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GEOMETRIES ORIGINATING FROM CERTAIN DISTANCE-REGULAR GRAPHS

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Geometries originating from certain distance-regular graphs*)

by

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ABSTRACT

Distance-regular graphs having intersection number $c_2=1$ are point graphs of linear incidence systems. This simple observation plays a crucial role in both the existence proof of a regular near octagon (associated with the Hall-Janko group) whose point graph is the unique automorphic graph with intersection array {10,8,8,2;1,1,4,5} and the non-existence proof of a distance-regular graph with intersection array {12,8,6,...;1,1,2,...}. These results imply a partial answer to a problem put forward by N.L. Biggs in 1976.

KEY WORDS & PHRASES: *finite geometries, distance-regular graphs, near polygons*

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*) This report will be submitted for publication elsewhere.

1. BASIC NOTIONS

Let $\Gamma = (V,E)$ be a connected graph of diameter d and let $\Gamma_i(\alpha)$ for $\alpha \in V$ denote the set of vertices at distance i from α. We recall from [2] that Γ is *distance-regular* if for any i $(0 \le i \le d)$ the numbers $b_i = |\Gamma_{i+1}(\alpha) \cap \Gamma_1(\beta)|$ and $c_i = |\Gamma_{i-1}(\alpha) \cap \Gamma_1(\beta)|$ do not depend on the choice of α, β such that $\beta \in \Gamma_i(\alpha)$. Of course, $b_d = 0$ and $c_1 = 1$. Write $k = |\Gamma_1(\alpha)|$. The array $\{k, b_1, b_2, \ldots, b_{d-1}; c_1, c_2, \ldots, c_d\}$ is called the *intersection array* of Γ.

The graph Γ is called *distance-transitive* if its automorphism group $\mathrm{Aut}(\Gamma)$ is transitive on each of the classes $\{\{\alpha,\beta\} \subset V \mid \beta \in \Gamma_i(\alpha)\}$ $(0 \le i \le d)$, and Γ is called *automorphic* (cf. [3]) whenever it is distance-transitive, not a complete graph or a line graph, and has an automorphism group which is primitive on V.

Given a linear incidence system (V,L) of points V and lines L, its *point graph* has the points V for vertices and the pairs of adjacent points for edges. For any two points of (V,L) the distance is meant to be the distance in the point graph. The notion of a regular near 2d-gon is introduced in [9]. A linear incidence system (V,L) is called a *regular near* 2d-*gon of order* $(s, t_d, t_2, \ldots, t_{d-1})$ and of *diameter* d if

(i) a line is incident with s+1 points; $s \ge 2$.

(ii) For any point $p \in V$ and line $\ell \in L$ there is a unique point q on ℓ nearest p.

(iii) The point graph of (V,L) is connected with diameter d.

(iv) For any i $(1 \le i \le d)$ and any two points p,q of distance i, there are precisely $1+t_i$ lines through q bearing a point of distance i-1 to p.

Note that $1+t_d$ is the number of lines through a point. A regular near 2d-gon has a distance-regular point graph. The following lemma (in the spirit of [4] and [7]) provides a partial converse to this phenomenon.

2. <u>LEMMA</u>. *Let* L *be the set of maximal cliques in a distance-regular graph* Γ *with intersection array* $\{k, b_1, b_2, \ldots, b_{d-1}; 1, 1, c_3, \ldots, c_d\}$. *Then* (V,L) *is a linear incidence system with constant line size* $k-b_1+1$. *Moreover, if there are natural numbers* $s, t, t_3, \ldots, t_{d-1}$ *such that* Γ *has intersection array* $\{s(t+1), st, st, s(t-t_3), \ldots, s(t-t_{d-1}); 1, 1, t_3+1, \ldots, t_{d-1}+1, t+1\}$, *then* (V,L) *is a regular near* 2d-*gon of order* $(s, t, 0, t_3, \ldots, t_{d-1})$.

PROOF. Given two mutually adjacent points $\alpha,\beta \in V$, there are $k-b_1-1$ points adjacent to both α,β. If γ,δ are two such points, they must be adjacent, for otherwise $\alpha,\beta \in \Gamma_1(\delta) \cap \Gamma_1(\gamma)$ would contradict $c_2=1$. This implies that all maximal cliques have size $k-b_1+1$ and that any two adjacent points are contained in precisely one maximal clique. As to the second statement of the lemma, fix $\alpha \in V$. By induction, we may assume that each line bearing points in both $\Gamma_{i-1}(\alpha)$ and $\Gamma_i(\alpha)$ has exactly one point in $\Gamma_{i-1}(\alpha)$. Thus there are precisely $1+t_i$ lines through a point $\beta \in \Gamma_i(\alpha)$ bearing a point of $\Gamma_{i-1}(\alpha)$. On these lines there are $(1+t_i)(s-1)$ points in $\Gamma_i(\alpha) \cap \Gamma_1(\beta)$. According to the intersection numbers these are all points of $\Gamma_i(\alpha) \cap \Gamma_1(\beta)$. Hence the remaining $t-t_i$ lines through β have all points but β in $\Gamma_{i+1}(\alpha)$. ⊠

3. THE NEAR OCTAGON ASSOCIATED WITH THE HALL-JANKO GROUP

Hall-Janko's simple group HJ of order 604800 has a conjugacy class V_0 of 315 involutions whose centralizers contain Sylow 2-subgroups. Setting E_0 for the pairs of involutions of V_0 whose product is again an element of V_0, a distance-regular graph (V_0,E_0) with intersection array $\{10,8,8,2;1,1,4,5\}$ results. By the previous lemma, the maximal cliques are the lines of a regular near octagon of order $(2,4,0,3)$.

It is possible to present this near octagon without assuming any previous knowledge on HJ. In order to do so we exploit the 'root system' given in [5]. Let $\mathbb{H} = \mathbb{R}1+\mathbb{R}i+\mathbb{R}j+\mathbb{R}k$ be the skew field of real quaternions whose multiplication is determined by $ij=k=-ji$ and $i^2=j^2=k^2=-1$. The multiplicative subgroup $Q = \{\pm1,\pm i,\pm j,\pm k\}$ is the well-known quaternion group. Set $\tau = (1+\sqrt{5})/2$ and $\zeta = (-1-i-j-k)/2$.

The projective plane $\mathbb{P}$ over $\mathbb{H}$ is represented by homogeneous coordinates $[x]$ for $x \in \mathbb{H}^3\backslash\{0\}$. Here $\mathbb{H}^3$ stands for the right vector space over $\mathbb{H}$ of dimension 3. Let A be the collineation group of $\mathbb{P}$ of order 1152 generated by the permutations of the homogeneous coordinates, scalar multiplication by ζ (acting on the left) and the transformations

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \to \begin{bmatrix} px_1 \\ qx_2 \\ rx_3 \end{bmatrix} \quad \text{for } p,q,r \in Q \text{ with } pqr \in \{\pm1\}.$$

Write

$$V_1 = A\begin{bmatrix}1\\0\\0\end{bmatrix} \cup A\begin{bmatrix}1\\1\\0\end{bmatrix} \cup A\begin{bmatrix}1\\\zeta^2+\tau\\1\end{bmatrix} \cup A\begin{bmatrix}1\\\zeta(1-\tau)\\\zeta^2\tau\end{bmatrix} .$$

Then V_1 consists of four A-orbits having lengths 3,24,192,96 respectively, so $|V_1| = 315$. Furthermore, the angles between elements $[x]$, $[y]$ of V_1 (i.e. $\sqrt{\frac{(x|y)(y|x)}{(x|x)(y|y)}}$, where $(x|y)$ stands for the standard unitary inner product on $\mathbb{H}^3$ linear in y) attain a limited number of values. The first A-orbit of V_1 consists of an *orthogonal triple*, i.e. a triple in which the angle of each pair is 0. Denote by L_1 the set of all orthogonal triples of V_1.

3.1 PROPOSITION. *Let* (V_1,L_1) *be as above. Then* (V_1,L_1) *is a regular near octagon of order* (2,4,0,3).

PROOF. In view of the lemma, it suffices to establish that the point graph Γ^1 of (V_1,L_1) is distance-regular with intersection array $\{10,8,8,2;1,1,4,5\}$. Consider the collineation s_X of $\mathbb{P}$ for $X = [x] \in V_1$ defined by

$$s_X(y) = y - 2x(x|y)/(x|x) \qquad (y\in\mathbb{H}^3).$$

Each of the s_X $(X\in V_1)$ stabilizes V_1 and leaves the angles invariant, whence $s_X \in \mathrm{Aut}(\Gamma^1)$. Let H be the subgroup of $\mathrm{Aut}(\Gamma^1)$ generated by the s_X $(X\in V_1)$ and A. It is straightforward that H acts transitively of rank 6 on V_1, the orbitals being indexed by the angles. In fact, the distances and angles of two points $X,Y \in V_1$ are related to the order of s_Xs_Y as indicated in the table

distance	0	1	2	3	4
angle	1	0	$1/\sqrt{2}$	1/2	$\tau/2, (\tau-1)/2$
order of s_Xs_Y	1	2	4	3	5

The proof of the proposition is thus reduced to the check whether the intersection number c_4 for a pair of points from the orbital indexed by angle $\tau/2$ coincides with the number c_4 for a pair from the $(\tau-1)/2$-orbital. This is left for the reader to verify. ⊠

3.2 COROLLARY. *The point graph* Γ^1 *of the near octagon in* (3.1) *is an automorphic graph.*

PROOF. The group H of the proof of (3.1) is the simple group of Hall-Janko (cf. [5] or [11]). It has index 2 in its automorphism group (cf. [8]). But Γ^1 can be described fully in terms of the conjugacy class of involutions $\{s_X | X \in P_1\}$ which is left invariant by Aut(H), so Aut(H) is a subgroup of Aut(Γ^1). In the permutation character of H on V_1 (cf. [8]) the two irreducible constituents of degree 14 are permuted by an outer automorphism of H. Thus Aut(H) has rank 5 on V_1. Finally, inspection of the subdegrees (which are 1, 10, 80, 160, 64) yields that the action of Aut(H) on V_1 is primitive. ⊠

3.3 THEOREM. *The automorphic graph* Γ^1 *of* (3.1) *is the unique automorphic graph with intersection array* $\{10,8,8,2;1,1,4,5\}$ *and has automorphism group* Aut(HJ).

PROOF. Let $\Gamma = (V,E)$ be an automorphic graph with intersection array as given above and set $G = \mathrm{Aut}(\Gamma)$. By the lemma $G = \mathrm{Aut}(V,L)$ for L the set of maximal cliques in Γ. Fix $\alpha \in V$. As G_α permutes the five lines through α, the order of its restriction to $\Gamma_1(\alpha)$ divides $5! \times 2^5$. From a theorem by JORDAN (cf. [12]; Theorem 18.4]) and from distance transitivity of Γ, it follows that $|G| = 2^a.3^b.5^c7$ with $a \geq 6$, $b \geq 2$, $c \geq 2$. Let N be a minimal normal subgroup of G. Since G is primitive, N is transitive on V, so 7 divides $|N|$. But 7^2 does not, so N is nonabelian simple. The centralizer in G of N, being normal in G, must be trivial, whence up to isomorphism $N \leq G \leq \mathrm{Aut}(N)$ holds. Fix $\delta \in \Gamma_4(\alpha)$. There is a 1-1 correspondence between the lines through α and the lines through δ such that a line ℓ through α corresponds to the unique line through δ that bears no point of distance 2 to any point of ℓ (this is a consequence of $c_3=4$). Denote by K the subgroup of $G_{\alpha,\delta}$ fixing the lines through α. We claim that K is trivial. For, K fixes the lines through δ (because of the 1-1 correspondence with the fixed lines through α), but each line through δ bears a unique point nearest α. These points are therefore fixed by K and so are the third points on the lines through δ. It follows that all neighbours within $\Gamma_4(\alpha)$ of any fixed point of K in $\Gamma_4(\alpha)$ are K-fixed points, too. As there are at least 46 points in the connected component of δ within the induced subgraph on $\Gamma_4(\alpha)$ of Γ and as this connected component is a block of imprimitivity for G_α in $\Gamma_4(\alpha)$, it follows that all points of $\Gamma_4(\alpha)$ are fixed by K. But G_α is faithful on $\Gamma_4(\alpha)$, whence K=1, as claimed.

The result is that G has order $[G:G_\alpha][G_\alpha:G_{\alpha,\delta}]\ |G_{\alpha,\delta}|$ dividing $315 \times 64 \times 5! = 2^9.3^3.5^2.7$. On the other hand $|N|$ is a multiple of $3^2.5^2.7$, as N_α must be transitive on the five lines through α. From [6], it is readily seen that the only possible order for N less than 10^6 is $2^7.3^3.5^2.7$, that N must be isomorphic to HJ, and that the corresponding permutation representation of degree 315 is unique. Aut(HJ) is then a subgroup of G (cf. 3.2) of index at most 2. Consequently G = Aut(HJ). The remaining values for $|N|$ are $2^a.3^3.5^2.7$ with $a \in \{8,9\}$. According to the work of BEISIEGEL [1] and STINGL [10], there is no simple group of such an order. ⊠

4. PROPOSITION. *There does not exist a distance-regular graph with intersection array* $\{12,8,6,\ldots;1,1,2,\ldots\}$.

PROOF. Let (V,L) be the linear incidence system obtained from such a distance-regular graph in the way described by the lemma. Fix $\alpha \in V$ and $\gamma \in \Gamma_2(\alpha)$. As there are 3 points on the unique line through γ containing a point of $\Gamma_1(\alpha)$ and as $k-b_2-c_2 = 5$ there are two lines through γ containing points of $\Gamma_3(\alpha)$. From $c_3 = 2$, it results that each of these two lines contains precisely one point distinct from γ inside $\Gamma_3(\alpha)$. Thus through any given point $\delta \in \Gamma_3(\alpha)$, there is exactly one line through δ bearing a point in $\Gamma_2(\alpha)$ and this line has precisely two points, say z_1, z_2 in $\Gamma_2(\alpha)$. By symmetry of the argument, there is precisely one line through α bearing two distinct points w_1, w_2 in $\Gamma_1(\alpha)$ of distance 2 to δ. Now w_i must be adjacent to z_i ($i = 1,2$; up to a permutation of indices). This implies that w_1, w_2, z_1, z_2 span a minimal 4-circuit, conflicting $c_2 = 1$. ⊠

5. CONCLUDING REMARKS

(i) The results of Gordon and Levingston put together with those above account for a complete solution of the existence problems connected with Biggs' list in [3, pp. 125,126].

(ii) The author is grateful to P. Rowlinson for bringing to his attention the problems dealt with in 3.3 and 4 and for explaining to him some of the known techniques.

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[1] B. BEISIEGEL, *Über einfache endliche Gruppen mit Sylow 2-Gruppen der Ordnung höchstens* 2^{10}, Comm. Algebra 5 (1977), 113-170.

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