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BOUNDED DISCREPANCY SETS

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Bounded discrepancy sets*)

by

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ABSTRACT

Let $\omega = \{\xi_j\}_{j=1}^{\infty}$ be a sequence in $[0,1)$. We define the discrepancy function D_n by $D_n(\omega,\alpha) = Z_n(\omega,\alpha) - n\alpha$, where $Z_n(\omega,\alpha)$ is the number of elements in $[0,\alpha)$ among the first n terms of ω. It is known that $\sup_{\alpha,n} D_n(\omega,\alpha) = \infty$ for every sequence ω. In this paper sets S are characterized for which an ω exists such that $\sup_n D_n(\omega,\alpha) < \infty$ for every $\alpha \in S$. Furthermore we investigate sets S such that $\sup_{\alpha\in S, n\in \mathbb{N}} D_n(\omega,\alpha) < \infty$ for some ω. In particular, we show in Corollary 1 of Theorem 5 that such sets S have relatively large gaps. Theorems 1-4 are based on Lemma 1, which provides a construction for sequences with small discrepancy at specific points. Theorems 5 and 6 are applications of Lemma 3 which is proved by a method of W.H. Schmidt.

KEY WORDS & PHRASES: *Discrepancy, irregularities of distribution, uniform distribution.*

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*) This report will be submitted for publication elsewhere.

1. INTRODUCTION

Let U be the unit interval consisting of numbers ξ with $0 \le \xi < 1$, and let $\omega = \{\xi_1, \xi_2, \ldots\}$ be a sequence of numbers in this interval. Given an α in U and a positive integer n, we write $Z_n(\omega,\alpha)$ for the number of integers i with $1 \le i \le n$ and $0 \le \xi_i < \alpha$ and we put $D_n(\omega,\alpha) = Z_n(\omega,\alpha) - n\alpha$. For convenience we define $D_n(\omega,1) = 0$ and $D_0(\omega,\alpha) = 0$ for all α, n and ω. Put $D(\omega,\alpha) = \sup_n |D_n(\omega,\alpha)|$.

In answering a question of J.G. van der CORPUT [2], Mrs. T. van AARDENNE-EHRENFEST [1] showed that there is no sequence ω in U for which $\sup_{\alpha\in U} D(\omega,\alpha)$ is bounded. P. ERDÖS [3] wondered whether for every sequence ω there exist numbers α such that $D(\omega,\alpha) = \infty$. This was answered by W.M. SCHMIDT [4] in the affirmative. Later SCHMIDT [7, p.40] proved that for every sequence ω even

$$\limsup_{n\to\infty} \frac{|D_n(\omega,\alpha)|}{\log\log n} > \frac{1}{2000}$$

for almost all α. SCHMIDT [5] also investigated sets at which D can remain bounded. He demonstrated that the set $S(\infty) := \{\alpha\colon D(\omega,\alpha) < \infty\}$ is countable for every sequence ω. Theorem 1 gives the opposite result that for every countable subset S of U there exists a sequence ω such that $D(\omega,\alpha) < \infty$ for every α in S. In the special case $S = \mathbb{Q}$ Theorem 3 gives a quantitative result which is in a sense the best possible. We remark that SCHMIDT [6] generalized his result on the countability of $S(\infty)$ in a very remarkable manner. See also L. SHAPIRO [8].

We call S a κ-discrepancy set if there exists a sequence ω such that $D(\omega,\alpha) < \kappa$ for every α in S. A bounded discrepancy set (BDS) is a set which is a κ-discrepancy set for some κ. Theorem 2 states that every finite set is a BDS. Recall that a number γ is a limit point of a set S if there is a sequence of distinct elements of S which converges to γ. The derivative $S^{(1)}$ of S consists of all the limit points of S. The higher derivatives are defined inductively by $S^{(d)} = (S^{(d-1)})^{(1)}$ $(d = 2,3,\ldots)$. SCHMIDT [5] proved that $S^{(d)}$ is empty if s is a κ-discrepancy set and if $d > 4\kappa$. Furthermore he showed that $S^{(d)}$ need not be empty if S is a d-discrepancy set. This provides a necessary condition for being a BDS. The fact that $S = \{n^{-1}\}_{n=2}^{\infty}$ is not a BDS while $S^{(2)} = \emptyset$ shows that the condition is not sufficient. The corollary of Theorem 5 gives a property of a BDS which this set does not fulfill: if S is a BDS then there is an $\varepsilon > 0$ such that every interval of length ℓ contains a subinterval J of length $\varepsilon\ell$ with $J \cap S = \emptyset$. It seems a difficult problem to characterize BDS' in a simple way, if possible at all. In Section 4 we argue that the essential problem already occurs for a monotonic decreasing sequence with limit 0. Theorem 4 gives a sufficient condition for being a BDS and in Theorem 6 we show that in a certain case the necessary and suffcient conditions coincide.

2. The basic tool for construction BDS' is the following lemma.

LEMMA 1. *Let* α, β, γ *be real numbers with* $0 \le \alpha < \beta < \gamma \le 1$. *Let* $V \subseteq U$. *Assume there is a sequence* $\omega = \{\xi_n\}_{n=1}^{\infty}$ *in* V *such that* $D(\omega,\alpha) \le A$ *and* $D(\omega,\gamma) \le C$. *Then there exists a sequence* $\omega' = \{\xi'_n\}_{n=1}^{\infty}$ *in* $V \cup \{\alpha\} \cup \{\beta\}$ *such that*

(i) $\xi'_n = \xi_n$ if $\xi_n \in [0,\alpha) \cup [\gamma,1)$,

(ii) $\xi'_n \in \{\alpha,\beta\}$ if $\xi_n \in [\alpha,\gamma)$,

(iii) $D(\omega',x) = D(\omega,x)$ for $x \in [0,\alpha] \cup [\gamma,1)$,

(iv) $D(\omega',\beta) \le \frac{\gamma-\beta}{\gamma-\alpha} A + \frac{\beta-\alpha}{\gamma-\alpha} C + \frac{1}{2}$.

PROOF. We may assume without loss of generality that $\xi_n = \alpha$ if $\xi_n \in [\alpha,\gamma)$, since $D(\omega,x)$ for $x \in (\alpha,\gamma)$ is of no importance for the lemma. We shall prove by induction on m that we can define $\xi'_m \in \{\alpha,\beta\}$ in such a way that

$$(1) \qquad -\frac{1}{2} \le \Delta_m \le \frac{1}{2}$$

where

$$(2) \qquad \Delta_m = D_m(\omega',\beta) - \frac{\gamma-\beta}{\gamma-\alpha} D_m(\omega,\alpha) - \frac{\beta-\alpha}{\gamma-\alpha} D_m(\omega,\gamma).$$

It is obvious that $\Delta_0 = 0$ and that (1) holds for $m = 0$. Suppose that m is some non-negative integer for which the induction hypothesis holds. If $\xi_{m+1} \in [0,\alpha) \cup [\gamma,1)$, then we put $\xi'_{m+1} = \xi_{m+1}$. It follows that

$$\Delta_{m+1} = \Delta_m + (1-\beta) - \frac{\gamma-\beta}{\gamma-\alpha}(1-\alpha) - \frac{\beta-\alpha}{\gamma-\alpha}(1-\gamma) = \Delta_m$$

if $\xi_{m+1} \in [0,\alpha)$ and that

$$\Delta_{m+1} = \Delta_m - \beta + \frac{\gamma-\beta}{\gamma-\alpha}\alpha + \frac{\beta-\alpha}{\gamma-\alpha}\gamma = \Delta_m$$

if $\xi_{m+1} \in [\gamma,1)$. Hence (1) holds in this case. If $\xi_{m+1} = \alpha$ then put $\xi'_{m+1} = \alpha$ if $\Delta_m \le (\beta-\alpha)/(\gamma-\alpha) - \frac{1}{2}$ and $\xi'_{m+1} = \beta$ otherwise. If $\xi'_{m+1} = \alpha$, then

$$\Delta_{m+1} = \Delta_m + (1-\beta) + \frac{\gamma-\beta}{\gamma-\alpha}\alpha - \frac{\beta-\alpha}{\gamma-\alpha}(1-\gamma) = \Delta_m + 1 - \frac{\beta-\alpha}{\gamma-\alpha}$$

and hence, by (1), $-\frac{1}{2} \le \Delta_{m+1} \le \frac{1}{2}$. If $\xi'_{m+1} = \beta$, then

$$\Delta_{m+1} = \Delta_m - \frac{\beta-\alpha}{\gamma-\alpha}$$

and hence, by (1), $-\frac{1}{2} \le \Delta_{m+1} \le \frac{1}{2}$. Thus (1) is valid with m+1 in place of m.

By the above construction a sequence $\omega' = \{\xi'_n\}_{n=1}^{\infty}$ is defined which satisfies (i) and (ii). Further (iii) is an immediate consequence of (i) and (ii). Finally it follows from (1) and (2) that

$$|D_m(\omega',\beta)| \le \frac{\gamma-\beta}{\gamma-\alpha}|D_m(\omega,\alpha)| + \frac{\beta-\alpha}{\gamma-\alpha}|D_m(\omega,\gamma)| + \frac{1}{2}$$

for $m = 1,2,\ldots$. This implies (iv).

REMARK. Note that the discrepancy of ω' is bounded in both α and β and γ. Hence ω' assumes both values in $[\alpha,\beta)$ and in $[\beta,\gamma)$. By (i) and (ii) this implies that both α and β occur as terms of ω'.

3. SCHMIDT [5] proved that every $S(\infty)$ set is countable. The following theorem shows that every countable set is a $S(\infty)$-set.

THEOREM 1. *For every countable set* $S = \{\alpha_1,\alpha_2\ldots\}$ *in* U *there exists a sequence* ω *such that* $D(\omega,\alpha_j) < \infty$ *for* $j = 1,2,\ldots$.

PROOF. Without loss of generality we may assume that $0,\alpha_1,\alpha_2,\ldots$ are distinct numbers. We shall prove by induction on m that there exists a sequence $\omega_m = \{\xi_{m,1},\xi_{m,2},\ldots\}$ in $\{0,\alpha_1,\ldots,\alpha_m\}$ such that

(i) $D(\omega_m,\alpha_j) = D(\omega_{m-1},\alpha_j)$ for $j = 1,2,\ldots,m-1$,

(ii) $D(\omega_m,\alpha_j) < \infty$ for $j = 1,2,\ldots,m$,

(iii) If $1 \leq j < m$ and $\xi_{m-1,n}$ is the first element of ω_{m-1} with $\xi_{m-1,n} = \alpha_j$, then $\xi_{m,n} = \alpha_j$.

For $m = 1$ we apply Lemma 1 with $\alpha = 0$, $\beta = \alpha_1$, $\gamma = 1$, $A = C = 0$, $V = \{0\}$. Suppose that m is a non-negative integer for which the induction hypothesis holds. Let α be the largest element of the set $\{0,1,\alpha_1,\ldots,\alpha_m\}$ which is smaller than α_{m+1} and let γ be the smallest element of this set which is larger than α_{m+1}. Apply Lemma 1 with this α and γ and with $\beta = \alpha_{m+1}$. This gives a sequence ω'_{m+1} in $\{0,\alpha_1,\ldots,\alpha_{m+1}\}$ satisfying (i) and (ii). Let n be the smallest integer with $\xi_{m,n} = \alpha$. If $\xi'_{m+1,n} = \alpha$, then put $\omega_{m+1} = \omega'_{m+1}$. If $\xi'_{m+1,n} = \beta$, then we form ω_{m+1} by interchanging the first α and the first β in ω'_{m+1}. This change does only affect the discrepancy in $(\alpha,\beta]$, in fact by at most 1 in absolute value. Since ω_{m+1} is derived from ω_m by merely replacing some α's by β's, the other α_j's in ω_m remain unaltered. Thus ω_{m+1} satisfies (i) - (iii) and the induction step is complete.

By (iii) the sequence $\{\xi_{m,n}\}_{m=1}^{\infty}$ is constant from some $m_0 = m_0(n)$ on. Put $\xi_n = \xi_{m_0,n}$. This induces a sequence $\omega = \{\xi_1,\xi_2,\ldots\}$. By the construction $\xi_n < \alpha_j$ if $\xi_{j,n} < \alpha_j$ and $\xi_n \geq \alpha_j$ if $\xi_{j,n} \geq \alpha_j$, for all j and n. Hence $D(\omega,\alpha_j) = D(\omega_j,\alpha_j) < \infty$ for $j = 1,2,\ldots$.

The following theorem shows that every finite set is a BDS and gives an upper bound which can only be improved by a constant factor in view of Corollary 2.

<u>THEOREM 2</u>. *For every finite set* $S = \{\alpha_1,\alpha_2,\ldots,\alpha_m\}$ *in* U *there exists a sequence* ω *such that*

$$D(\omega,\alpha_j) \leq \frac{\log(2m)}{2\log 2} \qquad \text{for } j = 1,2,\ldots,m.$$

<u>PROOF</u>. We prove by induction on t that for every finite set $\{\alpha_1,\alpha_2,\ldots,\alpha_{2^t-1}\}$ in U there exists a sequence ω_t such that $D(\omega_t,\alpha_j) \leq t/2$ for $j = 1,2,\ldots,2^t-1$. For $t = 1$ we apply Lemma 1 with $\alpha = 0$, $\beta = \alpha_1$, $\gamma = 1$, $A = C = 0$. Suppose the induction hypothesis is true for t. Let $\{\alpha_1,\alpha_2,\ldots,\alpha_{2^{t+1}-1}\} \subset U$. We may assume without loss of generality that $0 < \alpha_1 < \alpha_2 < \ldots < \alpha_{2^{t+1}-1}$. Put

$\alpha_0 = 0$. There exists a sequence ω_t' in $\{\alpha_0,\alpha_2,\alpha_4,\ldots,\alpha_{2^{t+1}-2}\}$ such that $D(\omega_t',\alpha_{2i}) \le t/2$ for $i = 0,1,\ldots,2^t-1$. On applying Lemma 1 with $\alpha = \alpha_{2i}$, $\beta = \alpha_{2i+1}$, $\gamma = \alpha_{2i+2}$, $A = C = t/2$ for $i = 0,1,\ldots,2^t-1$ and combining the resulting sequences in an obvious way, we obtain a sequence ω_{t+1} such that $D(\omega_{t+1},\alpha_i) \le (t+1)/2$ for $i = 0,1,\ldots,2^{t+1}-1$. This proves the induction hypothesis for all values of t.

Let a set $S = \{\alpha_1,\alpha_2,\ldots,\alpha_m\}$ be given. Let t be the integer with $2^{t-1} \le m < 2^t$. We have shown that there exists a sequence $\omega = \omega_t$ with

$$D(\omega,\alpha_j) \le \frac{1}{2}\, t < \frac{1}{2}\left(1 + \frac{\log m}{\log 2}\right) \qquad \text{for } j = 1,2,\ldots,m.$$

The following result gives a quantitative form of Theorem 1 in the special case $S = \mathbb{Q}$ which is best possible in a similar way as Theorem 2 is.

<u>THEOREM 3</u>. *There exists a sequence* ω *such that*

$$D\left(\omega,\frac{p}{q}\right) \le 1+4 \log q$$

for every p/q *with* $p,q \in \mathbb{Z}$ *and* $0 < p < q$.

<u>PROOF</u>. We prove by induction on t that there exists a sequence $\omega_t = \{\xi_{t,n}\}_{n=1}^{\infty}$ in a finite set V_t of at most 2^{3t} rational numbers with the following properties:

(i) $V_{t-1} \subset V_t$ for $t \ge 2$,

(ii) V_t contains all numbers $p2^{-2t}$ with $p \in \mathbb{Z}$ and $0 \le p < 2^{2t}$,

(iii) V_t contains all numbers pq^{-1} with $p,q \in \mathbb{Z}$ and $0 < p < q \le 2^t$,

(iv) if $\alpha \in V_{t-1}$ and $\xi_{t-1,n}$ is the first element of ω_{t-1} with $\xi_{t-1,n} = \alpha$, then $\xi_{t,n} = \alpha$,

(v) $D(\omega_t,\alpha) \le \frac{5}{2}\, t - \frac{3}{2}$ for every α in V_t.

For $t = 1$ we take $V_1 = \{0,\frac{1}{4},\frac{1}{2},\frac{3}{4}\}$ and by a double application of Lemma 1 there exists a sequence ω_1 in V_1 such that $D(\omega_1,\alpha) \le 1$ for $\alpha \in V_1$. Suppose t is a positive integer for which the induction hypothesis is true. We construct V_{t+1} in three steps:

$$V_t' = V_t \cup \{\frac{k}{2^{2t+1}}: k \in \mathbb{Z},\ 0 < k < 2^{2t+1}\},$$

$$V_t'' = V_t' \cup \{\frac{k}{2^{2t+2}}: k \in \mathbb{Z},\ 0 < k < 2^{2t+2}\},$$

$$V_{t+1} = V_t'' \cup \{\frac{p}{q}: p,q \in \mathbb{Z},\ 0 < p < q \le 2^{t+1}\}.$$

Observe that at each step any two "new" points are separated by an "old" point. Hence we can apply Lemma 1 as we did in the proof of Theorem 2 and we obtain sequences ω_t', ω_t'', ω_t''' with discrepancy at V_t', V_t'', V_{t+1} at most $\frac{5}{2}t - 1$, $\frac{5}{2}t - \frac{1}{2}$, $\frac{5}{2}t$ respectively. Clearly (i) - (iii) are fulfilled with t+1 in place of t. For every $\alpha \in V_t$ with the property that $\xi_{t+1,n} \ne \alpha$ where n is the smallest integer with $\xi_{t,n} = \alpha$ we make an interchange like in the proof of Theorem 1. In such a case $\xi_{t+1,n}$ is a number $\beta \in V_{t+1} \setminus V_t$ which is smaller than the smallest element of V_t which is larger than α. By interchanging the first α and the first β in ω_t''' the discrepancy function remains unchanged outside the interval $(\alpha,\beta]$ and changes by at most 1 in $(\alpha,\beta]$ in absolute value. Since these intervals $(\alpha,\beta]$ are disjoint, the sequence ω_{t+1} which results after all interchanges have been made, satisfies (iv) with t+1 in place of t and moreover $D(\omega_{t+1},\alpha) \le \frac{5}{2}t+1$ for every $\alpha \in V_{t+1}$. This completes the induction step.

By (iv) the sequence $\{\xi_{t,n}\}_{n=1}^{\infty}$ is constant from some $t_0 = t_0(n)$ on. Put $\xi_n = \xi_{t_0,n}$. This induces a sequences $\omega = \{\xi_1,\xi_2,\dots\}$. By the construction $\xi_n < \alpha$ if $\xi_{t,n} < \alpha$ and $\xi_n \ge \alpha$ if $\xi_{t,n} \ge \alpha$ for every α, n and t with $\alpha \in V_t$. Let $p/q \in \mathbb{Z}$ with $0 < p < q \le 2^t$. Let t be the integer with $2^{t-1} < q \le 2^t$. Then $p/q \in V_t$. Hence

$$D(\omega,\frac{p}{q}) = D(\omega_t,\frac{p}{q}) \le \frac{5}{2}t - \frac{3}{2} < 1 + 5\log q/2\log 2 < 1 + 4\log q.$$

4. Suppose we want to decide whether a set S is a BDS. If it is, there exists a sequence ω and an integer d such that

(3) $\qquad D(\omega,\alpha) \le d \qquad$ for every $\alpha \in S$.

It follows from a result of SCHMIDT [5] that S has to be countable and $S^{(4d+1)} = \emptyset$. Note that $D_n(\omega,\alpha) = \lim_{\varepsilon\uparrow 0} D_n(\omega,\alpha+\varepsilon)$ for every α and n. Hence if α_0 is the limit of an increasing sequence in S and S satisfies (3) then $D(\omega,\alpha_0) \le d$. If $\alpha_0 > 0$ is a limit point of S but not the limit of an increasing sequence in S, then we can replace every α_0 in ω by $\alpha_0 - \varepsilon$ for a sufficiently small $\varepsilon > 0$ without changing $D(,\alpha)$ for $\alpha \in S \cup S^{(1)}\backslash\{\alpha_0\}$. For this new sequence ω' we have $D(\omega',\alpha) = \lim_{\varepsilon\downarrow 0} D(\omega,\alpha+\varepsilon) \le d$. Since we can do so for all such $\alpha_0 \in S^{(1)}\backslash S$ simultaneously, we conclude that S is a BDS if and only if $S \cup S^{(1)}$ is a BDS. We may therefore assume without loss of generality that S is closed. It further follows that $S^{(j)}$ $(j = 1,2,\ldots)$ as a subsequence of S is also a BDS. So it is sufficient to be able to decide whether a set S is a BDS if it is known that $S^{(1)}$ is a BDS, for then one can apply the argument to make the transitions $S^{(4d+1)} \to S^{(4d)} \to \ldots \to S^{(1)} \to S$.

Let S be a set such that $S^{(1)}$ is a BDS. For $\alpha \in S$ let $\phi(\alpha)$ denote an element in $S^{(1)}$ with $|\alpha - \phi(\alpha)|$ minimal. Let $\beta \in S^{(1)}$ and let $\alpha_1,\alpha_2\ldots$ be all elements of S with $\phi(\alpha_j) = \beta$ and $\alpha_j > \beta$ ordered in such a way that $\alpha_1 > \alpha_2 > \alpha_3 > \ldots$. It is obvious that $\alpha_1,\alpha_2,\ldots$ is a BDS if and only if $\alpha_1-\beta,\alpha_2-\beta,\ldots$ is a BDS. For the points $\alpha \in S$ with $\phi(\alpha) = \beta$ and $\alpha < \beta$ a similar argument applies. So the essential difficulty is to decide whether a monotonic sequence $\alpha_1,\alpha_2,\ldots$ in U with limit 0 is a κ-discrepancy set or not. If $S^{(1)}$ is a BDS and there exists a constant κ such that for every $\beta \in S^{(1)}$ both the points $\alpha \in S$ with $\phi(\alpha) = \beta$, $\alpha < \beta$ and the points $\alpha \in S$ with $\phi(\alpha) = \beta$, $\alpha > \beta$ are κ-discrepancy sets, then S is a BDS itself.

The following result gives a sufficient condition for a monotonic decreasing sequence with limit 0 to be a BDS. Necessary conditions for such sequences are given in Theorems 5 and 6.

<u>THEOREM 4</u>. *Let* $\{\alpha_n\}_{n=1}^{\infty}$ *be a monotonic decreasing sequence in* U *with* $\alpha_n \to 0$ *as* $n \to \infty$. *If there exist a positive integer* h *and a constant* c *with* $c < 1$ *such that* $\alpha_{n+h} < c\alpha_n$ *for* $n = 1,2,\ldots$, *then there exists a sequence* ω *such that*

$$D(\omega,\alpha_n) \le \frac{1}{2-2c} + \frac{\log 2h}{2\log 2} \qquad \textit{for } n = 1,2,\ldots .$$

<u>PROOF</u>. We prove by induction on t that there exists a sequence

$\omega_t = \{\xi_{t,n}\}_{n=1}^{\infty}$ in $\{0,\alpha_{th},\alpha_{th-1},\ldots,\alpha_1\}$ such that

(4) $$D(\omega_t,\alpha_{jh}) \leq \frac{1}{2-2c} \qquad \text{for } j = 1,2,\ldots,t$$

and

(5) $$D(\omega_t,\alpha_j) \leq \frac{1}{2-2c} + \frac{\log 2h}{2\log 2} \qquad \text{for } j = 1,2,\ldots,th.$$

For $t = 0$ the assertion is true. Suppose t is a non-negative integer for which the induction hypothesis holds. First apply Lemma 1 with $\alpha = 0$, $\beta = \alpha_{(t+1)h}$, $\gamma = \alpha_{th}$ ($\gamma = 1$ if $t = 0$), $A = C = (2-2c)^{-1}$. Hence, there exists a sequence ω_t' in $\{0,\alpha_{(t+1)h},\alpha_{th},\alpha_{th-1},\alpha_{th-2},\ldots,\alpha_1\}$ such that

$$D(\omega_t',\alpha_{jh}) \leq \frac{c}{2-2c} + \frac{1}{2} = \frac{1}{2-2c} \qquad \text{for } j = 1,2,\ldots,t+1$$

and

$$D(\omega_t',\alpha_j) \leq \frac{1}{2-2c} + \frac{\log 2h}{2\log 2} \qquad \text{for } j = 1,2,\ldots,th.$$

Next we apply the argument used in the proof of Theorem 2 to the points $\alpha_{(t+1)h-1},\ldots,\alpha_{th+1}$. The only difference is that everywhere A and C have to be increased by $(2-2c)^{-1}$. So we obtain a sequence ω_{t+1} in $\{0,\alpha_{(t+1)h},\alpha_{(t+1)h-1},\ldots,\alpha_1\}$ which satisfies (4) and (5) with $t+1$ instead of t.

Every sequence $\{\xi_{t,n}\}_{t=1}^{\infty}$ is constant from some $t_0 = t_0(n)$ on. Let $\xi_n = \lim_{t\to\infty} \xi_{t,n}$. This defines the sequence $\omega = \{\xi_n\}_{n=1}^{\infty}$. As before we have

$$D(\omega,\alpha_j) = D(\omega_j,\alpha_j) \leq \frac{1}{2-2c} + \frac{\log 2h}{2\log 2} \qquad \text{for } j = 1,2,\ldots\ .$$

5. To derive further properties of a BDS we use a technique due to SCHMIDT [5]. Since we shall work from now on with one sequence ω only, we shall suppress the variable ω and write $D_n(\alpha)$, etc. Let I and J be real intervals. We shall use the following notations.

$$h_I(\alpha) = \max_{n\in I} D_n(\alpha) - \min_{n\in I} D_n(\alpha),$$

$$D_n(\alpha,\beta) = D_n(\beta) - D_n(\alpha) = Z(n,\beta) - Z(n,\alpha) - n(\beta-\alpha),$$

and

$$h_{I,J}(\alpha,\beta) =$$

$$= \max(\min_{n\in I} D_n(\alpha,\beta) - \max_{n\in J} D_n(\alpha,\beta), \min_{n\in I} D_n(\alpha,\beta) - \max_{n\in I} D_n(\alpha,\beta)).$$

The following lemma involves Schmidt's basic idea.

LEMMA 2. *Suppose* $\alpha,\beta \in U$ *and suppose that* J, K *are subintervals of an interval* I. *Then*

$$h_I(\alpha) + h_I(\beta) \geq h_{J,K}(\alpha,\beta) + \frac{1}{2}(h_J(\alpha) + h_J(\beta) + h_K(\alpha) + h_K(\beta)).$$

PROOF. [5, Lemma 5].

We use Lemma 2 to show that the average value of $h_I(\alpha)$ in a sequence of well-spaced points α cannot be very small.

LEMMA 3. *Let* λ *be a real number with* $0 < \lambda \leq \frac{1}{2}$. *Let* c *and* t *be positive integers with* $3\lambda c \leq 4$. *Put* $m = (4c)^t$. *Let* I *be a real interval* $[x,y)$ *with* $x \geq 0$ *of length at least* m/λ. *Let* $\alpha_0,\alpha_1,\ldots,\alpha_{m-1}$ *be real numbers satisfying* $0 < \alpha_j - \alpha_{j-1} \leq \lambda c/m$ *for* $j = 1,2,\ldots,m-1$ *and* $\alpha_{j+m/2} - \alpha_j \geq \lambda$ *for* $j = 0,1,\ldots,\frac{1}{2}m-1$. *Then, for any sequence* ω *in* U,

$$\frac{1}{m}\sum_{j=0}^{m-1} h_I(\alpha_j) > \frac{t}{64c}. \tag{6}$$

PROOF. Let $J = [v,w)$ be any interval of length $m/(4c\lambda)$ with $v \geq 0$. Take integers a and b such that $v \leq a < v+1$ and $w-1 \leq b < w$. Suppose

$$Z_b(\alpha_{m-1}) - Z_a(\alpha_{m-1}) - Z_b(\alpha_0) + Z_a(\alpha_0) \leq \frac{m}{8c}. \tag{7}$$

Then, for $j = 0,1,\ldots,\frac{1}{2}m-1$,

$$D_b(\alpha_{j+\frac{1}{2}m}) - D_a(\alpha_{j+\frac{1}{2}m}) - D_b(\alpha_j) + D_a(\alpha_j)$$

$$\leq Z_b(\alpha_{m-1}) - Z_a(\alpha_{m-1}) - Z_b(\alpha_0) + Z_a(\alpha_0) - (b-a)(\alpha_{j+\frac{1}{2}m} - \alpha_j)$$

$$\leq \frac{m}{8c} - (\frac{m}{4c\lambda} - 2)\lambda = -\frac{m}{8c} + 2\lambda.$$

Hence,

$$h_J(\alpha_{j+\frac{1}{2}m}) + h_J(\alpha_j)$$

$$= \max_{n\in J} D_n(\alpha_{j+\frac{1}{2}m}) - \min_{n\in J} D_n(\alpha_{j+\frac{1}{2}m}) + \max_{n\in J} D_n(\alpha_j) - \min_{n\in J} D_n(\alpha_j)$$

$$\geq \frac{m}{8c} - 2\lambda \geq \frac{m}{8c} - 1.$$

On summing over j we obtain that under the supposition (7)

(8) $$\frac{1}{m}\sum_{j=0}^{m-1} h_J(\alpha_j) \geq \frac{m}{16c} - \frac{1}{2}$$

for any positive interval J of length $m/(4c\lambda)$.

We use induction on t. For t = 1 we have $D_n(\alpha) + n\alpha \in \mathbb{Z}$. Let $j \in \{0,1,\ldots,\frac{1}{2}m-1\}$. By the conditions of the lemma we have

$$\lambda \leq \alpha_{j+\frac{1}{2}m} - \alpha_j \leq \frac{1}{2}\lambda c \leq \frac{2}{3}.$$

Since $\min(\frac{1}{6},\frac{1}{2}\lambda) \geq \frac{\lambda}{3}$, we have $\|\alpha_j\| \geq \lambda/3$ or $\|\alpha_{j+\frac{1}{2}m}\| \geq \lambda/3$, where $\|\alpha\|$ denotes the distance from α to the nearest integer. We can therefore choose integers $i\in\{j,j+\frac{1}{2}m\}$ and $r,s \in I$ such that $D_r(\alpha_i) - D_s(\alpha_i) \geq 1/4$. Hence $h_I(\alpha_i) \geq 1/4$ and therefore

$$\frac{1}{m}\sum_{j=0}^{m-1} h_I(\alpha_j) \geq \frac{1}{m}\cdot\frac{m}{2}\cdot\frac{1}{4} = \frac{1}{8}.$$

This proves the lemma in case t = 1.

We now assume that the assertion of the lemma holds for t-1 and we shall deduce it for t. Put

$$J_i = [x + \frac{(i-1)m}{4\lambda c}, x + \frac{im}{4\lambda c})] \qquad \text{for } i = 1,2,3,4.$$

Let z_j be the number of pairs (μ, ξ_μ) with $x+m/(4\lambda c) \le \mu < x+2m/(4\lambda c)$ and $\xi_\mu - p \in [\alpha_{j-1}, \alpha_j)$ for some integer p. Hence z_j is a non-negative integer. We distinguish two cases.

(a) Assume $\Sigma_{j=1}^{m-1} z_j \le m/(8c)$. Then (7) is fulfilled for $v = x+m/(4\lambda c)$, $w = x+m/(2\lambda c)$. Hence, by (8),

$$\frac{1}{m} \sum_{j=0}^{m-1} h_{J_2}(\alpha_j) \ge \frac{m}{16c} - \frac{1}{2} \ge \frac{t}{16c}.$$

Since $J_2 \subset I$, this implies inequality (6).

(b) Assume $\Sigma_{j=1}^{m-1} z_j > m/(8c)$. For every $r \in J_1$ and $s \in J_3$ we have

$$D_s(\alpha_{j-1}, \alpha_j) - D_r(\alpha_{j-1}, \alpha_j) \ge z_j - (s-r)(\alpha_j - \alpha_{j-1})$$

$$\ge z_j - \frac{3m}{4\lambda c} \cdot \frac{\lambda c}{m} = z_j - \frac{3}{4}.$$

Hence, for $j = 0,1,\ldots,m-1$, in case $z_j \ge 1$,

$$h_{J_1, J_3}(\alpha_{j-1}, \alpha_j) \ge \frac{1}{4} z_j.$$

By Lemma 2, or obviously if $z_j = 0$,

$$h_I(\alpha_{j-1}) + h_I(\alpha_j) \ge \frac{1}{4} z_j + \frac{1}{2}(h_{J_1}(\alpha_{j-1}) + h_{J_1}(\alpha_j) + h_{J_3}(\alpha_{j-1}) + h_{J_3}(\alpha_j)).$$

Since $h_I(\alpha_j) \ge \max(h_{J_1}(\alpha_j), h_{J_3}(\alpha_j)) \ge \frac{1}{2} h_{J_1}(\alpha_j) + \frac{1}{2} h_{J_3}(\alpha_j)$, we have

$$2 \sum_{j=0}^{m-1} h_I(\alpha_j) \ge \sum_{j=0}^{m-1} (h_{J_1}(\alpha_j) + h_{J_3}(\alpha_j)) + \frac{1}{4} \sum_{j=1}^{m-1} z_j + h_I(\alpha_0)$$

$$+ h_I(\alpha_{m-1}) - \frac{1}{2} h_{J_1}(\alpha_0) - \frac{1}{2} h_{J_1}(\alpha_{m-1}) - \frac{1}{2} h_{J_3}(\alpha_0) - \frac{1}{2} h_{J_3}(\alpha_{m-1}) \ge$$

$$\ge \sum_{j=0}^{m-1} h_{J_1}(\alpha_j) + \sum_{j=0}^{m-1} h_{J_3}(\alpha_j) + \frac{m}{32c}.$$

On applying the induction hypothesis to J_i and the point sets $\{\alpha_{4c\ell+k}\}_{\ell=0}^{m/(4c)-1}$, we obtain

$$\sum_{j=0}^{m-1} h_{J_i}(\alpha_j) = \sum_{k=0}^{4c-1} \sum_{\ell=0}^{m/(4c)-1} h_{J_i}(\alpha_{4c\ell+k})$$

$$> \sum_{k=0}^{4c-1} \frac{m}{4c}\cdot\frac{t-1}{64c} = \frac{m}{64c}(t-1)$$

for j = 1 and j = 3. Hence,

$$\frac{1}{m}\sum_{j=0}^{m-1} h_I(\alpha_j) > \frac{t-1}{64c} + \frac{1}{64c} = \frac{t}{64c} .$$

This proves Lemma 3.

6. As an application of Lemma 3 we derive the following theorem.

<u>THEOREM 5</u>. *Let* γ *and* δ *be real numbers with* $0 \le \gamma < \delta \le 1$. *Let* H *be some positive integer. Let* $\gamma = \alpha_1, \alpha_2, \ldots, \alpha_N = \delta$ *be real numbers satisfying* $0 < \alpha_{i+1} - \alpha_i \le (\delta-\gamma)/H$ *for* $i = 1,2,\ldots,N-1$. *Then for every sequence* ω

(8) $$\max_{i=1,2,\ldots,N} D(\omega,\alpha_i) \ge \frac{1}{2000}\log\frac{H}{48} .$$

<u>PROOF</u>. Put $\ell = \delta - \gamma$. Let $t = [\log(H/3)/\log 16]$. So $H/48 < 16^t \le H/3$. Split $[\gamma,\delta)$ into 3.16^t parts of equal lengths and choose in every third part a point from $\{\alpha_1,\alpha_2,\ldots,\alpha_N\}$. This is possible, since $\ell/3.16^t \ge \ell/H$. This gives $m = 16^t$ points $\beta_1,\beta_2,\ldots,\beta_m$ with $\beta_j - \beta_{j-1} \le 4\ell/(3m)$. Further $\beta_{j+\frac{1}{2}m} - \beta_j \ge \ell/3$. We apply Lemma 3 with $\lambda = \ell/3$ and $c = 4$. Hence

$$\frac{1}{m}\sum_{j=0}^{m-1} h_I(\beta_j) > \frac{t}{256} > \frac{\log(H/48)}{256\log 16} > \frac{1}{1000}\log\frac{H}{48} .$$

It follows that for any sequence ω

$$\max_{j=0,1,\ldots,m-1} D(\omega,\beta_j) > \frac{1}{2000}\log\frac{H}{48}.$$

In particular (8) holds.

COROLLARY 1. *Let* S *be a* BDS. *Then there exists an* $\varepsilon > 0$ *such that every subinterval of* U *of length* ℓ *contains a subinterval* J *of length at least* $\varepsilon\ell$ *with* $J \cap S = \emptyset$.

PROOF. Let S be any BDS. Let ω be a sequence and κ a positive number such that

$$D(\omega,\alpha) \le \kappa \qquad \text{for every } \alpha \in S.$$

Let $[\gamma,\delta)$ be any subinterval of U. Choose H so large that

$$\frac{1}{2000} \log \frac{H}{48} > \kappa.$$

Put $\varepsilon = H^{-1}$. Then, by Theorem 5, $\max_{i=1,\dots,N} D(\omega,\alpha_i) > \kappa$ for any set $\{\alpha_1,\dots,\alpha_N\}$ in $[\gamma,\delta)$ with $0 < \alpha_{j+1} - \alpha_j \le \varepsilon(\delta-\gamma)$ for $j = 1,2,\dots,N-1$. Thus S does not contain such a subset. This proves the corollary.

The following result shows that Theorems 2 and 3 cannot be improved by more than a constant factor. (The constant $(4000)^{-1}$ can be improved considerably.)

COROLLARY 2. *Let* $n > 48^2$. *Then for every sequence* ω

$$\max_{j=0,1,\dots,n-1} D(\omega,\tfrac{j}{n}) \ge \frac{1}{2000} \log \frac{n}{48} \ge \frac{1}{4000} \log n.$$

7. It follows from Corollary 1 that $S = \{\frac{1}{n}\}_{n=2}^{\infty}$ is not a BDS. This result is also a consequence of the following theorem which gives a necessary and sufficient condition for sequences satisfying a certain regularity condition.

THEOREM 6. *Let* $\alpha_1,\alpha_2,\dots$ *be a strictly decreasing sequence with limit* 0. *Suppose there exists a constant* c *such that* $\alpha_{n-1} - \alpha_n \le c(\alpha_{m-1} - \alpha_m)$ *for every* n *and* m *with* $n \ge m$. *Then* $S = \{\alpha_1,\alpha_2,\dots\}$ *is a* BDS *if and only if for some positive integer* h

$$\limsup_{n\to\infty} \log \frac{\alpha_{n+h}}{\alpha_n} < 0.$$

PROOF. Suppose $\limsup_{n\to\infty} \log \alpha_{n+h} \alpha_n^{-1} < 0$. Then there exists a constant $c < 1$ such that $\alpha_{n+h} < c\alpha_n$ for $n = 1,2,\ldots$. It follows from Theorem 4 that S is a BDS. (Here we did not use the regularity condition.)

Suppose S is a BDS. Then by Corollary 1 there exists a positive number ε such that every interval $[0,\alpha_n)$ contains an interval J of length $\varepsilon\alpha_n$ such that $S \cap J = \emptyset$. Let k be such that $J \subset (\alpha_{n+k}, \alpha_{n+k-1})$. Then

$$\max_{j=0,\ldots,k} \alpha_{n+j-1} - \alpha_{n+j} > c^{-1}(\alpha_{n+k-1} - \alpha_{n+k}) \geq \varepsilon\alpha_n c^{-1}.$$

Hence,

$$\alpha_n \geq \alpha_n - \alpha_{n+k} \geq \varepsilon k \alpha_n c^{-1}.$$

Thus $k \leq c\varepsilon^{-1}$ is bounded, which implies that for $h = [c\varepsilon^{-1}]$

$$\limsup_{n \to \infty} \log \frac{\alpha_{n+h}}{\alpha_n} < \log(1-\varepsilon) < 0.$$

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