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HIGHLY PROXIMAL EXTENSIONS AND RELATIVE DISJOINTNESS

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Highly proximal extensions and relative disjointness *)

by

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ABSTRACT

For discrete phase groups we prove that there are no non-finite distal maximally highly proximal minimal transformation groups. Without conditions on the group we show that if two homomorphisms of minimal transformation groups are relatively disjoint then their extensions to the maximally highly proximal transformation groups are relatively disjoint too.

KEY WORDS & PHRASES: *minimal transformation group, highly proximal, disjointness, extremally disconnectedness*

*) This report will be submitted for publication elsewhere.

A *topological transformation group* (*ttg*) is a triple (T,X,π) with T a topological group, X a compact Hausdorff topological space and $\pi: T \times X \to X$, the action, a continuous map such that $\pi(t,\pi(s,x)) = \pi(ts,x)$ and $\pi(e,x) = x$. Since T will be fixed for the rest of this paper and confusion with respect to the action will be unlikely, we shall denote the ttg only by its phase space and we shall write the action as a left multiplication of elements of X with elements of T. A *homomorphism of ttg's* $\phi: X \to Y$ is a continuous map between two ttg's such that $\phi(tx) = t\phi(x)$ for every $t \in T$ and $x \in X$.

We will assume basic knowledge about topological dynamics as can be found in $[G_1]$ and $[E_1]$ and we adopt the notation of $[G_1]$.
Let X be a ttg, then 2^X denotes the hyperspace of X with the usual (Vietoris) topology. For a homomorphism of ttg's $\phi: X \to Y$ we put $2^\phi :=$ $= \{A \in 2^X \mid |\phi[A]| = 1\}$. In order to circumvent confusion between the (extended) actions of the universal ambit A for T on X and on 2^X we will write the latter as the "circle operation", cf. $[G_1]$. Recall, that for a net $t_i \mathrel{T} p$ in T and $A \in 2^X$ $p\circ A = \lim t_i A$ (in 2^X); for arbitrary $A \subseteq X$ we put $p\circ A := p\circ\bar{A}$.

In this paper we will mainly be interested in irreducible maps (a homomorphism $\phi: X \to Y$ of ttg's is called *irreducible* if $\phi[A] = Y$ implies $\bar{A} = X$) and in the preservation of properties under irreducible extensions. Among the properties which are preserved are minimality, point-transitivity, ergodicity and the existence of a dense set of almost periodic points, proofs of which are easy exercises for the reader.

The following lemma shows the connection between the topological and dynamical properties of irreducible homomorphisms of minimal ttg's.

<u>LEMMA</u>. *Let* $\phi: X \to Y$ *be a homomorphism of minimal ttg's. Equivalent are*

<u>a</u>. ϕ *is irreducible,*

<u>b</u>. $\bar{\phi}: 2^\phi \to Y$ *is proximal* ($\bar{\phi}(A) := \phi[A]$),

<u>c</u>. 2^ϕ *has a unique minimal sub ttg,*

<u>d</u>. *There is a* $y \in Y$ *and a net* $\{t_i\}$ *in such that* $\{t_i\overleftarrow{\phi}(y)\}$ *converges in* 2^X *to a singleton,*

<u>e</u>. *For all* $y \in Y$, $p \in A$, $x \in \overleftarrow{\phi}(y)$ *we have* $p\circ\overleftarrow{\phi}(y) = \{px\}$.

PROOF. The equivalence of a,d and e is wellknown (e.g. [AG]) e ⇒ b ⇒ c is trivial and c ⇒ d follows from the fact that X and $T\overleftarrow{\phi}(y)$ are sub ttg's in 2^{ϕ}, and $T\overleftarrow{\phi}(y)$ contains a minimal sub-ttg which has to be X by the uniqueness. □

A homomorphism $\phi: X \to Y$ is called *highly proximal* (*h.p.*) if it satisfies one of the conditions in the lemma above. From b it is clear that a h.p. map is proximal and it is not difficult to see that it is even strongly proximal [G_2]. If X is metric then ϕ is h.p. iff it is almost-one-to-one ([S]).

We can define a h.p. equivalence relation on the family of minimal ttg's by defining X and Y *h.p. equivalent* if they have a common h.p. extension, i.e. there are a Z and h.p. extensions $\phi: Z \to X$ and $\psi: Z \to Y$. Every equivalence class has a unique maximal element, which projects onto every element of that class. The maximal element in the equivalence class that contains X will be denoted by X^*. Let M be the universal minimal ttg for T and J the collection of idempotents in M. If $\gamma: (M,u) \to (X,x_0)$ is an ambitmorphism (a homomorphism that preserves base points) for some $u \in J$ and $x_0 = ux_0 \in X$ then $X^* := QF(u \circ \overleftarrow{\gamma}(x_0), M)$, the minimal subset of 2^M (quasifactor of M) generated by $u \circ \overleftarrow{\gamma}(x_0)$. Such a maximal element in a h.p. equivalence class will be called a *maximally highly proximal ttg* (*m.h.p. ttg*). For more details and proofs we refer to [AG] or [S].

2. LEMMA.

a *Let* X *be a minimal ttg. Then* $X = X^*$ *iff* X *is an open image of* M (*and so iff every extension of* X *is open*).

b *If* T *is a discrete group then the m.h.p. ttg's are just the minimal ttg's with extremally disconnected phase space* (*A topological space is extremally disconnected if the closure of any open set is again open*).

The following theorem shows that there are nontrivial highly proximal extensions in case the group is discrete. Another example of that fact is, that the locally almost periodic minimal ttg's are just the minimal ttg's that are h.p. extensions of almost periodic minimal ttg's ([MW_1], the topology of the group is not important here).

3. THEOREM. *Let* T *be a discrete group. If* X *is distal and m.h.p. then* X *is finite.*

PROOF. Let X be distal and m.h.p., then the maximal almost periodic factor Y of X is extremally disconnected (any map ϕ: X $\to$ Z is distal and so open). Since Y as an almost periodic ttg has a homogeneous phase space ([E_1]p.23) it follows that Y is finite (e.g.[C]Thm.8.3).

By the Fürstenberg structure theorem [E_2] X can be built up by a succession of almost periodic extensions so there is a factor Y' of X which is an almost periodic extension of Y. But then by [MW_2] 2.1 Y' is an almost periodic minimal ttg and so Y' = Y and X = Y. □

Let ϕ: X $\to$ Y be a homomorphism of minimal ttg's, then we can define commutative diagrams

$$\begin{tikzcd} X' \arrow[r, "\sigma"] \arrow[d, "\phi'"'] \arrow[dr, phantom, "AG(\phi)"] & X \arrow[d, "\phi"] \\ Y' \arrow[r, "\tau"'] & Y \end{tikzcd} \qquad\qquad \begin{tikzcd} X^* \arrow[r, "\kappa_X"] \arrow[d, "\phi^*"'] \arrow[dr, phantom, "*(\phi)"] & X \arrow[d, "\phi"] \\ Y^* \arrow[r, "\kappa_Y"'] & Y \end{tikzcd}$$

where κ_X, κ_Y are the canonical maximally h.p. extensions of X and Y and (of course) ϕ^* is an open map.

For AG(ϕ) we define $Y' := QF(u\circ\phi^{\leftarrow}(y_0), X) = \{v\circ\phi^{\leftarrow}(y) \mid y = vy \in Y,\ v \in J\}$, the quasifactor of X generated by $u\circ\phi^{\leftarrow}(y_0)$ for some $u \in J$ and $y_0 = uy_0 \in Y$, and $X' = \{(x,A) \mid x \in A \in Y'\}$, ϕ' and σ are defined as the projections, τ: Y' $\to$ Y by $\tau(p\circ\phi^{\leftarrow}(y_0)) = py_0$. Then σ and τ are h.p. and ϕ' is open ([AG]), and $\phi = \phi'$ iff ϕ is open. In fact we could consider ϕ' and ϕ^* as "irreducible extensions" of ϕ. Several properties of ϕ are lifted to ϕ' and ϕ^*, for instance if ϕ is proximal then ϕ' and ϕ^* are, if P_ϕ is an equivalence relation then $P_{\phi'}$ and P_{ϕ^*} are and moreover if ϕ is distal (almost periodic) then $\phi' = \psi\circ\theta$ where ψ is distal (almost periodic) and θ h.p..
In order to study other "lifting properties" we use the following notation: For homomorphisms ϕ: X $\to$ Z and ψ: Y $\to$ Z define $R_{\phi\psi} := \{(x,y) \in X \times Y \mid \phi(x) = \psi(y)\}$ and $R_\phi := R_{\phi\phi}$. Clearly $R_{\phi\psi}$ is a ttg. We say $\phi \perp \psi$ iff $R_{\phi\psi}$ is minimal, $\phi \mathrel{\dot{\perp}} \psi$ iff $R_{\phi\psi}$ is ergodic (i.e. ϕ and ψ are *disjoint* respectively *weakly*

disjoint), (ϕ,ψ) satisfies the *generalized Bronstein condition* (*g.B.c*
$R_{\phi\psi}$ has a dense set of almost periodic points ([V]) and ϕ satisfies the *Bronstein condition* (*B.c.*) iff (ϕ,ϕ) satisfies g.B.c..

The following lemma will be usefull

4. <u>LEMMA</u>. *Let* $\phi\colon X \to Z$ *and* $\psi\colon Y \to Z$ *be homomorphisms of minimal ttg's and let one of them be open. Let* W *be an open subset of* $R_{\phi\psi}$ *then there are open sets* U *and* V *in* X *and* Y *such that*

<u>a</u> $U \times V \cap R_{\phi\psi} \subseteq W$ *and* $U \times V \cap R_{\phi\psi} \neq \emptyset$,

<u>b</u> *for every* $x \in U$ *there is a* $y \in V$ *with* $(x,y) \in R_{\phi\psi}$.

<u>PROOF</u>. Let W be an open subset of $R_{\phi\psi}$ and choose U and V open in X and Y such that $U \times V \cap R_{\phi\psi} \neq \emptyset$ and $U \times V \cap R_{\phi\psi} \subseteq W$.
If ψ is open, define $U' = U \cap \overleftarrow{\phi}\psi[V]$, then $U' \neq \emptyset$ and $U' \times V \cap R_{\phi\psi} \neq \emptyset$, $U' \times V \cap R_{\phi\psi} \subseteq W$, while $U' \subseteq \overleftarrow{\phi}\psi[V]$ so U' and V satisfy the lemma. Let ϕ be open, then define $V' := V \cap \overleftarrow{\psi}\phi[U]$, $V' \neq \emptyset$ and open, since Y is minimal it follows that $(\psi[V'])^{\circ} \neq \emptyset$. Define $U' := U \cap \overleftarrow{\phi}[(\psi[V'])^{\circ}]$, then $U' \neq \emptyset$, $U' \times V' \cap R_{\phi\psi} \neq \emptyset$, $U' \times V' \cap R_{\phi\psi} \subseteq W$ and $U' \subseteq \overleftarrow{\phi}[\psi[V']]$. Remark that if $(x,y) \in W$ and ψ is open we can find such a U and V with $(x,y) \in U \times V \cap R_{\phi\psi}$. □

5. <u>PROPOSITION</u>. *Let* $\phi\colon X \to Z$ *and* $\psi\colon Y \to Z$ *be homomorphisms of minimal ttg's and let one of them be open. If* ϕ *is point distal, then* (ϕ,ψ) *satisfies g.B.c..*

<u>PROOF</u>. First remark that $x \in X$ is a ϕ-distal point iff $J(x) = J(\phi(x))$ where $J(z) = \{u \in J \mid uz = z\}$.
Choose an open set W in $R_{\phi\psi}$ and open sets U and V in X and Y as in lemma 4. Then there is an $x \in U$ that is a ϕ-distal point. Choose $y \in V$ such that $\phi(x) = \psi(y)$ and let $v \in J$ be such that $vy = y$, then $v \in J(\psi(y)) = J(\phi(x)) = J(x)$ so $v(x,y) = (x,y)$ and (x,y) is an almost periodic point in W. □

6. <u>COROLLARY</u>. *An open point distal homomorphism of minimal ttg's is* RIC (*i.e. is disjoint from every proximal homomorphism of minimal ttg's with the same co-domain* [G_1] X.1.3). *In particular if* ϕ *is point distal then* ϕ' (*in* AG(ϕ)) *is* RIC *and point distal*.

PROOF. Let $\phi\colon X \to Z$ be open and point distal and $\psi\colon Y \to Z$ proximal, then by 5 $R_{\phi\psi}$ has a dense set of almost periodic points, but since ψ is proximal $R_{\phi\psi}$ has a unique minimal subset so $R_{\phi\psi}$ is minimal and $\phi \perp \psi$.
From the construction of $AG(\phi)$ it is clear that if x is a ϕ-distal point and $u \in J(x)$ then $(x, u\circ\overleftarrow{\phi}\phi(x))$ is a ϕ'-distal point in X'. Since ϕ' is open the corollary follows. □

This corollary in fact generalizes [V] 2.3.6 since it shows that for a point distal homomorphism ϕ of minimal ttg's the diagram of AUSLANDER and GLASNER ($AG(\phi)$) and that of ELLIS, GLASNER and SHAPIRO coincide. [$EGS(\phi)$ is constructed in the same way as $AG(\phi)$ but Y' is defined as $QF(u\circ u\overleftarrow{\phi}(y_0), X)$]. So the canonical PI tower for ϕ is an HPI tower and a point distal map is an HPI-extension iff it is a PI-extension. ([V],[G_1],[AG]).

Consider the following commutative diagram of homomorphisms of minimal ttg's

$$\begin{tikzcd}
X' \arrow[d, "\sigma"'] \arrow[dr, "\phi'"] & & Y' \arrow[dl, "\psi'"'] \arrow[d, "\tau"] \\
X \arrow[dr, "\phi"'] \arrow[r, phantom, "\simeq"] & Z' \arrow[d, "\xi"] \arrow[r, phantom, "\simeq"] & Y \arrow[dl, "\psi"] \\
& Z &
\end{tikzcd}$$

We shall refer to it as diagram 7 and always use the same symbols.

8. PROPOSITION. *Let in diagram 7 ϕ' or ψ' be open and σ and τ be h.p. then $\sigma \times \tau\colon R_{\phi'\psi'} \to \sigma \times \tau[R_{\phi'\psi'}]$ is an irreducible map.*

PROOF. It is obvious that $\sigma \times \tau[R_{\phi'\psi'}] \subseteq R_{\phi\psi}$. Now let $W \subseteq R_{\phi'\psi'}$ be an open set in $R_{\phi'\psi'}$. We shall prove that W contains a fiber under $\sigma \times \tau$; this will imply that $\sigma \times \tau\colon R_{\phi'\psi'} \to \sigma \times \tau[R_{\phi'\psi'}]$ is irreducible. Choose U^1 and V^1 as in Lemma 4 and define $U^2 := \overleftarrow{\sigma}[X\backslash\sigma[X'\backslash U^1]]$ then U^2 is open and nonempty for σ is irreducible, also $U^2 \subseteq U^1$. By the choice of U^1 and V^1 we know that

$(\phi'[U^2])^\circ \subseteq \psi'[V^1]$ and so $V^2 := \psi'^{\leftarrow}[\phi'[U^2]^\circ] \cap V^1$ is open and non empty. Define $V^3 := \tau^{\leftarrow}[Y\backslash\tau[Y'\backslash V^2]]$. Then $V^3 \subseteq V^2$ and V^3 is open and non empty, moreover $\psi'[V^3] \subseteq \phi'[U^2]$. So $\emptyset \neq U^2 \times V^3 \cap R_{\phi'\psi'} \subseteq W$, and since $U^2 \times V^3 = (\sigma\times\tau)^{\leftarrow}[\sigma\times\tau[U^2\times V^3]]$ it follows that W contains a fiber under $\sigma \times \tau(:R_{\phi'\psi'} \to \sigma\times\tau[R_{\phi'\psi}])$. □

In several cases we know that $\sigma \times \tau[R_{\phi'\psi'}] = R_{\phi\psi}$ for instance

a If $Z' = Z$

b If $\xi: Z' \to Z$ is proximal and $R_{\phi\psi}$ has a dense set of almost periodic points; for then, by proximality of ξ, $\sigma \times \tau[R_{\phi'\psi'}]$ contains the almost periodic points of $R_{\phi\psi}$.

9. COROLLARY. *Let in diagram* 7 ϕ' *or* ψ' *be open and* σ *and* τ *be h.p. and let* $\sigma \times \tau[R_{\phi'\psi'}] = R_{\phi\psi}$. *Then*

a $\phi \perp \psi$ *iff* $\phi' \perp \psi'$;

b $\phi \stackrel{.}{\perp} \psi$ *iff* $\phi' \stackrel{.}{\perp} \psi'$;

c (ϕ,ψ) *satisfies g.B.c. iff* (ϕ',ψ') *satisfies g.B.c.*.

PROOF. Follows immediately from 8, the discussion above and that just before lemma 1. □

There are two canonical ways to obtain diagrams as in 7, both of them with ξ h.p. and ϕ' and ψ' open

a $*(\phi,\psi)$ in this case $\phi' := \phi^*: X^* \to Z^*$ and $\psi' := \psi^*: Y^* \to Z^*$

b $AG(\phi,\psi)$ where $Z' := \{(p\circ\overleftarrow{\phi}(z_0), p\circ\overleftarrow{\psi}(z_0)) \mid p \in M\}$ for some $z_0 \in Z$ and $X' := \{(x,(A,B)) \mid x \in A, (A,B) \in Z'\}$, $Y' := \{(y,(A,B)) \mid y \in B, (A,B) \in Z'\}$. The maps σ,τ,ϕ' and ψ' are then the projections.

Clearly $*(\phi,\phi) = *(\phi)$ and $AG(\phi,\phi) = AG(\phi)$.
In fact we just proved the following theorem.

10. THEOREM. *With notation as before and* ϕ',ψ' *as in* $AG(\phi,\psi)$; ϕ^*,ψ^* *as in* $*(\phi,\psi)$:

a *if* $\phi \perp \psi$ *then* $\phi' \perp \psi'$ *and* $\phi^* \perp \psi^*$;

b *if* (ϕ,ψ) *satisfies g.B.c. then* (ϕ',ψ') *and* (ϕ^*,ψ^*) *do*.

If (ϕ,ψ) *satisfies g.B.c. or* $Z = Z^*$ *then*

c $\phi \perp \psi$ *iff* $\phi' \perp \psi'$ *iff* $\phi^* \perp \psi^*$;

d $\phi \bullet \psi$ *iff* $\phi' \bullet \psi'$ *iff* $\phi^* \bullet \psi^*$.

In particular this means for AG(ϕ) *and* $*(\phi)$ (ϕ: X $\to$ Z) *that*

e *if* ϕ *satisfies B.c. then* ϕ' *and* ϕ^* *do,*

f *if* ϕ *satisfies B.c., or if* $Z = Z^*$ *then* ϕ *is weakly mixing iff* ϕ' *is weakly mixing iff* ϕ^* *is weakly mixing,*

g *if* ϕ *is* RIC *then* ϕ^* *is* RIC, *moreover if* ϕ *is point distal then* ϕ' *is* RIC *and so* ϕ^* *is* RIC.

PROOF. a until f follows from g; g follows from b and the observation that if ϕ is RIC then (ϕ,ψ) satisfies g.B.c. ([V] page 814). □

Question: is 10 f true without the additional condition of B.c. or $Z = Z^*$.

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