimal distal ttg has also a non-trivial AP-quasifactor. For let $\{*\} \neq X \in \mathcal{D}$ and let Y be a non-trivial AP-factor of X, then Y is an open image of X and so Y is a quasifactor of X.

2.12. THEOREM. *Let* F *be a* τ*-closed subgroup of* $G = uM$, *and let* X *be minimal. The following statements are equivalent:*

a. $X \in M(F)^{\perp}$;

b. $X \perp QF(u\circ F, M)$;

c. X *has no non-trivial quasifactor, which is a factor of* $QF(u\circ F, M)$;

d. $u \circ Fx = X$ *for all* $x \in X$.

If F *is a* τ*-closed normal subgroup of* G *we may replace* d *by:*

d'. *There is an* $x \in X$ *with* $u \circ Fx = X$.

PROOF. The equivalence of a, b and c is trivial from 2.7 and the foregoing example (iii). c $\Rightarrow$ d For arbitrary $x \in X$, $QF(u \circ Fx, X)$ is a quasifactor of X. But also $QF(u \circ Fx, X)$ is a factor of $QF(u \circ F, M)$ by way of the homomorphism defined by $p \circ F \longmapsto p \circ Fx$. Our assumption forces $QF(u \circ Fx, X)$ to be trivial, it then follows that $u \circ Fx = X$. d $\Rightarrow$ c Suppose that $\mathcal{X}$ is a quasifactor of X and a factor of $QF(u \circ F, M)$. So we may assume that $\mathcal{X} = QF(D, X)$ for some $D \in 2^X$ with $u \circ D = D$ and that the homomorphism from $QF(u \circ F, M)$ onto $QF(D, X)$ is given by $p \circ F \longmapsto p \circ D$. Since $q \in p \circ F$ iff $q \circ F = p \circ F$, we know that $(p \circ F) \circ D = \bigcup\{q \circ D \mid q \in p \circ F\} = p \circ D$, so for every $p \in M$: $p \circ FD \subseteq (p \circ F) \circ D = p \circ D$. Choose $x \in D$; then $p \circ Fx \subseteq p \circ D$ and $p \circ Fx = p \circ (u \circ Fx) = p \circ X = X$. Therefore $X \subseteq D$ and $QF(D, X)$ is trivial.
Now let F be a τ-closed normal subgroup of G. Since d trivially implies d' we only have to check d' $\Rightarrow$ c as follows: Let $x_0 \in X$ be such that $u \circ Fx_0 = X$. F is normal, so for all $\alpha \in G$ $F = \alpha^{-1}F\alpha \subseteq \alpha^{-1} \circ F\alpha$, hence $\alpha \circ F \subseteq \alpha\alpha^{-1} \circ F\alpha = u \circ F\alpha$. With notation as in the proof of d $\Rightarrow$ c, we choose $x \in D$, say $x = px_0$. Then $p = v\alpha$ for some $v \in J$ and $\alpha \in G$ ([9], I.2.4). Since $\alpha \circ Fx_0 \subseteq u \circ F\alpha x_0 = u \circ Fv\alpha x_0 = u \circ Fx \subseteq u \circ FD$ and $\alpha \circ Fx_0 = \alpha \circ (u \circ Fx_0) = \alpha \circ X = X$ it follows that $QF(D, X)$ is trivial. $\square$

Recall the definition of a RIC extension in example (iii). We define a minimal ttg to be *incontractible* if the trivial homomorphism $\phi: X \longrightarrow \{*\}$ is a RIC-extension or equivalently X is incontractible iff $u \circ Gx = X$ for some $x \in X$.

2.13. COROLLARY.

a. *For a minimal ttg* X, *the following are equivalent:*

(i) $X \in \mathcal{P}^{\perp}$;

(ii) X *is incontractible;*

(iii) X *has no non-trivial proximal quasifactor.*

b. *Let* X *be minimal, then* $X \in \mathcal{P}^{\perp\perp}$ *iff every non-trivial quasifactor of* X *has a non-trivial proximal quasifactor.*

PROOF. Put $F = G$ in 2.12 and remember that G is a τ-closed normal subgroup of G.

§3. EXTENSIONS, $\mathcal{D}^{\perp}$ AND $\mathcal{P}^{\perp}$

3.1. THEOREM.

a. *If* $X \in \mathcal{D}^{\perp}$ *and* $\phi: X \longrightarrow Y$ *is distal then* $X^{\perp} = Y^{\perp}$.

b. *If* $\phi: X \longrightarrow Y$ *is distal then* $\mathcal{D}^{\perp} \cap X^{\perp} = \mathcal{D}^{\perp} \cap Y^{\perp}$.

PROOF.

a. Clearly $X^{\perp} \subseteq Y^{\perp}$, conversely let Z be minimal and $Z \perp Y$, and suppose that $Z \not\perp X$. Then by 2.4.b there exists a non-trivial quasifactor $\mathcal{X}$ of X, which is a factor of Z^{*}. Since $X \in \mathcal{D}^{\perp}$ it follows that $\mathcal{X}$ is not distal (2.9), hence 1.5 implies that $\mathcal{X} \not\perp Y$. So Y has a non-trivial quasifactor $\mathcal{Y}$, which is a factor of $\mathcal{X}^{*}$. Then by 2.4.a $\mathcal{Y}$ is a factor of Z^{*}, so by 2.4.b $Y \not\perp Z$, which contradicts the assumption.

b. Let $Z \in \mathcal{D}^{\perp} \cap Y^{\perp}$ and suppose $Z \not\perp X$. Then X has a non-trivial quasifactor $\mathcal{X}$, which is a factor of Z^{*}. From $Z \perp Y$ we conclude that $Z^{*} \perp Y$ and so $\mathcal{X} \perp Y$. By 1.5 $\mathcal{X}$ must be distal, but as a factor of an element of $\mathcal{D}^{\perp}$ this is impossible.

3.2. COROLLARY. (Theorem II.1 of [2]) $\mathcal{D}^{\perp\perp}$ *is closed under distal extensions.*

PROOF. Let $\phi: X \longrightarrow Y$ be distal, with $Y \in \mathcal{D}^{\perp\perp}$ then $\mathcal{D}^{\perp} \cap Y^{\perp} = \mathcal{D}^{\perp}$. By 3.1.b $\mathcal{D}^{\perp} = \mathcal{D}^{\perp} \cap X^{\perp} \subseteq X^{\perp}$ and $X \in \mathcal{D}^{\perp\perp}$. □

We will now obtain the same kind of result with distal replaced by proximal (3.5) thus generalizing a result of SHAPIRO ([15],2.4) by way of a generalization of [11], 4.5. First remember that for a (not necessarily minimal) ttg X, $x \in X$ is called an *almost periodic point* if its orbit closure is minimal.

3.3. LEMMA. *Let* X *and* Y *be minimal with Ellisgroups* H *respectively* F *in* $G = uM$. *Then* $X \perp Y$ *iff* $HF = G$ *and* $X \times Y$ *contains a dense subset of almost periodic points.*

PROOF. Recall that $x_0 \in X$ and $y_0 \in Y$ with $ux_0 = x_0$ and $uy_0 = y_0$, and that H and F are the Ellisgroups of X and Y relative x_0 respectively y_0. Assume $X \perp Y$, then $X \times Y$ is minimal and every element of $X \times Y$ is almost periodic.

In particular $X \times Y$ is the orbit closure of $(x_0,y_0) = u(x_0,y_0)$. Choose $\gamma \in G$; then there exists a $p \in M$ with $p(x_0,y_0) = (x_0,\gamma y_0)$, so $px_0 = x_0$ and $py_0 = \gamma y_0$. Since $up \in H$ and $up^{-1}\gamma \in F$ it follows that $\gamma = u\gamma = up.up^{-1}\gamma \in HF$, and $G \subseteq HF$.

Now suppose $HF = G$ and $X \times Y$ has a dense subset of almost periodic points. We prove that all almost periodic points are in the orbit closure of (x_0,y_0), which is a minimal ttg since $u(x_0,y_0) = (x_0,y_0)$ ([4],3.7). The minimality of $X \times Y$ is then obvious. Let $(x,y) = (px_0,qy_0)$ be almost periodic. Then there is a $v \in J$ with $vpx_0 = px_0$ and $vqy_0 = qy_0$. Choose $\alpha,\beta \in G$ with $vp = v\alpha$ and $vq = v\beta$. Then $(x,y) = v\beta(\beta^{-1}\alpha x_0,y_0)$, and as $\beta^{-1}\alpha \in G = HF$ we may choose $h \in H$ and $f \in F$ with $\beta^{-1}\alpha = fh$. Now $(x,y) = v\beta(fhx_0,y_0) =$
$= v\beta(fx_0,y_0) = v\beta f(x_0,f^{-1}y_0) = v\beta f(x_0,y_0)$ and (x,y) is an element of the orbit closure of (x_0,y_0). □

There are several situations where $X \times Y$ has a dense set of almost periodic points, for instance if $X \times Y$ is a distal extension of a minimal set. Another situation is the basis of the following theorem, with notations as above.

3.4. <u>THEOREM</u>. *If* $X \in P^{\perp}$ *and* Y *is minimal then* $X \perp Y$ *iff* $HF = G$.

<u>PROOF</u>. We only have to prove, that the incontractiblity of X implies, that $X \times Y$ has a dense set of almost periodic points. This follows immediately from the remark on page 814 of [16]. □

We define a topological group T to be *strongly amenable* if there does not exist any non-trivial minimal proximal ttg with phase group T, or equivalently, if every non-trivial minimal ttg on T is incontractible ([9],II.3). For instance if T is abelian of nilpotent then T is strongly amenable. From 3.4 it is obvious, that for strongly amenable groups the disjointness of X and Y is equivalent with $HF = G$. (For T abelian see [11],4.5).

3.5. <u>THEOREM</u>.

<u>a</u>. *If* $X \in P^{\perp}$ *and* $\phi: X \longrightarrow Y$ *is proximal then* $X^{\perp} = Y^{\perp}$.

<u>b</u>. *If* $\phi: X \longrightarrow Y$ *is proximal then* $P^{\perp} \cap X^{\perp} = P^{\perp} \cap Y^{\perp}$.

PROOF.

a. Assume $Z \perp Y$ and Z minimal. Then $\phi \times 1_Z : X \times Z \longrightarrow Y \times Z$ is proximal. Since $Y \times Z$ is minimal it follows from [9], II.1 that $X \times Z$ contains a unique minimal sub-ttg. By the proof of 3.4 $X \times Z$ has a dense set of almost periodic points and so $X \times Z$ is minimal and $X \perp Z$.

b. Suppose $Z \in \mathcal{P}^{\perp}$ and $Z \perp Y$. Let the Ellisgroups of X,Y and Z in G be respectively H,F and K. By 3.4 $KF = G$ and [9], I.4.1(2) implies that $H = F$ for a suitable choice of $x_0 = ux_0 \in X$ and $y_0 = uy_0 \in Y$. But then it follows, that $KH = G$ and $Z \perp X$.

3.6. COROLLARY. *$\mathcal{P}^{\perp\perp}$ is closed under proximal extensions.*

PROOF. Similar to 3.2. □

§4. DISJOINTNESS AND WEAK MIXING

In [14] PETERSEN characterizes the weakly mixing minimal ttg's with abelian phase group as the minimal ttg's which admit no non-trivial almost periodic factor. We shall generalize this result slightly: see 4.3 below. Recall that a ttg X is *ergodic* if X is the only closed invariant subset of X with non-empty interior, and that X is *weakly mixing* if $X \times X$ is ergodic. We denote the collection of weakly mixing minimal ttg's with $\mathcal{WM}$.
We need the following two results:

4.1. THEOREM. (ELLIS [6],1.9). *Every distal and ergodic ttg is minimal.*

4.2. THEOREM. *Let $X \in \mathcal{P}^{\perp} \cap \mathcal{D}^{\perp}$ and let Y be ergodic having a dense set of almost periodic points; then $X \times Y$ is ergodic. In particular this applies to the case that Y is minimal.*

PROOF. This is a reformulation of [16], 2.1.6, using the fact that $\mathcal{D}^{\perp} = \mathcal{AP}^{\perp}$, 2.9 and 2.13.(a). □

4.3. THEOREM. *$\mathcal{WM} \subseteq \mathcal{D}^{\perp}$ and $\mathcal{P}^{\perp} \cap \mathcal{D}^{\perp} = \mathcal{P}^{\perp} \cap \mathcal{WM}$.*

PROOF. Let $X \in \mathcal{WM}$ and let Z be a distal factor of X. Then $Z \times Z$ is distal and ergodic (for $X \times X$ is ergodic and ergodicity is preserved under factors). So $Z \times Z$ is minimal by 4.1, hence Z is trivial and $X \in \mathcal{D}^{\perp}$ by 2.9. Now let $X \in \mathcal{P}^{\perp} \cap \mathcal{D}^{\perp}$, then by 4.2 $X \times X$ is ergodic, so $X \in \mathcal{WM}$, and since $\mathcal{P}^{\perp} \cap \mathcal{WM} \subseteq \subseteq \mathcal{P}^{\perp} \cap \mathcal{D}^{\perp} \subseteq \mathcal{WM}$ it follows that $\mathcal{P}^{\perp} \cap \mathcal{D}^{\perp} = \mathcal{P}^{\perp} \cap \mathcal{WM}$. □

4.4. COROLLARY. *Let* T *be strongly amenable; then* $\mathcal{WM} = \mathcal{D}^{\perp} = \mathcal{AP}^{\perp}$.

We conclude this section with an observation about distal extensions of weakly mixing minimal ttg's but first:

4.5. LEMMA.
a. $\mathcal{D}^{\perp}$ *is closed under proximal extensions.*
b. $\mathcal{P}^{\perp}$ *is closed under distal extensions.*

PROOF.
a. Let $\phi: X \longrightarrow Y$ be a proximal homomorphism of minimal ttg's and $Y \in \mathcal{D}^{\perp}$. Suppose that $X \notin \mathcal{D}^{\perp}$ then by 2.9 X has a non-trivial distal factor Z. By 2.5 Z is a factor of Y, which contradicts $Y \in \mathcal{D}^{\perp}$.
b. Let $\phi: X \longrightarrow Y$ be a distal homomorphism of minimal ttg's and $Y \in \mathcal{P}^{\perp}$. Choose $Z \in \mathcal{P}$ and suppose $X \not\perp Z$, then there is a non-trivial quasifactor $\mathcal{X}$ of X, which is a factor of Z^*. If $\mathcal{X} \not\perp Y$, then there is a non-trivial quasifactor $\mathcal{Y}$ of Y, which is a factor of $\mathcal{X}^*$, hence of Z^*. This contradicts $Z \perp Y$, so $\mathcal{X} \perp Y$. Then $\mathcal{X}$ is distal by 1.5 so Z^* has a distal factor, but this is impossible since $Z^* \in \mathcal{P} \subseteq \mathcal{D}^{\perp}$. □

4.6. COROLLARY. *Let* $Y \in \mathcal{P}^{\perp} \cap \mathcal{WM}$. *Then every minimal distal extension of* Y *without distal factor is weakly mixing.*

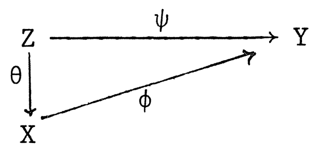
PROOF. Let $\phi: X \longrightarrow Y$ be distal and $X \in \mathcal{D}^{\perp}$. Since $Y \in \mathcal{P}^{\perp}$ it follows from 4.5.b that $X \in \mathcal{P}^{\perp}$, so $X \in \mathcal{P}^{\perp} \cap \mathcal{D}^{\perp} = \mathcal{P}^{\perp} \cap \mathcal{WM}$. □

§5. DISJOINTNESS AND $(\mathcal{H})\mathcal{PI}$

Let $\phi: X \longrightarrow Y$ be a homomorphism of minimal ttg's, ϕ is called *strictly*-PI or X is called a *strictly*-PI *extension* of Y if there exist an ordinal ν and for every ordinal $\alpha \leq \nu$ a minimal ttg (W_α, w_α) with u-invariant base point w_α such that

1. $W_0 = Y$ and $W_\nu = X$;
2. for every $\alpha \leq \nu$ (W_α, w_α) is a factor of $(W_{\alpha+1}, w_{\alpha+1})$ under a homomorphism ϕ_α which is either proximal or almost periodic;
3. if α is a limit ordinal then $(W_\alpha, w_\alpha) = \vee\{(W_\beta, w_\beta) \mid \beta < \alpha\}$.

Here $\vee\{(W_\beta, w_\beta) \mid \beta < \alpha\}$ denotes the (minimal!) orbit closure of $(w_\beta)_{\beta<\alpha}$ in $\Pi\{W_\beta \mid \beta < \alpha\}$. We shall refer to such a system $\{(W_\alpha, w_\alpha), \phi_\alpha\}$ as a *tower*.
The homomorphism $\phi\colon X \longrightarrow Y$ of minimal ttg's is called PI or X is a PI *extension* of Y, if there exist a minimal ttg Z, a strictly-PI homomorphism $\psi\colon Z \longrightarrow Y$ and a proximal homomorphism $\theta\colon Z \longrightarrow X$, such that the next diagram commutes.



Observe, that in [16] a PI extension is what we called a strictly-PI extension. The reason for our denomination is the following:
A minimal ttg X is called (*strictly*-)PI if X is a (strictly-)PI extension of the trivial ttg $\{*\}$, and this is equivalent to the definition of a (strictly)-PI ttg in [7], [9]. In the same way we define (*strictly*-) HPI homomorphisms and -ttg's, by replacing proximal by highly proximal in 2. of the description of the tower. For more details see [7], [9], [16] and [2].

We intend to determine $PI^\perp$ and $HPI^\perp$, where PI and HPI denote the collections of all PI- and all HPI ttg's respectively.

First, define for a τ-closed subgroup F of G, H(F) as the smallest τ-closed normal subgroup of F, such that F/H(F) with the quotient topology is a compact Hausdorff topological group. Let F_∞ be the τ-closed normal subgroup of F that is the inverse limit of the sequence H(F), H(H(F)),... .

5.1. THEOREM. *Let* $\phi\colon X \longrightarrow Y$ *be a homomorphism of minimal ttg's. Then* ϕ *is* PI *iff* $H \supseteq F_\infty$, *where* H *and* F *are the Ellisgroups of* X *and* Y *respectively.*

PROOF. This is a relativized version of X.4.2 of [9]. □

From 5.1 it follows immediately that if $\phi = \theta \circ \psi$ is PI then also θ is PI.

5.2. LEMMA.

a. *$PI = [PI] = M(G_\infty)$, and every* PI ttg *is a factor of a strictly*-PI *ttg.*

b. *$HPI = [HPI]$, and every* HPI *ttg is a factor of a strictly*-HPI *ttg.*

c. $X \in PI^\perp$ *iff* $u \circ G_\infty x_0 = X$.

d. *$PI^\perp \subseteq P^\perp \cap D^\perp$ and $HPI^\perp \subseteq D^\perp$.*

PROOF.

a. [9], X.4.2.

b. [2], Corollary III.1.

c. Follows from a and 2.12(d').

d. Since $G \supseteq G_\infty$ it follows from c and 2.13 a that $PI^\perp \subseteq P^\perp$. In [6] ELLIS proved that every minimal distal ttg is an inverse limit of almost periodic extensions and so $D \subseteq HPI \subseteq PI$, thus $PI^\perp \subseteq HPI^\perp \subseteq D^\perp$ and $PI^\perp \subseteq P^\perp \cap D^\perp$. □

We shall now prove a PI-analogue of 3.1 and 3.5. But first observe that if K is a collection of minimal ttg's, then $K^\perp$ is closed under inverse limits, as $X \times \vee Y_\alpha \cong V(X \times Y_\alpha)$ for $X \in K$ and $Y_\alpha \in K^\perp$.

5.3. THEOREM.

a. *If $X \in PI^\perp$ and $\phi: X \longrightarrow Y$ is strictly*-PI *then $X^\perp = Y^\perp$.*

b. *If $\phi: X \longrightarrow Y$ is strictly-PI, then $PI^\perp \cap X^\perp = PI^\perp \cap Y^\perp$.*

PROOF.

a. Clearly every (W_α, w_α) in the tower of ϕ is in $PI^\perp$. Since $PI^\perp \subseteq P^\perp \cap D^\perp$ and every ϕ_α is either proximal or almost periodic (hence distal) it follows from 3.1.a and 3.5.a, that for every α, $W_\alpha^\perp = Y^\perp$ and so $X^\perp = Y^\perp$.

b. Follows in a similar way from 3.1.b and 3.5.b. □

5.4. COROLLARY. *$PI^{\perp\perp}$ is closed under* PI *extensions.*

PROOF. Similar to 3.2 it can be shown that $PI^{\perp\perp}$ is closed under strictly-PI extensions. Let $\phi: X \longrightarrow Y$ be a PI extension, then there exist a minimal ttg Z and a strictly-PI extension $\psi: Z \longrightarrow Y$, such that X is a factor of Z. Now we may conclude that if $Y \in PI^{\perp\perp}$ then also $Z \in PI^{\perp\perp}$ and so $X \in PI^{\perp\perp}$. □

5.5. THEOREM.

a. $\mathcal{D}^{\perp} = \mathcal{HPI}^{\perp}$.

b. $\mathcal{P}^{\perp} \cap \mathcal{D}^{\perp} = \mathcal{PI}^{\perp}$.

PROOF. We only have to prove the converse of 5.2.d. Let $W \in \mathcal{D}^{\perp}$ and let X be a strictly-HPI ttg. If we apply 3.1.b to the almost periodic steps in the tower of X and 2.3 to the highly proximal ones, it follows that $X \perp W$. Since 5.2.b it is clear that this implies $\mathcal{HPI}^{\perp} \subseteq \mathcal{D}^{\perp}$.

Let $W \in \mathcal{D}^{\perp} \cap \mathcal{P}^{\perp}$ and let X be a strictly-PI ttg. Using 3.1.b for the almost periodic steps and 3.5.b for the proximal steps, we see that $X \perp W$, therefore $\mathcal{P}^{\perp} \cap \mathcal{D}^{\perp} \subseteq \mathcal{PI}^{\perp}$. □

5.6. COROLLARY.

a. $\mathcal{PI}^{\perp} = \mathcal{P}^{\perp} \cap \mathcal{WM} = \mathcal{P}^{\perp} \cap \mathcal{D}^{\perp} = \mathcal{P}^{\perp} \cap \mathcal{AP}^{\perp} = \mathcal{P}^{\perp} \cap \mathcal{HPI}^{\perp}$.

b. *If* T *is strongly amenable then* $\mathcal{PI}^{\perp} = \mathcal{WM} = \mathcal{D}^{\perp} = \mathcal{AP}^{\perp} = \mathcal{HPI}^{\perp}$.

c. *If* $X \in \mathcal{P}^{\perp}$*, then* $X \in \mathcal{WM}$ *iff* $HG_{\infty} = G$ *(*H *is the Ellisgroup of* X*).*

PROOF. a,b are clear c follows from 5.2.a and 3.4. □

§6. DISJOINTNESS AND RELATIVE-PRIMENESS

We will now turn to some variations on the theme of whether relative prime implies disjointness. It is wellknown, that in general this is not true, so the problem is to search for conditions, which are sufficient for this implication to hold. As before G = uM, and Ellisgroups are subgroups of G. Let $\mathcal{K}^{\perp c}$ denote the complement of $\mathcal{K}^{\perp}$ in the collection of all minimal ttg's and remember that two minimal ttg's are called *relative prime* if they have no non-trivial common factor.

6.1. LEMMA. *Let* X *and* Y *be minimal with Ellisgroups* H *and* F *and let* K *be the smallest* τ*-closed subgroup of* G *containing* $H \cup F$*. If* $QF(u \circ K, M) \in \mathcal{D}^{\perp c}$ *then* X *and* Y *are not relative prime by a non-trivial common distal factor.*

PROOF. Since $QF(u \circ K, M) \in \mathcal{D}^{\perp c}$ it has a non-trivial distal factor $\tilde{\phi}: QF(u \circ K, M) \longrightarrow Z$. Define $\phi: X \longrightarrow Z$ by $px_0 \longmapsto \tilde{\phi}(p \circ K)$ and $\psi: Y \longrightarrow Z$ by $py_0 \longmapsto \tilde{\phi}(p \circ K)$.

It suffices to prove that ϕ and ψ are well defined, for then they are obviously continuous, equivariant surjections (preserving basepoints). Let p and q in M be such that $px_0 = qx_0$. Then $up^{-1}q \in H \subseteq K$, so $p \circ K$ and $q \circ K$ are proximal in $QF(u \circ K, M)$. Since Z is distal it follows that $\tilde{\phi}(p \circ K) = \tilde{\phi}(q \circ K)$ and $\phi(px_0) = \phi(qx_0)$. Similarly ψ is well defined. □

6.2. <u>LEMMA</u>. *Each of the following conditions implies that* $QF(u \circ K, M) \in \mathcal{D}^{\perp c}$.
<u>a</u>. $\mathcal{D}^{\perp\perp} \cap M(K)^{\perp c} \neq \emptyset$.
<u>b</u>. $M(G_\infty K) \cap \mathcal{P}^{\perp}$ *contains a non-trivial ttg.*

<u>PROOF</u>. Let $Z \in \mathcal{D}^{\perp\perp} \cap M(K)^{\perp c}$. By 2.12 there exists a $\bar{z} \in Z$ with $u \circ K\bar{z} \neq Z$. So $QF(u \circ K\bar{z}, Z)$ is non-trivial and by 2.10 it has a non-trivial distal factor. Obviously this implies that $QF(u \circ K, M)$ has a non-trivial distal factor and $QF(u \circ K, M) \in \mathcal{D}^{\perp c}$. Let $Z \in M(G_\infty K) \cap \mathcal{P}^{\perp}$ be non-trivial. Its Ellisgroup contains $G_\infty K$ and so it contains G_∞. Therefore Z is an incontractible PI ttg (5.2.<u>a</u>) and has a non-trivial almost periodic factor ([9],X.4.4.). This is also a factor of $QF(u \circ K, M)$, hence $QF(u \circ K, M) \in \mathcal{D}^{\perp c}$. □

For the following theorems we need the introduction of regular minimal ttg's. We call a minimal ttg X *regular* if its Ellisgroup H is a normal subgroup of G. In that case for all $x \in X$ with $ux = x$ we have $\mathcal{G}(X,x) = H$. For an explicit treatment of regular minimal ttg's we refer to [1].

6.3. <u>THEOREM</u>. *Let* X *and* Y *be minimal ttg's with* X *or* Y *regular and* $X \in \mathcal{D}^{\perp\perp}$. *Then* $X \perp Y$ *iff* X *and* Y *are relatively prime.*

<u>PROOF</u>. Suppose X or Y is regular, $X \in \mathcal{D}^{\perp\perp}$ and $X \not\perp Y$. With notation as in 6.1 it is clear that $K = HF = FH$ is a τ-closed subgroup of G ([9], IX.1.10). As $X \not\perp Y$ we have $X \not\perp QF(u \circ F, M)$, since Y is a factor of $QF(u \circ F, M)$. In the case that F is a normal subgroup $X \neq u \circ Fx_0 = u \circ FHx_0 = u \circ Kx_0$ (2.12.<u>d</u>'). If H is normal then $\alpha H \alpha^{-1} = H$ for all $\alpha \in G$. In this case, there exists by 2.12.<u>d</u> an $\bar{x} \in X$ with $u \circ F\bar{x} \neq X$. Let $w \in J$ and $\alpha \in G$ be such that $\bar{x} = w\alpha x_0$. Then $\mathcal{G}(X, u\bar{x}) = \alpha H \alpha^{-1} = H$ and $X \neq u \circ F\bar{x} = u \circ Fu\bar{x} = u \circ FH\bar{x} = u \circ K\bar{x}$. So in both cases we can find an $\bar{x} \in X$, such that $\mathcal{X} = QF(u \circ K\bar{x}, X)$ is a non-trivial quasifactor of X. Since $X \not\perp \mathcal{X}$ (1.3) and $\mathcal{G}(\mathcal{X}, u \circ K\bar{x}) \supseteq K$ it follows that $\mathcal{X} \in M(K)$ and $X \in M(K)^{\perp c}$. The proof is finished by applying 6.1 and 6.2.<u>a</u>. The other way around is trivial. □

The following consequence of 6.2.b can also be found in [8] in a somewhat weaker version.

6.4. THEOREM. *Let* X *and* Y *be minimal with Ellisgroups* H *and* F, *with* X *or* Y *regular, such that* $G_\infty \subseteq HF$ *and every quasifactor of* X *is incontractible. Then* $X \perp Y$ *iff* X *and* Y *are relatively prime.*

PROOF. Let the notation be as before. Similar to the proof of 6.3 we get a non-trivial quasifactor $\mathcal{X} = QF(u \circ K\bar{x}, X)$ of X. Our assumptions guarantee its incontractibility. Since $\mathfrak{G}(\mathcal{X}, u \circ K\bar{x}) \supseteq G_\infty$ it follows that $\mathcal{X} \in M(G_\infty K) \cap \mathcal{P}^\perp = M(K) \cap \mathcal{P}^\perp$. Now apply 6.1 and 6.2.b.

Observe, that the condition of each quasifactor of X being incontractible is trivially fulfilled if T is strongly amenable.

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