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ON THE SIZE OF A MAXIMUM TRANSVERSAL IN A STEINER TRIPLE SYSTEM

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On the size of a maximum transversal in a Steiner triple system *)

by

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ABSTRACT

We show that a partial parallel class of maximum size in a Steiner triple system on v points leaves not more than $O(v^{2/3})$ points uncovered.

KEY WORDS & PHRASES: *transversal, partial parallel class, Steiner triple system*

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*) This report will be submitted for publication elsewhere.

Let $(X,\mathcal{B})$ be a Steiner triple system on $v = |X|$ points, and suppose that $\mathcal{F} \subset \mathcal{B}$ is a partial parallel class (transversal, clear set, set of pairwise disjoint blocks) of maximum size $t = |\mathcal{F}|$. We want to derive a bound on $r = |X\setminus \cup\mathcal{F}| = v - 3t$. (I conjecture that in fact r is bounded, e.g., $r \le 4$ - 4 is attained for the Fano plane -, but all that has been proved so far (cf. LINDNER & PHELPS [1], WANG [2]) are bounds $r < C.v$ for some C. Here we shall prove $r < 5v^{2/3}$.)

Define a sequence of positive real numbers by $q_0 = Q.\frac{r^2}{v}$, $q_1 = \frac{1}{2} q_0, \ldots$ $q_i = \frac{1}{2} q_{i-1}, \ldots, q_\ell$, where ℓ is determined by $q_\ell \ge 6$, $\frac{1}{2} q_\ell < 6$, i.e., $\ell = [\log(Qr^2/6v)/\log 2]$. (The constant Q will be chosen later.) Define inductively sets A_i, K_i and collections $\mathcal{B}_i$, $\mathcal{F}_i$ as follows. Let

$$A_0 = X\setminus \cup\mathcal{F},$$

and for $0 \le i \le \ell$, let

$$\mathcal{B}_i = \{T \in \mathcal{B} \mid |T \cap A_i| \ge 2\},$$

$$K_i = \{x \in X\setminus A_i \mid \#\{T \in \mathcal{B}_i \mid x \in T\} \ge q_i\},$$

$$\mathcal{F}_i = \{T \in \mathcal{F} \mid |T \cap K_i| \ge 1\},$$

$$A_{i+1} = A_0 \cup \cup\mathcal{F}_i \setminus K_i.$$

One verifies immediately that each of these series is increasing: $A_i \subset A_{i+1}$, $K_i \subset K_{i+1}$ etc. Also that $A_i \cap K_j = \emptyset$ $(\forall i,j)$. It is convenient to set $\mathcal{F}_{-1} = \emptyset$.
{The numbers q_i are chosen in such a way that an exchange process works. If B is an arbitrary block and I want to add it to $\mathcal{F}$, I must discard at most three members of $\mathcal{F}$ in order to maintain disjointness. But if the discarded triples are in $\mathcal{F}_i$ for some i then they are of the form $\{a,b,x\}$ with $x \in K_i$, and now that we no longer use x (supposing that $x \notin B$) we may add new triples $\{x,c,d\} \in \mathcal{B}_i$ to $\mathcal{F}$. In order to be able to add three pairwise disjoint triples $\{x_j,c_j,d_j\} \in \mathcal{B}_i$ $(j = 1,2,3)$ we must be sure that each x_j is incident with sufficiently many blocks in $\mathcal{B}_i$. (In fact it suffices if x_1 is incident with 1 block, x_2 with 3 blocks and x_3 with 5 blocks.) If $i = 0$ we are

finished and have increased the size of our transversal. If i > 0 then we must continue, discard the at most six members of $\mathcal{F}_{i-1}$ containing the points c_j, d_j and add again members of $\mathcal{B}_{i-1}$ etc.}

CLAIM.

(i) A_i does not contain a block $B \in \mathcal{B}$ $(0 \le i \le \ell+1)$.

(ii) No block $T \in \mathcal{F}$ intersects K_i in more than one point $(0 \le i \le \ell)$.

PROOF. Ad (i): If $B \subset A_0$ for some block $B \in \mathcal{B}$ then $\mathcal{F} \cup \{B\}$ would be a larger partial parallel class, a contradiction. If $B \subset A_{i+1}$ then we can enlarge $\mathcal{F}$ by an exchange process:

Define $\mathcal{N}_j$, $\mathcal{R}_j$ by backward induction on j $(i+1 \ge j \ge 0)$:

$$\mathcal{R}_{i+1} = \emptyset, \qquad \mathcal{N}_{i+1} = \{B\},$$

$$\mathcal{R}_j = \{T \in \mathcal{F}_j \setminus \mathcal{F}_{j-1} \mid T \cap \bigcup_{k=j+1}^{i+1} \cup\mathcal{N}_k \ne \emptyset\}.$$

Choose for $\mathcal{N}_j$ some collection of $|\mathcal{R}_j|$ blocks from $\mathcal{B}_j$ such that each $T \in \mathcal{R}_j$ meets exactly one of them, and such that $\mathcal{N}_j \cup \mathcal{N}_{j+1} \cup \ldots \cup \mathcal{N}_{i+1}$ is a collection of pairwise disjoint blocks. That the latter is possible follows from

$$|(\bigcup_{k=j}^{i+1} \cup\mathcal{N}_k) \cap A_j| \le 3.2^{i-j}$$

and

$$q_j \ge 6.2^{i-j} - 1.$$

Now $\mathcal{F}' = (\mathcal{F} \cup \bigcup_{j=0}^{i+1} \mathcal{N}_j) \setminus \bigcup_{j=0}^{i} \mathcal{R}_j$ is a layer partial parallel class, a contradiction.

Ad (ii): This is proved using an almost identical argument. □

Let $a_i = |A_i|$, so that $r = a_0$, and let $k_i = |K_i|$. By (ii) it follows that

$$(1) \qquad a_{i+1} = 2k_i + r.$$

From (i) it follows that

$$\binom{a_i}{2} \le k_i \cdot \frac{a_i}{2} + (v - k_i - a_i) \cdot q_i ,$$

hence

(2) $$a_i < k_i + \frac{2q_i v}{a_i} ,$$

and, using (1) and $a_j \ge a_0$, $q_j \le q_0$,

(3) $$a_{i+1} > 2a_i + r(1-4Q).$$

Now $v \ge a_{\ell+1} + k_\ell = r + 3k_\ell$ so that

$$\frac{1}{3} v > a_\ell - 2Qr$$

$$> 2a_{\ell-1} + r(1-6Q)$$

$$> 4a_{\ell-2} + r(3-14Q)$$

$$> \ldots$$

$$> 2^\ell a_0 + r(2^\ell - 1 - (2^{\ell+2}-2)Q)$$

$$= r(2^{\ell+1}-1)(1-2Q)$$

$$> r(\frac{Qr^2}{6v} - 1)(1-2Q).$$

Take $Q = \frac{1}{4}$. Then we have for large r:

$$(16+\varepsilon)v^2 > r^3$$

and one verifies immediately that $r \ge 5v^{2/3}$ leads to a contradiction for all r. In this proof we implicitly assumed that $\ell \ge 0$. But $\ell < 0$ means $Qr^2 < 6v$ so that again $Q = \frac{1}{4}$, $r \ge 5v^{2/3}$ leads to a contradiction. Thus we proved:

THEOREM. *A maximum transversal of an* STS(v) *has size at least*

$$\frac{1}{2}v - \frac{5}{3}v^{\frac{2}{3}}.$$

It is easy to improve the constant 5 (a minor change in this proof gives 3, and further improvement is possible) but I am presently unable to improve on the exponent $\frac{2}{3}$.

Note. An almost identical proof works for Steiner quadruple systems, and again gives $r = O(v^{2/3})$.

REFERENCES

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