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AFDELING ZUIVERE WISKUNDE
(DEPARTMENT OF PURE MATHEMATICS)

ZW 120/78 DECEMBER

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ON A SUFFICIENT CONDITION FOR THE EXISTENCE OF G-COMPACTIFICATIONS

Preprint

2e boerhaavestraat 49 amsterdam

Printed at the Mathematical Centre, 49, 2e Boerhaavestraat, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O).

AMS(MOS) subject classification scheme (1970): 54H15, 54D35, 54E15

On a sufficient condition for the existence of G-compactifications*)

by

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ABSTRACT

In this paper it is shown that if a topological transformation group is pointwise equicontinuous with respect to some uniformity of the phase space, then it can equivariantly be embedded in a topological transformation group with a compact phase space and the same acting group.

KEY WORDS & PHRASES: *topological transformation group, compactification, (uniform) equicontinuity*

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*) This report will be submitted for publication elsewhere.

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INTRODUCTION

In this paper we show that if the transition group of a topological transformation group is pointwise equicontinuous with respect to some uniform structure of the phase space, then the topological transformation group has a G-compactification (i.e. it can be embedded isomorphically in a topological transformation group with compact phase-space). This result is not covered by [6], where it has been shown that every topological transformation group on a completely regular space and with a locally compact phase group G has a G-compactification. As a consequence of our result we get that each topological transformation group with compact phase group G admits a G-compactification, a result which also follows from [6].

1. PRELIMINARIES

A *topological transformation group* (*ttg*) is a triple (G,X,π) where G is a topological group, X a topological space and $\pi : G \times X \to X$ is a continuous map, satisfying the following conditions:

(i) $\pi(e,x) = x$ for every $x \in X$ (e denotes the unit of G),

(ii) $\pi(t_1,\pi(t_2,x)) = \pi(t_1 t_2,x)$ for every $t_1, t_2 \in G$ and $x \in X$.

For a given ttg (G,X,π) we shall define for every $t \in G$ and every $x \in X$ the applications π^t resp. π_x by the formula $\pi^t(x): = \pi(t,x) = :\pi_x(t)$. Every π^t is called a *transition* and every π_x is called a *motion* of the ttg (G,X,π). We recall that the group $\{\pi^t | t \in G\}$ is a subgroup of the group H(X) of all the authomeomorphisms of the space X and the map $t \mapsto \pi^t$ (called also the *natural homomorphism* associated to (G,X,π)) is a group-homomorphism. We shall say that the ttg (G,X,π) is *effective* if for each $t \in G \setminus \{e\}$ there is $x \in X$ with $\pi(t,x) \neq x$.

1.1. LEMMA. *Let* (G,X,π) *be a ttg with* X *Hausdorff and let* $H_\pi: = \{t | \pi(t,x) = x$ *for every* $x \in X\}$. *Then:*

(a) H_π *is a closed invariant subgroup of* G,

(b) *Putting* $\tilde{G} = G/H_\pi$ *and* $\tilde{\pi}(tH_\pi,x): = \pi(t,x)$ *for* $t \in G$ *and* $x \in X$, *the triple* $(\tilde{G},X,\tilde{\pi})$ *is an effective ttg.*

(c) $\{\pi^t | t \in G\} = \{\tilde{\pi}^{\tilde{t}} | \tilde{t} \in \tilde{G}\}$.

PROOF. (a) and (b) follow from remark 1.3 in [3], and (c) is an immediate consequence of the definition of $\tilde{\pi}$. We notice here also that (G,X,π) is effective iff $t \mapsto \pi^t$ is injective or iff $H = \{e\}$; in this case (G,X,π) and $(\tilde{G},X,\tilde{\pi})$ are in fact identical. □

1.2. Let (G,X,π) and (G,Y,σ) be two ttg's; a continuous function $f : X \to Y$ is called a *homomorphism* (of ttg's) if $f(\pi(t,x) = \sigma(t,f(x))$ for every $(t,x) \in G \times X$. If Y is compact Hausdorff and f a topological embedding, then (G,Y,σ) is called a G-*compactification of* (G,X,π).

It is obvious that a ttg (G,X,π) can have a G-compacification only if X is completely regular (i.e. uniformizable and Hausdorff); this condition imposed on X will be considered henceforth as being automatically fullfilled. In [5] (see Theorem 7.3.12) the following fundamental result is proved concerning G-compactifications. It is a generalization of a result of R.B. Brook [2].

1.3. PROPOSITION. *The ttg* (G,X,π) *admits a* G-*compactification iff there is a uniformity* $\mathcal{U}$ *compatible with the topology of* X *so that for every* $U \in \mathcal{U}$ *there is a neighbourhood* V *of* e *with:*

$$t \in V \Rightarrow (\pi(t,x),x) \in U, \quad \textit{for each } x \in X. \quad \square$$

A ttg satisfying the above condition is said to be $\mathcal{U}$-*bounded*. It is obvious that the $\mathcal{U}$-boundedness of (G,X,π) is equivalent with the $\mathcal{U}$-equicontinuity of the family $\{\pi_x \mid x \in X\}$ at e.

We recall here the terminology concerning equicontinuity resp. uniform equicontinuity. Let X be a topological space, let $(Y,\mathcal{V})$ be a uniform space and let $F \subseteq Y^X$. We say that F is $\mathcal{V}$-*equicontinuous* at $x_0 \in X$ if for each $W \in \mathcal{V}$ there is a neigbourhood Γ of x_0 in X with $x \in \Gamma \Rightarrow (f(x),f(x_0)) \in W$, for every $f \in F$. The set F is said to be (*pointwise*) $\mathcal{V}$-*equicontinuous on* X if it is $\mathcal{V}$-equicontinuous at every $x \in X$. Finally, assuming that $(X,\mathcal{U})$ is a uniform space, F is said to be $(\mathcal{U},\mathcal{V})$-*uniformly equicontinuous* if for each $W \in \mathcal{V}$ there is $U \in \mathcal{U}$ such that $(x,y) \in U \Rightarrow (f(x),f(y)) \in W$ for every $f \in F$.

1.4. LEMMA. *Let* (G,X,π) *be a ttg,* Φ *its transition group and* $\phi : G \to \Phi$ *the natural homomorphism associated to* (G,X,π)*. Further let* $\mathcal{U}$ *be a uniformity of* X *and let us consider on* Φ *the topology of* $\mathcal{U}$*-convergence. Then* (G,X,π) *is* $\mathcal{U}$*-bounded iff* ϕ *is continuous at* e.

PROOF. As is well-known, the uniformity of $\mathcal{U}$-convergence on Φ is defined by the base $\{W(U) \mid U \in \mathcal{U}\}$, where

$$W(U) := \{(\phi,\psi) \mid \phi, \psi \in \Phi, (\phi(x), \psi(x)) \in U \text{ for all } x \in X\};$$

the corresponding uniform topology is called the topology of $\mathcal{U}$-convergence on Φ.

The continuity of ϕ at e can be written as follows: for each symmetric $U \in \mathcal{U}$ there is a neighbourhood V of e in G so that:

$$t \in V \Rightarrow \phi(t) \in W(U)[\phi(e)]$$

that is,

$$t \in V \Rightarrow (\phi(e), \phi(t)) \in W(U)$$

or equivalently,

$$(\pi^t(x),x) \in U \text{ for all } x \in X \text{ and } t \in V. \quad \square$$

2. MAIN RESULT

We begin this section with a lemma which has some intrinsic interest.

2.1. LEMMA. *Let* $\mathcal{U}$ *be a uniformity of the topological space* X *and let* $\Phi \subseteq X^X$ *be a semigroup (by composition), which contains* 1_X *(the identity on* X*). If* Φ *is pointwise* $\mathcal{U}$*-equicontinuous on* X*, then there is a uniformity* $\mathcal{V}$ *of* X *making* Φ $(\mathcal{V},\mathcal{V})$*-uniformly equicontinuous.*

PROOF. For each $U \in \mathcal{U}$, define $\Phi(U) := \{(x,y) \mid (\phi(x),\phi(y)) \in U$ for every $\phi \in \Phi\}$. We have for every $U, V \in \mathcal{U}$:

(a) $\Phi(U) \supseteq \Delta(X)$ ($\Delta(X)$ denotes the diagonal of $X \times X$),
(b) $\Phi(U \cap V) = \Phi(U) \cap \Phi(V)$,
(c) $[\Phi(U)]^{-1} = \Phi(U^{-1})$,
(d) $[\Phi(U)]^2 \subseteq \Phi(U^2)$.

The relations (a) - (d) show that $\{\Phi(U) \mid U \in \mathcal{U}\}$ is a base for a uniformity on X; this uniformity will be designated by $\Phi(\mathcal{U})$. Since $1_X \in \Phi$ it follows that $\Phi(U) \subseteq U$ for every $U \in \mathcal{U}$, hence $\mathcal{U} \subseteq \Phi(\mathcal{U})$. In particular, the topology induced by $\Phi(\mathcal{U})$ is finer than the original topology on X.

Conversely, let us consider $U \in \mathcal{U}$ and $x_0 \in X$; by $\mathcal{U}$-equicontinuity of Φ there is a neighbourhood Γ of x_0 such that:

$$x \in \Gamma \Rightarrow \phi(x) \in U[\phi(x_0)] \quad \text{for every } \phi \in \Phi,$$

hence $(x_0,x) \in \Phi(U)$ and therefore $x \in \Phi(U)[x_0]$. In other words $\Gamma \subseteq \Phi(U)[x_0]$, that is $\Phi(U)[x_0]$ is a neighbourhood of x_0 in the original topology of X. This means that the topology induced by $\Phi(\mathcal{U})$ is weaker than that induced by $\mathcal{U}$. Consequently, $\Phi(\mathcal{U})$ is a uniformity of X.

The procedure applied to $\mathcal{U}$ can be repeated in the case of $\Phi(\mathcal{U})$ giving rise to the uniformity $\Phi(\Phi(\mathcal{U}))$. Because $1_x \in \Phi$ and because Φ is closed under composition we have for each $U \in \mathcal{U}$:

$$\Phi(\Phi(U)) = \{(x,y) \mid (\phi(x),\phi(y)) \in \Phi(U)\}$$

$$= \{(x,y) \mid (\psi(\phi(x)),\psi(\phi(y))) \in U \text{ for all } \phi,\psi \in \Phi\} = \Phi(U),$$

so $\Phi(\mathcal{U}) = \Phi(\Phi(\mathcal{U}))$. Obviously Φ is $(\Phi(\Phi(U)),\Phi(U))$-uniformly equicontinuous, i.e. Φ is $(\Phi(\mathcal{U}),\Phi(\mathcal{U}))$-uniformly equicontinuous. □

2.2. PROPOSITION. *Let* (G,X,π) *be a ttg and let* Φ *be its transition group. If* Φ *is* $\mathcal{U}$*-equicontinuous on* X *(with respect to a certain uniformity* $\mathcal{U}$ *of* X*), then* (G,X,π) *has a* G*-compactification.*

PROOF. By lemma 2.1 we can assume that Φ is $(\mathcal{U},\mathcal{U})$-uniformly equicontinuous. Let $\mathcal{U}_d$ be the right uniformity on G, i.e. the uniformity having as a base the sets $V_\Theta = \{(t,s) \mid st^{-1} \in \Theta\}$, where Θ is an arbitrary neighbourhood of e. Let us observe that

(2) $(t,s) \in V_\Theta \Rightarrow (tu,su) \in V_\Theta$ for each $u \in G$.

For each $(V,U) \in \mathcal{U}_d \times \mathcal{U}$ we define now the set

(3) $\bar{W}(V,U) := \{(\pi^t(x),\pi^s(y)) \mid (t,s) \in V \text{ and } (x,y) \in U\}$.

The following inclusions are immediate:

(B1) $\bar{W}(V,U) \supseteq \Delta(X)$,

(B2) $\bar{W}(V_1 \cap V_2, U_1 \cap U_2) \subseteq \bar{W}(V_1,U_1) \cap \bar{W}(V_2,U_2)$,

(B3) $\bar{W}(V,U)^{-1} \supseteq \bar{W}(V^{-1},U^{-1})$,

and they hold for every (V,U), (V_1,U_1), $(V_2,U_2) \in \mathcal{U}_d \times \mathcal{U}$.

If now $(V_0,U_0) \in \mathcal{U}_d \times \mathcal{U}$, we choose $(V,U_1) \in \mathcal{U}_d \times \mathcal{U}$ with $V^2 \subseteq V_0$ and $U_1^2 \subseteq U_0$. By the uniform equicontinuity of Φ there is $U \in \mathcal{U}$ with $(x,y) \in U \Rightarrow (\pi^t(x),\pi^t(y)) \in U_1$ for every $t \in G$. A straightforward reasoning using (2), shows that:

(B4) $\bar{W}(V,U)^2 \subseteq \bar{W}(V_0,U_0)$.

From (B1) - (B4) it follows that $\{\bar{W}(V,U) \mid V \in \mathcal{U}_d, U \in \mathcal{U}\}$ is a base for a uniformity $\mathcal{W}$ on X. In order to prove that $\mathcal{W}$ is a uniformity of X we shall denote the original topology on X by $\mathcal{T}_1$, whereas the topology induced by $\mathcal{W}$ will be denoted by $\mathcal{T}_2$.

The inequality $\mathcal{T}_2 \leq \mathcal{T}_1$ follows readily, observing that $U \subseteq \bar{W}(V,U)$ for $(V,U) \in \mathcal{U}_d \times \mathcal{U}$. We prove the converse inequality. To this end, let $x_0 \in X$ and let Γ be a $\mathcal{T}_1$-neighbourhood of x_0. By the continuity of π there is a neighbourhood Θ of e and a $\mathcal{T}_1$-neighbourhood Γ_1 of x_0 such that:

$$(t,x) \in \Theta \times \Gamma_1 \Rightarrow \pi(t,x) \in \Gamma.$$

As $\mathcal{T}_1$ is a uniform topology there is $U_1 \in \mathcal{U}$ such that $U_1[x_0] \subseteq \Gamma_1$. Now we choose $U \in \mathcal{U}$ having the property:

$$(x,y) \in U \Rightarrow (\pi^t(x),\pi^t(y)) \in U_1 \quad \text{for every } t \in G.$$

We will prove:

(4) $\bar{W}(V_\Theta,U)[x_0] \subseteq \Gamma$.

Let $y \in \bar{W}(V_\Theta,U)[x_0]$; then $(x_0,y) = (\pi^s(u),\pi^r(v))$, where $(s,r) \in V_\Theta$, and $(u,v) \in U$. But $(x_0,y) = (\pi^s(u),\pi^{rs^{-1}}(\pi^s(v)))$, and by the choice of U: $(\pi^s(u),\pi^s(v)) \in U_1$, hence $\pi^s(v) \in U_1[\pi^s(u)] = U_1[x_0] \subseteq \Gamma_1$. On the other hand $rs^{-1} \in \Theta$, so that $\pi(rs^{-1},\pi^s(v)) \in \Gamma$. But this means that $\pi^r(v) = y \in \Gamma$, i.e. (4) is proved. Consequently, Γ is also a T_2-neighbourhood of x_0, that is $T_1 \leq T_2$. The compatibility of $\mathcal{W}$ and T_1 is proved.

The uniformity of the $\mathcal{W}$-convergence on Φ has as base all the sets of the form

$$\lambda(V,U) := \{(f,g) \mid f,g \in \Phi, (f(x),g(x)) \in \bar{W}(V,U) \text{ for all } x \in X\},$$

with $(V,U) \in \mathcal{U}_d \times \mathcal{U}$.

We consider the natural homomorphism $\phi : G \to \Phi$ and $(V,U) \in \mathcal{U}_d \times \mathcal{U}$. If $(t,s) \in V$, then $(\phi(t),\phi(s)) = (\pi^t,\pi^s) \in \lambda(V,U)$, because $(\pi^t(x),\pi^s(x)) \in \bar{W}(V,U)$ for each $x \in X$. It follows that ϕ is uniformly continuous with respect to the uniformity $\mathcal{U}_d$ on G and the uniformity of $\mathcal{W}$-convergence on Φ. In particular, ϕ is continuous with respect to the corresponding topologies. By lemma 1.4. we get that (G,X,π) is $\mathcal{W}$-bounded. □

<u>2.3. COROLLARY</u>. *Let* (G,X,π) *be a ttg with* G *compact. Then* (G,X,π) *has at least one* G-*compactification.*

<u>PROOF.</u> We must find a uniformity $\mathcal{U}$ of X so that the transition group Φ of (G,X,π) is $\mathcal{U}$-equicontinuous on X. It is not a restriction if we assume that (G,X,π) is effective otherwise we consider $(\tilde{G},X,\tilde{\pi})$ defined in lemma 1.1. and notice that it is effective and has the same transition group as (G,X,π). In this case the natural homomorphism $\phi : G \to \Phi$ is a bijection.

Considering on Φ the image through ϕ of the topology of G we get that Φ is a compact topological group. Because π is continuous on $G \times X$ the function $\sigma : X \times \Phi \to X$, given by: $\sigma(x,f) = f(x)$, is continuous as well (we have: $\sigma(x,f) = \pi(\phi^{-1}(f),x))$). Using a well known theorem about function spaces (e.g. [4], ch. VII) it follows that Φ is $\mathcal{U}$-equicontinuous on X for every uniformity $\mathcal{U}$ of the space X. □

2.4. EXAMPLE. Let us consider the ttg $(G, LUC_u(G), \tilde{\rho})$, where $LUC_u(G)$ is the space of all left-uniformly continuous real-valued functions on the topological group G endowed with the topology of the uniform convergence (on $\mathbb{R}$ we consider the usual uniformity). The action $\tilde{\rho}$ is defined by $\tilde{\rho}^t f(s) = f(st)$ for $f \in LUC_u(G)$ and $s, t \in G$. Then $\tilde{\rho}$ is continuous ([5], prop.2.2.4) and obviously $\{\tilde{\rho}^t | t \in G\}$ is uniformly equicontinuous with respect to the uniformity of uniform convergence. Thus $(G, LUC_u(G), \tilde{\rho})$ has a G-compactification.

The example given above shows us that our result is effectively not covered by [6], because we do not require the local compactness of G.

In the next example we point out that considering the uniformities $\Phi(\mathcal{U})$ (lemma 2.1.) resp. $\mathcal{W}$ (proposition 2.2.) is not superfluous; as we shall see the $\mathcal{U}$-equicontinuity of the transition group of a ttg generally does not imply the $(\mathcal{U},\mathcal{U})$-uniform equicontinuity, nor the $\mathcal{U}$-boundedness of the respective ttg.

2.5. EXAMPLE. Let $\mathbb{C}$ be the complex plane and let S^1 be the unit circle in $\mathbb{C}$. Further we consider $X_n := \{z | z \in C, |z| = 1 + \frac{n}{n+1}\}$, $X := \bigcup_{n=1}^{\infty} X_n$ and define $\pi : S^1 \times X \to X$ as follows: $\pi(t,z) := t^n z$ if $(t,z) \in S^1 \times X_n$. Notice that $\{\pi^t | t \in S^1\}$ is $\mathcal{V}$-equicontinuous on X for every uniformity $\mathcal{V}$ of X (S^1 being compact). We shall consider on X the uniformity induced by the additive uniformity of $\mathbb{C}$; let us designate it by $\mathcal{U}_0$.

First, notice that $\Phi = \{\pi^t | t \in S^1\}$ is not $(\mathcal{U}_0,\mathcal{U}_0)$-uniformly equicontinuous, otherwise on Φ the topology of the $\mathcal{U}_0$-convergence would coincide with that of the pointwise convergence ([1], theorem 1, ch.X, §2). But this is not the case; indeed we choose $t_n := \exp\left(\frac{i\pi}{n}\right)$ we have $t_n \to 1$ i.e. $\pi^{t_n} \to \pi^1$ (pointwise). Taking now $z_n \in X_n$ we get $|\pi^{t_n}(z_n) - z_n| = |-2z_n| > 2$, hence π^{t_n} does not converge to π^1 in the topology of the $\mathcal{U}_0$-convergence. This implies that in our case $\mathcal{U}_0 \neq \Phi(\mathcal{U}_0)$. Finally (S^1,X,π) is not $\mathcal{U}_0$-bounded because in the contrary case lemma 1.4. would imply: $\pi^{t_n} \to \pi^1$ ($\mathcal{U}_0$-uniformly) which is obviously false. It follows that (S^1,X,π) is not $\Phi(\mathcal{U}_0)$-bounded, because $\mathcal{U}_0 \subsetneq \Phi(\mathcal{U}_0)$.

Hence in the given example the uniformities $\mathcal{U}_0$, $\Phi(\mathcal{U}_0)$ and $\mathcal{W}$ (build from $\Phi(\mathcal{U}_0)$, cf. prop.2.2) are pairwise distinct.

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