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ON AN IDENTITY OF T. BANG

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by

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ABSTRACT

Let $\phi_{n,k}$ be the class of n×n matrices with nonnegative integer coefficients such that each row and column sum equals k. By the "tensor product identity" of T. Bang we prove that

$$\min\{\text{per }\alpha \mid \alpha \in \phi_{n,4}\} \geq 2.(\tfrac{3}{2})^{n}$$

$$\min\{\text{per }\alpha \mid \alpha \in \phi_{n,6}\} \geq 5.(\tfrac{20}{9})^{n-1}.$$

KEY WORDS & PHRASES: *permanents*

1. INTRODUCTION

Let $\phi_{n,k}$ be the class of n×n matrices α with nonnegative integer coefficients such that each row and column sum of α equals k. We denote the permanent of α by per α and define

$$\lambda_k(n) := \min_{\alpha\in\phi_{n,k}} \text{per}\ \alpha; \qquad \theta_k := \inf_n (\lambda_k(n))^{1/n}.$$

Trivially, $\lambda_1(n) = 1$ and $\lambda_2(n) = 2$, so $\theta_1 = \theta_2 = 1$. ERDÖS and RÉNYI [2] conjectured that $\theta_k > 1$ for $k \geq 3$. This was proved by M. Voorhoeve and T. Bang and S. Friedland. VOORHOEVE [5] showed that $\theta_3 \geq 4/3$ by elementary methods, whereas BANG [1] and FRIEDLAND [3] proved that the permanent of a doubly stochastic matrix is at least e^{-n}, which implies that $\theta_k \geq k/e$. The basic tool in Bang's and Friedland's papers is an identity due to Bang.

In this paper we use Bang's identity to derive the estimates

$$\theta_4 \geq 3/2; \qquad \theta_6 \geq 20/9.$$

These bounds are slightly better than 4/e and 6/e respectively. SCHRIJVER and VALIANT [4] proved

$$(1) \qquad \theta_k \leq (k-1)^{k-1}/k^{k-2},$$

showing, by VOORHOEVE [5], that $\theta_3 = 4/3$. We conjecture that (1) gives the correct value for all θ_k's.

2. THE BASIC IDENTITY

<u>THEOREM 1</u>. (T. Bang). *Let* J_k *be the* k×k *matrix with all entries* 1/k. *Then for any* n×n *matrix* $A = (a_{ij})$, *the permanent of the tensor product* $A \otimes J_k$ *satisfies*

$$\text{per}(A \otimes J_k) = \frac{(k!)^{2n}}{k^{kn}} \sum_{n,k}^{(\alpha)} \prod_{i,j=1}^{n} (a_{ij}^{\alpha_{ij}}/\alpha_{ij}!),$$

where the sum $\Sigma_{n,k}^{(\alpha)}$ *ranges over all matrices* $\alpha = (\alpha_{ij})_{i,j=1}^{n}$ *in the class* $\phi_{n,k}$.

PROOF. See FRIEDLAND [3], Theorem 2.1, or rather BANG [1].

COROLLARY 1. *Using the same notation, we have*

$$\operatorname{per}(A \otimes J_{k-1})\operatorname{per}(A) = \frac{((k-1)!)^{2n}}{(k-1)^{(k-1)n}} \sum_{n,k}^{(\alpha)} \operatorname{per}\alpha \cdot \prod_{i,j=1}^{n} (a_{ij}^{\alpha_{ij}}/\alpha_{ij}!).$$

PROOF. We use throughout the notation $\Sigma_{n,k}^{(\alpha)}$ for sums ranging over all matrices α in $\phi_{n,k}$. Moreover, any product Πg_{ij} ranges over all possible g_{ij}'s. By Theorem 1 and the definition of the permanent,

$$\operatorname{per}(A \otimes J_{k-1})\operatorname{per}(A) = \frac{((k-1)!)^{2n}}{(k-1)^{(k-1)n}} \sum_{n,k-1}^{(\beta)} \Pi(a_{ij}^{\beta_{ij}}/\beta_{ij}!) \cdot \sum_{n,1}^{(\gamma)} a_{ij}^{\gamma_{ij}} =$$

$$= \frac{(k-1)!^{2n}}{(k-1)^{(k-1)n}} \sum_{n,k}^{(\alpha)} q(\alpha) \prod_{i,j=1}^{n} (a_{ij}^{\alpha_{ij}}/\alpha_{ij}!),$$

where

$$q(\alpha) = \sum_{\substack{\beta+\gamma=\alpha \\ \beta\in\phi_{n,k-1},\gamma\in\phi_{n,1}}} \Pi(\alpha_{ij}!/\beta_{ij}!).$$

Clearly, $q(\alpha) = \operatorname{per}\alpha$. $\square$

3. DERIVATION OF THE ESTIMATES FOR θ_4 AND θ_6

THEOREM 2. $\lambda_4(m) \geq 2.(\frac{3}{2})^m$.

COROLLARY 2. $\theta_4 \geq 3/2$.

PROOF. Take $k = 2$ in Theorem 1 and Corollary 1. Since $\lambda_2(n) = 2$, we find

(2) $$\operatorname{per}(A \otimes J_2) \ll \frac{1}{2}(\operatorname{per} A)^2.$$

Here the sign $\ll$ means term-by-term inequality: if we look upon per $A \otimes J_2$ and $\frac{1}{2}(\operatorname{per} A)^2$ as polynomials in the variables a_{ij}, each coefficient of the former is majorated by the latter.

Now take $A = B \otimes J_2$ for some $m\times m$ matrix $B = (b_{ij})$. If we consider both sides of (2) as polynomials in the b_{ij}'s, the same term-by-term inequality remains valid, hence

(3) $$\operatorname{per}(B \otimes J_4) = \operatorname{per}((B \otimes J_2) \otimes J_2) \ll \frac{1}{2}(\operatorname{per}(B \otimes J_2))^2$$

$$\ll \frac{1}{4} \operatorname{per}(B \otimes J_2)(\operatorname{per} B)^2.$$

By a reasoning similar to Corollary 1, we find

$$\operatorname{per}(B \otimes J_2)(\operatorname{per} B)^2 = \sum_{m,4}^{(\alpha)} r(\alpha) \prod b_{ij}^{\alpha_{ij}}/\alpha_{ij}! \ ,$$

where

$$r(\alpha) = \sum_{\substack{\beta+\gamma+\delta=\alpha \\ \beta\in\Phi_{m,2},\gamma,\delta\in\Phi_{m,1}}} \prod(\alpha_{ij}!/\beta_{ij}!) =$$

(5) $$= \sum_{\substack{\phi+\psi=\alpha \\ \phi\in\Phi_{m,3},\psi\in\Phi_{m,1}}} \operatorname{per} \phi . \prod(\alpha_{ij}!/\phi_{ij}!) \leq \max(\operatorname{per} \phi)\operatorname{per} \alpha$$

$$< (\operatorname{per} \alpha)^2.$$

By Theorem 1, the correct interpretation of the term-by-term inequality yields for each $\alpha \in \phi_{m,4}$

$$(\tfrac{9}{4})^m \leq \frac{1}{4} r(\alpha) \leq \frac{1}{4}(\operatorname{per} \alpha)^2. \qquad \Box$$

THEOREM 3. $\lambda_6(m) \geq 5.(\frac{20}{9})^{m-1}$.

COROLLARY 3. $\theta_6 \geq \frac{20}{9}$.

PROOF. By VOORHOEVE [5] we have for $n \geq 2$

$$\lambda_3(n) \geq 6.(\tfrac{4}{3})^{n-3} = \frac{81}{32} (\tfrac{4}{3})^n.$$

Hence, by Theorem 1 and Corollary 1, for $n \ge 2$

$$\operatorname{per}(A \otimes J_3) \ll \frac{32}{81} \operatorname{per}(A \otimes J_2)\operatorname{per}(A).$$

This implies for any $m\times m$ matrix B

$$\operatorname{per}(B \otimes J_6) \ll \frac{32}{81} \operatorname{per}(B \otimes J_4)\operatorname{per}(B \otimes J_2)$$

$$\ll \frac{16}{81} \operatorname{per}(B \otimes J_4)(\operatorname{per} B)^2.$$

As before,

$$\operatorname{per}(B \otimes J_4)(\operatorname{per} B)^2 \ll (\tfrac{9}{4})^m \cdot \sum_{m,6}^{(\alpha)} (\operatorname{per} \alpha)^2 \Pi(b_{ij}^{\alpha_{ij}}/\alpha_{ij}!)$$

So for all $\alpha \in \Phi_{m,6}$

$$(\tfrac{9}{4})^m (\operatorname{per} \alpha)^2 > \frac{81}{16} \cdot (\frac{100}{9})^m,$$

which proves the theorem. □

4. REMARKS

In (5) we used the estimate $\operatorname{per} \phi < \operatorname{per} \alpha$, a very crude approach, since $\phi \in \Phi_{n,3}$ and $\alpha \in \Phi_{n,4}$. It is not unreasonable to assume that if $\operatorname{per} \alpha = \lambda_4(m)$, $\operatorname{per} \phi$ is "about" $\lambda_3(m)$. This assumption then would imply that $\theta_4 = 27/16$. A similar heuristic reasoning holds for θ_6. This supports our conjecture.

<u>CONJECTURE 1</u>. $\theta_k = (k-1)^{k-1}/k^{k-2}$.

A stronger conjecture is

<u>CONJECTURE 2</u>. $\operatorname{per}(A \otimes J_k) \ll \operatorname{per}(A \otimes J_{k-1})\operatorname{per} A$.

<u>CONJECTURE 3</u>. For any $k\times k$ doubly stochastic matrix B and any $n\times n$ matrix A

$$\operatorname{per}(A \otimes B) \gg \operatorname{per}(A \otimes J_k)$$

Conjecture 3 implies both Conjecture 2 and the so-called van der Waerden conjecture, which is the case n = 1 of Conjecture 3.

REFERENCES

[1] BANG, T., *Matrixfunktioner som med et numerisk lille deficit viser v.d. Waerdens permanenthypothese,* to appear.

[2] ERDÖS, P. & A. RÉNYI, *On random matrices II,* Studia Sci. Math. Hungar. 3 (1968), 459-464.

[3] FRIEDLAND, S., *A lower bound for the permanent of a doubly stochastic matrix,* Annals of Math. 110, (1979), 167-176.

[4] SCHRIJVER, A. & W.G. VALIANT, *On lower bounds for permanents,* to appear.

[5] VOORHOEVE, M., *A lower bound for the permanents of certain* (0,1)-*matrices,* Proc. Kon. Ned. Akad. Wet., Ser. A 82 = Indag. Math. 41 (1979), 83-86.