A (57,14,1) STRONGLY REGULAR GRAPH DOES NOT EXIST
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A (57,14,1) strongly regular graph does not exist

by

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ABSTRACT

We show that a strongly regular graph with parameters

\[ n = 57, \quad k = 14, \quad \lambda = 1, \quad \mu = 4 \]

(0,1)-eigenvalues: 1*14, 38*2, 18*(-5);
(1,-1)-eigenvalues: 1*28, 38*(-5), 18*9 ) does not exist.

KEY WORDS & PHRASES: Strongly regular graph.
1. TWO LEMMAS

LEMMA 1. Let $G$ be a strongly regular graph with parameters $n,k,\lambda,\mu$. Let $H$ be an induced subgraph with $N$ points, $M$ edges and degree sequence $d_1,\ldots,d_N$. Then

$$(kN - 2M) - \left(\lambda M + \mu \left(\frac{N}{2} - M\right) - \sum_{i=1}^{N} \frac{d_i}{2}\right) \leq n - N$$

and equality holds iff exactly $(kN - 2M) - (n - N)$ points in $G \setminus H$ are adjacent to precisely two points of $H$, while the remaining points in $G \setminus H$ are adjacent to precisely one point of $H$.

PROOF. Let there be $x_i$ points in $G \setminus H$ adjacent to $i$ points of $H$. We have

$$\sum x_i = n - N,$$
$$\sum ix_i = kN - 2M,$$
$$\sum \binom{i}{2} x_i = \lambda M + \mu \left(\frac{N}{2} - M\right) - \sum_{i=1}^{N} \frac{d_i}{2}.$$ 

Since $\sum \binom{i}{2} x_i - \sum ix_i + \sum x_i = x_0 + \sum_{i=3}^{N} \binom{i-1}{2} x_i \geq 0$ this proves the lemma.  

LEMMA 2. Let $G$ be a strongly regular graph with parameters $n,k,\lambda,\mu$. Let $s$ be the smallest eigenvalue of the $(0,1)$-adjacency matrix of $G$, i.e., the negative root of the equation $x^2 + (\mu - \lambda)x + \mu - k = 0$. Then if $S$ is a coclique in $G$ we have

$$V := |S| \leq \frac{n* (-s)}{k-s}$$

and equality holds iff each point outside $S$ is adjacent to exactly

$$K := \frac{k \cdot V}{n - V}$$

points in $S$. In this case we find a $2-(V,K,\mu)$ design with point set $S$ and blocks $B_z = \{y \in S \mid y$ adjacent to $z\}$ for $z \in G \setminus S$.

PROOF. Let there be $x_i$ points in $G \setminus S$ adjacent to $i$ points of $S$. We have
\[ \sum x_i = n - V, \]
\[ \sum ix_i = k \cdot V, \]
\[ \sum (i^2)x_i = \mu \cdot \binom{n}{2}, \]
so that
\[ \sum (i-K)^2x_i = \mu V(V-1) + kV - \frac{k^2V^2}{n-V} \geq 0. \]

Writing \( x = \frac{kV}{V-n} \) and simplifying (using \( 0 < V < n \)) we see that this inequality is equivalent with
\[ x^2 + \left( \mu \cdot \frac{n-1}{k} - k+1 \right)x + \mu - k \leq 0 \]
which is exactly the desired inequality (note that the largest possible \( V \) corresponds to the smallest possible \( x \), and that the middle coefficient equals \( \mu - \lambda \) since \( n = 1 + k + k(k-1-\lambda)/\mu \)). □

2. THE NONEXISTENCE OF \((57,14,1)\)

Let \( G \) be a strongly regular graph with parameters \( n = 57, k = 14 \) and \( \lambda = 1 \). Then \( \mu = 4 \) and the smallest eigenvalue of the \((0,1)\)-adjacency matrix of \( G \) is \( s = -5 \). By Lemma 2 a coclique in \( G \) can have at most 15 points. We first derive a contradiction under the assumption that \( G \) contains a coclique of size 15, and then under the opposite assumption.

2.1. \( G \) has a 15-coclique

Let \( S \) be a 15-coclique in \( G \). If we identify a point \( z \) not in \( S \) with the set \( B_z = \{ y \in S \mid y \sim z \} \) (where \( \sim \) denotes adjacency) then the points of \( G \) are the points and blocks of a \( 2-(15,5,4) \) design \((S,B)\). Choose a block \( B_0 \), and investigate the intersection numbers
\[ x_i := x_i(B_0) := \# \{ B \in B \mid |B \cap B_0| = i \}. \]

Obviously, since \( \lambda, \mu \leq 4 \) we have \( x_5 = 1 \), i.e., there are no repeated blocks.
Since $\lambda = 1$, each edge is in a unique triangle, and each point is incident with 7 triangles. Of the seven triangles incident with $B_0$, five contain a point of $S$ and two consist of blocks only. But if a triangle consists of three blocks, these blocks must be mutually disjoint, because $\lambda = 1$. This proves $x_0 \geq 4$.

We have the equations

\[
x_0 + x_1 + x_2 + x_3 + x_4 = 41,
\]
\[
x_1 + 2x_2 + 3x_3 + 4x_4 = 5 \cdot 13 = 65,
\]
\[
x_2 + 3x_3 + 6x_4 = \binom{5}{2} \cdot 3 = 30.
\]

Consequently,

\[
x_0 + x_3 + 3x_4 = 6.
\]

Since $x_0 \geq 4$ it follows that $x_4 = 0$ and thus $x_0 + x_3 = 6$. But this soon leads to a contradiction:

Let $B_0, B_1, B_2$ and $B_0, B_3, B_4$ be two triangles containing $B_0$. Since intersections of size 4 do not occur we may suppose $|B_3 \cap B_1| = 3$, and then

\[
B_0 : \ 11111 \ 00000 \ 00000
\]

Let $B_1, B_5, B_7$ be another triangle containing $B_1$. W.l.o.g. $|B_5 \cap B_0| = 3$.

\[
B_1 : \ 00000 \ 11111 \ 00000
\]

Let $B_2, B_6, B_8$ be another triangle containing $B_2$. W.l.o.g. $|B_6 \cap B_0| = 3$.

\[
B_2 : \ 00000 \ 00000 \ 11111
\]

Finally, let $B_3, B, B'$ be another triangle containing $B_3$. W.l.o.g. $|B \cap B_0| = 3$.

\[
B_3 : \ 00000 \ 11000 \ 11000
\]

\[
B_4 : \ 00000 \ 00111 \ 00111
\]

\[
B_5 : \ 00000 \ 00000 \ 00000 \ <3*1>
\]

\[
B_6 : \ 00000 \ 00000 \ 00000 \ <2*1>
\]

\[
B : \ <3*1> \ 000..00...
\]

Since $x_3 \leq 2$ and $B_5 \neq B_6$, $B$ must coincide with either $B_5$ or $B_6$. But then $B$ and $B_0$ have at least five common neighbours: $B_1$ or $B_2$, $B_3$, and the three points in $B \cap B_0$. Contradiction, for $\lambda, \mu \leq 4$.

2.2. $G$ does not contain a 15-coclique

**Lemma.** $G$ does not contain a regular subgraph $H$ with 6 points and valency 3 (i.e., $K_{3,3}$ or the prism).
PROOF. Apply Lemma 1 with \( N = 6, M = 9, d_1 = \ldots = d_6 = 3 \).

We find \( 66 - 15 = 51 \). Since equality holds, exactly 15 points outside \( H \) are connected with two points in \( H \). If \( z \) is a point in \( G \setminus H \) adjacent to two points of \( H \), then let \( H' \) be the graph induced by \( G \) on \( H \cup \{ z \} \). Again apply Lemma 1, now with \( N = 7, M = 11, d_1 = 2, d_2 = d_3 = d_4 = d_5 = 3, d_6 = d_7 = 4 \).

We find \( 76 - 26 = 50 \). Since equality holds again, no point in \( G \setminus (H \cup \{ z \}) \) is adjacent to three points in \( H \cup \{ z \} \). It follows that if \( S \) is the set of 15 points adjacent to two points in \( H \), then \( S \) is a 15-coclique.

Contradiction.

In the previous section we considered \( G \) as a 2-(15,5,4) design; now we shall consider \( G \) as a GD[4,3,2;14] group divisible design: Let \( \infty \) be some fixed point, \( \Gamma := \Gamma(\infty) \) the set of its neighbours and \( \Delta \) the set of its non-neighbours. Then \( |\Gamma| = 14 \) and \( |
\Delta| = 42 \). \( G \) induces on \( \Gamma \) a regular graph with valency \( \lambda = 1 \), so that we find seven disjoint pairs in \( \Gamma \), the groups. For each point \( z \in \Delta \) we find a block \( B_z = \{ x \in \Gamma \mid x \sim z \} \) of size \( \mu = 4 \). One verifies immediately that \( \Gamma \) with these groups and blocks is a group divisible design GD[4,3,2;14] (in HANANI's notation).

(A) Let \( T \) be the union of two groups in \( \Gamma \). The set \( R \) of the six points in \( \Delta \) not joined to any point of \( T \) is a 6-coclique.

**Proof.** For \( u \in R \), let \( x_1 := x_1(u) := \# \{ z \in \Delta \mid z \sim u \text{ and } |\Gamma(z) \cap T| = i \} \). Then

\[
x_0 + x_1 + x_2 = k - \mu = 10
\]

and

\[
x_1 + 2x_2 = \mu \cdot |T| = 16
\]

so that \( x_2 - x_0 = 6 \). Suppose that \( u, v \in R \) and \( u \sim v \). Then \( x_0 \geq 1 \), so \( x_2 \geq 7 \) and hence both \( u \) and \( v \) have at least 7 neighbours in the set (of size 12) of points with two neighbours in \( T \). But then they must have at least two common neighbours. Contradiction with \( \lambda = 1 \). \( \square \)

(B) Let \( U = U(B) \) be the union of the three groups that do not intersect \( B \).

Let \( x_1 := x_1(U) := \# \{ z \in \Delta \mid |\Gamma(z) \cap U| = i \} \). Then
\[ x_0 + x_1 + x_2 + x_3 = |\Delta| = 42, \]
\[ x_1 + 2x_2 + 3x_3 = |U| \cdot (k-2) = 72, \]
\[ x_2 + 3x_3 = 12 \cdot (\mu-1) = 36, \]
so that \( x_0 + x_3 = 6. \)

Let \( y_i := y_i(B) := \# \{ z \in \Delta \mid z \sim B \text{ and } |\Gamma(z) \cap U(B)| = i \}. \) Then
\[ y_0 + y_1 + y_2 + y_3 = k-\mu = 10 \]
and
\[ y_1 + 2y_2 + 3y_3 = \mu \cdot |U| = 24. \]

From (A) it follows that \( y_0 = y_1 = 0 \) and hence \( y_2 = 6, y_3 = 4. \) We can identify these four neighbours of \( B \) intersecting \( U \) in three points: they are the blocks \( B_p \) where \( p \in \mathbb{N} \) and \( p \cap B_p \) is a triangle.

[For: suppose \( B_p \) intersects \( U \) in less than three points. Then there is a second group \( \{r,s\} \) intersecting both \( B \) and \( B_p \). Of course \( r \in B \cap B_p \) is impossible since \( \lambda = 1 \), so we would have \( r \in B \) and \( s \in B_p \). But now we find a prism on the set \( \{B,B_p,p,r,s,\infty\} \). Contradiction.

There are 42 blocks, but only \( \binom{7}{4} = 35 \) sets of 4 groups. Therefore, there must be two blocks, say \( B' \) and \( B'' \), intersecting the same four groups (i.e., \( U = U(B') = U(B'') \)). Now \( x_0(U) \geq 2 \) and \( x_3(U) \geq y_3 = 4, \) so \( x_3(U) = y_3(B') = y_3(B'') = 4; \) the four blocks intersecting \( U \) in three points are common neighbours of \( B' \) and \( B'' \), so \( B' \cap B'' = \emptyset \) since \( \mu = 4 \).

But for \( p \in B' \) the block \( B_p \) intersects \( \Gamma \setminus U \) only in the point \( p, \) i.e., \( B_p \neq B'' \) for \( p \in B', q \in B'' \). Contradiction.

Hence no graph \( G \) exists.