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POLYNOMIAL VALUES AND ALMOST POWERS

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by

J.W.M. Turk

ABSTRACT

An almost power is an integer of the form ay^m, where $a \in \mathbb{Z}$ is given and y and m are arbitrary with $y \in \mathbb{Z}$ and $m \in \mathbb{N}$, $m \geq 2$. We prove some results on almost powers which appear as a value $F(x)$ of a given polynomial $F \in \mathbb{Z}[X]$ at some integer x.

KEY WORDS & PHRASES: *(almost) power, prime divisor, polynomial*

1. INTRODUCTION

For $n \in \mathbb{Z}$ and $m \in \mathbb{N}$ with $m \geq 2$ the m-free part of n is the smallest positive integer a with the property that $\pm n = ay^m$ for some $y \in \mathbb{Z}$. Let $F \in \mathbb{Z}[X]$ have at least two distinct zeros. Let $a \in \mathbb{N}$ with $a > 1$ be the m-free part of $F(x)$ for some $x \in \mathbb{Z}$ and some $m \in \mathbb{N}$ with $m \geq 2$. We prove that a has a certain multiplicative structure (see Corollary 1), e.g. the greatest prime divisor $P(a)$ of a exceeds $\varepsilon_1 \log\log a$, where ε_1 is a positive number depending only on F. This includes the well-known fact that $P(F(x)) > \varepsilon_1 \log\log |F(x)|$ for $x \in \mathbb{Z}$ with $|F(x)| > 1$.

Let $F \in \mathbb{Z}[X]$ have at least three simple zeros. Let $x \in \mathbb{Z}$ with $|F(x)| \geq 16$ and suppose that $F(x) = \pm ab$, where $a \in \mathbb{N}$ and b is some power, i.e. $b \in \{y^m \mid y \in \mathbb{Z},\ m \in \mathbb{N},\ m \geq 2\}$. We prove that then $a > \varepsilon_2(\log\log\log |F(x)|)^{\varepsilon_3}$ where ε_2 and ε_3 are positive numbers depending only on F. We also give such bounds for $P(a)$, thereby giving a quantitative version of the well-known fact that for every P there exist only finitely many $x \in \mathbb{Z}$ such that $F(x) = ab$ for some $a \in \mathbb{Z}$ with $P(a) \leq P$ and some power b.

2. Let $F \in \mathbb{Z}[X]$ have at least two distinct zeros and let $a \in \mathbb{Z}$, $a \neq 0$. It is well-known (see, e.g., [1]) that there exist positive numbers ε_F and $c_F(a)$ such that $P(F(x)) > \varepsilon_F \log\log |F(x)|$ for $|F(x)| > 1$ and that if $F(x) = ay^m$, for certain $x,y \in \mathbb{Z}$ with $|y| > 1$ and some $m \in \mathbb{N}$, then $m \leq c_F(a)$. In the existing proofs for the second result, the first result is used. This is unnecessary, in fact the first result can be proved in the same manner as the second one, as can be seen in the proof of Theorem 1 in this section. We also give an upper bound $c_F(a)$ for m which is explicit in a. For completeness we state the results that we use in the proof of Theorem 1. These results can be found in [1] as Theorem A (= Proposition), Lemma C (= Lemma 1,) while Lemma 2 is implicit in the proof of Theorem 1 in [1].

<u>PROPOSITION</u>. *Let $\alpha_1,\ldots,\alpha_N$, where $N \geq 2$, be non-zero algebraic numbers. Let K be the smallest normal field containing $\alpha_1,\ldots,\alpha_N$ and put $D = [K:\mathbb{Q}]$. Let $A_1,\ldots,A_N$ (≥ 3) be upper bounds for the heights of $\alpha_1,\ldots,\alpha_N$, respectively. Put $\Omega' = \prod_{j=1}^{N-1} \log A_j$, $\Omega = \Omega' \log A_N$. There exist positive numbers C_1 and C_2 such that for every $B \geq 2$ the inequalities*

$$0 < |\alpha_1^{b_1} \ldots \alpha_N^{b_N} - 1| < \exp(-(C_1 ND)^{C_2 N} \Omega \log \Omega' \log B)$$

have no solution in rational integers $b_1, \ldots, b_N$ *with absolute values at most* B.

LEMMA 1. *Let* γ_1 *and* γ_2 *be algebraic integers in a field* K *of degree* D. *Then*

$$H(\frac{\gamma_1}{\gamma_2}) \leq 2D \cdot 2^D \prod_\sigma \max\{|\sigma\gamma_1|, |\sigma\gamma_2|\}$$

where σ *runs through all isomorphic injections of* K *into* $\mathbb{C}$ *and* $H(\alpha)$ *denotes the height of* α.

LEMMA 2. *Let* K *be a field of degree* $[K:\mathbb{Q}] = D$. *By definition, the units of* K *are the algebraic integers* ε *in* K *with* $|N\varepsilon| = 1$, *where* $N = N_{K/\mathbb{Q}}$ *is the norm map from* K *to* $\mathbb{Q}$. *There exist an integer* $r = r(K) \in \{0,1,\ldots,D-1\}$ *and units* $\eta_0, \ldots, \eta_r$ *of* K, *with* η_0 *a root of unity, such that every unit* ε *of* K *is of the form* $\varepsilon = \prod_{j=0}^r \eta_j^{b_j}$ *for certain* $b_j \in \mathbb{Z}$ $(0 \leq j \leq r)$, *while* $|b_0| \leq c_0(K)$, *some suitable number depending only on* K. *Moreover, there exists a number* $c = c(K)$ *such that for every* $\alpha \in K$ *there exists a unit* ε *of* K *such that* $\beta = \varepsilon\alpha$ *satisfies* $c^{-1}|N\beta|^{1/D} \leq |\sigma\beta| \leq c|N\beta|^{1/D}$ *for every isomorphic injection* σ *of* K *into* $\mathbb{C}$.

THEOREM 1. *Let* $F \in \mathbb{Z}[X]$ *have at least two distinct zeros and let* $a \in \mathbb{Z}$, $a \neq 0$. *There exist positive numbers* x_F, $c_1 = c_1(F)$ *and* $c_2 = c_2(F)$, *depending only on* F, *such that if*

(1) $$F(x) = ay^m$$

with $x, y, m \in \mathbb{N}$ *and* $x \geq x_F$, *with the proviso that* $m \leq \log|F(x)|$ *if* $y = 1$, *then*

(2) $$m \leq (2(\omega(a)+1))^{c_1(\omega(a)+1)} (\prod_{p|a} \log p)^{c_2} =: c_F(a),$$

where $\omega(a)$ *denotes the number of distinct primes dividing* a.

<u>PROOF</u>. Write $F(X) = a_n \prod_{i=1}^{\nu} (X-\alpha_i)^{k_i}$ with $\alpha_1,\dots,\alpha_\nu$ distinct and $k_1,\dots,k_\nu \in \mathbb{N}$. We may assume that F is monic ($a_n=1$) in view of the following argument. It follows from (1) that $G(a_n x) := a_n^{n-1}F(x) = (a_n^{n-1}a)y^m$, where n is the degree of F. If the theorem has been proved for monic polynomials then, since G is monic, $m \le c_G(a_n^{n-1}a)$ and the contribution of the primes dividing a_n can be incorporated in $c_1 = c_1(F)$. Since $a_n = 1$ the $\alpha_1,\dots,\alpha_\nu$ are algebraic integers. Let K be the (normal) field generated by $\alpha_1,\dots,\alpha_\nu$, put $d = [K:\mathbb{Q}]$ and let $\mathcal{P}_1,\dots,\mathcal{P}_s$ be the distinct prime ideals of K which divide the ideal generated by $a\cdot\prod_{i<j} (\alpha_i-\alpha_j)$. Assume that (1) holds for some $x \ge x_F$, where $x_F \ge 2$ is sufficiently large, depending only on F (how large will be apparent from the sequel). Then the prime ideal decomposition of the integral ideal $[x-\alpha_i]$ generated by $x-\alpha_i$ has the form

$$[x-\alpha_i] = \prod_{k=1}^{s} \mathcal{P}_k^{w_k(i)} \Gamma_i^{m_i}, \qquad (i = 1,\dots,\nu)$$

for certain $w_k(i) \ge 0$, $w_k(i) \in \mathbb{Z}$, where $m_i = m/(m,k_i)$, with $\Gamma_i = [1]$ if $y = 1$. Choose α_i and α_j with $\alpha_i \ne \alpha_j$, say α_1 and α_2. Note that m_1 and m_2 are divisible by $m^* = m/(m,[k_1,k_2])$. Hence there exist integral ideals Γ_1^*, Γ_2^* (= [1] if y=1) in K with

$$(3) \qquad [x-\alpha_i] = \prod_{k=1}^{s} \mathcal{P}_k^{w_k(i)} (\Gamma_i^*)^{m^*} \qquad (i = 1,2).$$

Note that it follows from (1) and $y > 1$ that $m^* \le m \le (\log|F(x)|)(\log 2)^{-1} \le c_3 \log x$ for some $c_3 = c_3(F)$. If $y = 1$ in (1) then

$$(4) \qquad m^* \le m \le c_3 \log x$$

also holds, by assumption. Taking $x_F \ge 2 \max_{1\le i\le\nu} |\alpha_i|$, we have, by (3) and $x \ge x_F$,

$$(5) \qquad \sum_{k=1}^{s} w_k(i) \log N\mathcal{P}_k + m^* \log N\Gamma_i^* = \log N[x-\alpha_i] \le \log(2x)^d \ll_F \log x, \qquad (i = 1,2).$$

Let h be the class number of K. By Lemma 2 there exist algebraic integers

$\pi_1,\dots,\pi_s,\gamma_1,\gamma_2$ in K (with $\gamma_i=1$ if $\Gamma_i^* = [1]$) and a number $c = c(K)$ such that

$$(6)\qquad \begin{cases} [\pi_k] = \mathcal{P}_k^h, \quad [\gamma_i] = (\Gamma_i^*)^h, & \text{for } k = 1,\dots,s \text{ and } i = 1,2 \text{ and} \\ c^{-1} \le |\sigma\alpha|\,|N\alpha|^{-1/d} \le c & \text{for every automorphism } \sigma \text{ of } K \text{ and } \alpha \in \{\pi_1,\dots,\pi_s,\gamma_1,\gamma_2\}. \end{cases}$$

It follows from (3) and (6) that $(x-\alpha_i)^h = \varepsilon_i \prod_{k=1}^{s} \pi_k^{w_k(i)} \gamma_i^{m^*}$ for some unit ε_i of K (i = 1,2). Hence, by Lemma 2,

$$(7)\qquad (x-\alpha_i)^h = \prod_{j=0}^{r} \eta_j^{b_j(i)} \prod_{k=1}^{s} \pi_k^{w_k(i)} \gamma_i^{m^*} \qquad (i = 1,2),$$

for certain $b_j(i) \in \mathbb{Z}$, with $|b_0(i)| \le c_0(K)$. We now show that

$$(8)\qquad |b_j(i)| \ll_F \log x \qquad \text{for } j = 0,1,\dots,r \text{ and } i = 1,2.$$

Since $\sigma x = x$ and $|\sigma\eta_0| = 1$ for $\sigma \in \mathrm{Aut}(K)$ we infer from (7) that

$$(9)\qquad \sum_{j=1}^{r} b_j(i)\log|\sigma\eta_j| = h\log|x-\sigma\alpha_i| - \sum_{k=1}^{s} w_k(i)\log|\sigma\pi_k| - m^*\log|\sigma\gamma_i|$$

$$=: \lambda_\sigma(i) \qquad \text{for } \sigma \in \mathrm{Aut}(K) \text{ and } i = 1,2.$$

It follows from (6), $N\mathcal{P}_k \ge 2$, $N\Gamma_i^* \ge 1$ that $\log|\sigma\pi_k| \ge -\log c$ and $\log|\sigma\gamma_i| \ge -\log c$. With the use of (5) we obtain that $|\lambda_\sigma(i)| \ll_F \log x$ for every σ and $i = 1,2$. From the equations (9) with r distinct σ's and Cramer's rule it follows that (8) holds for $j = 1,\dots,r$ and $i = 1,2$. As observed already, (8) also holds for $j = 0$. From (7) we conclude that

$$(10)\qquad (x-\alpha_i)^h = \prod_{j=0}^{r} \eta_j^{\beta_j(i)} \prod_{k=1}^{s} \pi_k^{\omega_k(i)} \delta_i^{m^*} \qquad (i = 1,2),$$

where $\omega_k(i) \in \{0,1,\dots,m^*-1\}$ with $\omega_k(i) \equiv w_k(i) \bmod m^*$ and $|\beta_j(i)| < m^*$ with $\beta_j(i) \equiv b_j(i) \bmod m^*$ and $\mathrm{sgn}(\beta_j(i)) = \mathrm{sgn}(b_j(i))$, with

$$\delta_i = \gamma_i \Big\{ \prod_{j=0}^{r} \eta_j^{b_j(i)-\beta_j(i)} \prod_{k=1}^{s} \pi_k^{w_k(i)-\omega_k(i)} \Big\}^{1/m^*} \qquad (i = 1,2).$$

We now show that the algebraic integers $\delta_i \in K$ satisfy, for some $c_4 = c_4(F)$,

(11) $$|\sigma\delta_i| \le \exp((c_4 \log x)/m^*) \qquad \text{for } \sigma \in \mathrm{Aut}(K) \text{ and } i = 1,2.$$

By (6) and (5) we have $\log|\sigma\gamma_i| \ll_F \log N\Gamma_i^* \ll_F (\log x)/m^*$. By (8) we have

$$\sum_{j=0}^{r} (b_j(i) - \beta_j(i)) \log|\sigma\eta_j| \ll_F \log x.$$

By (6) and (5) we have

$$\sum_{k=1}^{s} (w_k(i) - \omega_k(i)) \log|\sigma\pi_k| \ll_F \sum_{k=1}^{s} (w_k(i) - \omega_k(i)) \log N\mathcal{P}_k$$
$$\le \sum_{k=1}^{s} w_k(i) \log N\mathcal{P}_k \ll_F \log x.$$

This proves (11). From (10) we obtain

(12) $$\left(\frac{x-\alpha_2}{x-\alpha_1}\right)^h - 1 = \prod_{j=0}^{r} \eta_j^{\beta_j(2)-\beta_j(1)} \prod_{k=1}^{s} \pi_k^{\omega_k(2)-\omega_k(1)} \left(\frac{\delta_2}{\delta_1}\right)^{m^*} - 1.$$

Taking $x_F > (\alpha_2-\alpha_1\zeta)(1-\zeta)^{-1}$ for every $\zeta \ne 1$ with $\zeta^h = 1$, we have, for $x \ge x_F$, that the expression in (12) is non-zero. Assuming that $m^* > 1$ we apply the proposition to the right hand side of (12), with $N = r+s+2$, $\alpha_i = \eta_{i-1}$ for $1 \le i \le r+1$, $\alpha_{r+2},\dots,\alpha_{N-1} = \pi_1,\dots,\pi_s$, $\alpha_N = \delta_2/\delta_1$ and $B = m^*$. We have $N = r+s+2 \le d-1+d\omega+2 \le (d+1)(\omega+1)$, where $\omega = \omega(a)$. Also $H(\alpha_i) \le c_5 = c_5(K) =: A_i$ for $1 \le i \le r+1$. By Lemma 1 and (6) we have $H(\pi_k) \le 2d\cdot 2^d \prod_\sigma \max\{|\sigma\pi_k|,1\} \le 2d\cdot 2^d (N\mathcal{P}_k)^h \le p(k)^{c_6} =: A_{r+k+1}$ $(1 \le k \le s)$ for some $c_6 = c_6(K)$, where $p(k)$ is the rational prime in $\mathcal{P}_k$. Note that the number of distinct k with $p(k) = p$ is at most d for every prime p. By Lemma 1 and (11) we have

$$H(\delta_2/\delta_1) \le 2d\cdot 2^d \prod_\sigma \max\{|\sigma\delta_2|,|\sigma\delta_1|\} \le 2d\cdot 2^d \exp((dc_4 \log x)/m^*)$$
$$\le \exp((c_7 \log x)/m^*) =: A_N,$$

where we used (4) in the last inequality. It follows from the proposition that

$$\left|\left(\frac{x-\alpha_2}{x-\alpha_1}\right)^h - 1\right| > \exp\left(-(C_1 d(d+1)(\omega+1))^{C_2(d+1)(\omega+1)} c_8\left(\prod_{p|a} \log p\right)^d \cdot\right.$$

$$\left.\cdot\ (\log c_8 + d \sum_{p|a} \operatorname{loglog} p)\cdot(\log x)\,(m^*)^{-1}\log m^*\right)$$

for some $c_8 = c_8(K)$. On the other hand, for $x \geq x_F$, with x_F sufficiently large,

$$\left|\left(\frac{x-\alpha_2}{x-\alpha_1}\right)^h - 1\right| = \left|\left(1+\frac{\alpha_1-\alpha_2}{x-\alpha_1}\right)^h - 1\right| < c_9 x^{-1} < \exp\left(-\tfrac{1}{2}\log x\right).$$

Combining these estimates for $|((x-\alpha_2)/(x-\alpha_1))^h - 1|$ we obtain, for some large $c_{10} = c_{10}(F)$,

$$m^* = 1 \quad\text{or}\quad m^*/\log m^* < (2(\omega+1))^{c_{10}(\omega+1)}\left(\prod_{p|a}\log p\right)^d\left(1+\sum_{p|a}\operatorname{loglog} p\right).$$

Finally observe that $m \leq [k_1,k_2]m^* = c_{11}m^*$. This implies that (2) holds for every $c_2 > d$, provided that $c_1 = c_1(F)$ is sufficiently large. □

REMARK. If $y = 1$ in (1) then one can, more naturally, apply the proposition to the right hand side of (7) with $\gamma_i = 1$, with $N = r+s+1$, $B = c_F \log x$ (cf. (8) and (5)).

3. DEFINITION. Let $m \in \mathbb{N}$ with $m \geq 2$. The m-free part of an integer n is the smallest positive integer a with the property that $\pm n = ay^m$ for some $y \in \mathbb{Z}$.

COROLLARY 1. *Let $F \in \mathbb{Z}[X]$ have at least two distinct zeros. There exist positive numbers ε_1 and δ_1, depending only on F, with the following property. Let $a \in \mathbb{N}$ with $a \geq 3$ be the m-free part of $F(x)$ for some $x \in \mathbb{Z}$ and some $m \in \mathbb{N}$ with $m \geq 2$. Then*

$$(13)\qquad \begin{cases} \text{(i)}\quad \omega(a) > \delta_1(\operatorname{loglog} a)(\operatorname{logloglog} a)^{-1} \\ \qquad\qquad \text{or} \\ \text{(ii)}\quad P(a) > \exp(\delta_1(\operatorname{loglog} a)(\operatorname{logloglog} a)^{-1}). \end{cases}$$

In particular

(14) $P(a) > \varepsilon_1 \log\log a.$

PROOF. Since $\omega(a) \geq 1$ and $P(a) \geq 2$ we may assume that $a \geq a_0$, where a_0 is some large number depending only on F, since for the remaining values of a the inequalities (13) and (14) are valid if we take $\delta_1 > 0$ and $\varepsilon_1 > 0$ sufficiently small. Observe that $c_F(a)$ in (2) satisfies

(15) $$c_F(a) \leq ((\omega(a)+1)\log 3P(a))^{c_0(\omega(a)+1)} \qquad (\text{for every } a \in \mathbb{Z},\ a \neq 0)$$

for some $c_0 = c_0(F)$. Since a is m-free we have

(16) $$a \leq \prod_{p|a} p^{m-1} \leq P(a)^{(m-1)\omega(a)}.$$

We have $F(x) = \pm ay^m$ for some $x,y \in \mathbb{Z}$. We may assume that $F(x) = ay^m$ with $x \geq 0$ and $y \in \mathbb{N}$ (by considering also $G(x) = \pm F(-x)$). Since $a \geq a_0$, we have $x \geq x_F$. If $y > 1$ then, by Theorem 1, $m \leq c_F(a)$ and it follows from (15) and (16) that

(17) $$((\omega(a)+1)\log 3P(a))^{c(\omega(a)+1)} \geq \log a,$$

where $c = c_0 + 1$. If $y = 1$ then it follows from (2) with $m = [\log F(X)] = [\log a]$ and (15), that (17) also holds in this case. If $\omega(a) \leq \log P(a)$ then it follows from (17) that (13)(ii) holds and if $\omega(a) > \log P(a)$ then 13(i) follows from (17). The inequality (14) is a direct consequence of (13) in view of $\omega(a) \leq \pi(P(a)) \ll P(a)(\log P(a))^{-1}$ for $a > 1$. □

REMARK. Let $a \in \mathbb{N}$ be m-free. Since $a \leq \prod_{p|a} p^{m-1} \leq e^{2(m-1)P(a)}$ we have $P(a) > \frac{1}{2}(\log a)(m-1)^{-1}$. (On the other hand, if $a = \prod_{p\leq P} p^{m-1}$, where $P \geq P_0$, then $a \geq 2^{(m-1)P}$, hence $P(a) < \frac{3}{2}(\log a)(m-1)^{-1}$.) It follows that if $a > 1$ is m-free for some $m \in \mathbb{N}$ with $m < (2\varepsilon)^{-1}(\log a)(\log\log a)^{-1}$, then $P(a) > \varepsilon \log\log a$. However, if a is m-free for some larger m, then such a bound for P(a) does not follow (as it should be: a may be equal to 2^{m-1}). By Corollary 1, if $a > 2$ is the m-free part of some F(x), where F has at least two distinct zeros, then $P(a) > \varepsilon \log\log a$ holds regardless of the value of m (≥ 2), with $\varepsilon = \varepsilon_1(F)$.

As observed already, the lower bounds for $\omega(a)$ and $P(a)$ in (13) and (14) are trivial for small values of $a \in \mathbb{N}$, i.e. for $(2<)\ a \le a_0(F)$. By the well-known theorem of Schinzel and Tijdeman these values for a can only occur for $|x| \le x_F$, provided that F has at least three simple zeros. Hence:

COROLLARY 2. *Let* $F \in \mathbb{Z}[X]$ *have at least three simple zeros. There exist positive numbers* x_F *and* ε_F *with the following property. Let* $a \in \mathbb{N}$ *be the m-free part of* $F(x)$ *for some* $x \in \mathbb{Z}$ *with* $|x| \ge x_F$ *and some* $m \in \mathbb{N}$ *with* $m \ge 2$. *Then*

$$P(a) > \varepsilon_F \log\log a \ge 1.$$

The assertion of Corollary 2 includes that F takes only finitely many powers as values (take a = 1) and also that $P(F(x)) > \varepsilon_F \log\log |F(x)|$ for $|x| \ge x_F$ (take $a = |F(x)|$ and $m = [(\log|F(x)|)(\log 2)^{-1}]+1$).

4. Let $F \in \mathbb{Z}[X]$ have at least three simple zeros. Suppose that

(18) $$F(x) = \pm a y^m$$

where $x,y,a,m \in \mathbb{N}$ with $m \ge 2$. Let a_m denote the m-free part of a. SPRINDŽUK (see [1]) obtained the following upper bound for x when m = 2 in (18):

(19) $$x < \exp((2a_2)^{C(F)}) \qquad \text{when } m = 2 \text{ in } (18).$$

BAKER (see [1]) has proved the upper bound $\exp\exp((5m)^{10} n^{10n^3} H^{n^2})$ for x if $G(x) = y^m$, $m \ge 3$, G has at least two simple zeros, where n is the degree of G and H is the maximum of the absolute values of the coefficients of G. Applying this to $G(X) = \pm a_m^{m-1} F(X)$ we obtain

(20) $$x < \exp\exp((2a_m)^{C(F)m}) \qquad \text{when } m \ge 3(2) \text{ in } (18).$$

The dependency of the bound in (20) on m can be removed with Theorem 1.

THEOREM 2. *Let* $F \in \mathbb{Z}[X]$ *have at least three simple zeros. There exist positive numbers* $\varepsilon_1, \varepsilon_2, \varepsilon_3, c$, *depending only on* F, *with the following properties.*

Let $x \in \mathbb{Z}$ *with* $|F(x)| \geq 16$ *and suppose that*

$$F(x) = \pm ab$$

where $a \in \mathbb{N}$ *and* b *is some power. Then*

$$(21) \qquad ((\omega(a)+1)\log 3P(a))^{c(\omega(a)+1)} > \log\log\log|F(x)|.$$

In particular

$$(22) \qquad 2a > (\log\log\log|F(x)|)^{\varepsilon_1}$$

$$(23) \qquad \text{if } a = p^{\alpha} \text{ is a prime power then } \log 2p > (\log\log\log|F(x)|)^{\varepsilon_2}$$

$$(24) \qquad P(a) > \varepsilon_3 \log\log\log\log|F(x)|.$$

PROOF. To prove (21) we may assume that $x \geq x_F$ (by taking $c = c(F)$ sufficiently large) and that $F(x) = \pm ay^m$ for some $m, y \in \mathbb{N}$ with $y > 1$ and $m > 1$ (if $y = 1$ then (21) follows from Theorem 1 and (15)). The inequality (21) follows now from (20), $a_m \leq P(a)^{m\omega(a)}$ (see (16)), (2) and (15). To prove (22) it is sufficient to observe that $P(a) \leq a$ and $\omega(a) \ll (\log a)(\log\log a)^{-1}$ for $a > 2$, while (23) follows immediately from (21). Finally, (24) follows from (21) noting that $\omega(a) \leq \pi(P(a)) \ll P(a)(\log P(a))^{-1}$ for $a > 1$. □

Finally we formulate (22) somewhat differently.

DEFINITION. For $n \in \mathbb{Z}$ let $a(n)$ be the smallest positive integer a with the property that $\pm n = ab$ for some power b, i.e. for some $b = y^m$ with $y \in \mathbb{Z}$ and $m \in \mathbb{N}$ with $m \geq 2$.

Obviously, $1 \leq a(n) \leq n$ for $n \in \mathbb{N}$, with equality if and only if n is a power or square-free.

COROLLARY 3. *Let* $F \in \mathbb{Z}[X]$ *have at least three simple zeros. There exists a positive number* $\varepsilon(F)$*, depending only on* F*, such that for every* n *in* $\{F(x) \mid x \in \mathbb{Z}\}$ *with* $|n| \geq 16$

$$a(n) > \frac{1}{2}(\text{logloglog}|n|)^{\varepsilon(F)}.$$

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