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AFDELING ZUIVERE WISKUNDE ZW 145/80 NOVEMBER
(DEPARTMENT OF PURE MATHEMATICS)

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THE REPRESENTATION THEORY OF SL(2,$\mathbb{R}$), A GLOBAL APPROACH

Preprint

kruislaan 413 1098 SJ amsterdam

Printed at the Mathematical Centre, 413 Kruislaan, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O.).

1980 Mathematics subject classification: 22-02, 22E46, 33A30, 33A75, 43A90, 44A15.

The representation theory of $SL(2,\mathbb{R})$, a global approach *)

by

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ABSTRACT

The representation theory of $SL(2,\mathbb{R})$ is developed by the use of global (i.e. non-infinitesimal) methods. This approach is based on an explicit knowledge of the matrix elements of the principal series with respect to the K-basis. The irreducible subquotient representations of the principal series are determined, and also their Naimark equivalences and unitarizability. All irreducible K-unitary, K-finite representations of $SL(2,\mathbb{R})$ are classified, where an inversion formula for the generalised Abel transform provides an important tool. Most theorems are first proved for more general groups G, such that future applications will be possible to other semisimple groups G for which all irreducible representations are K-multiplicity free. Each section concludes with extensive bibliographic notes.

KEY WORDS & PHRASES: *representation theory of semisimple Lie groups; representation theory of* $SL(2,\mathbb{R})$; *principal series; K-unitary, K-multiplicity free representations; canonical matrix elements; Jacobi functions; irreducibility; Naimark equivalence; classification of irreducible representations; Gelfand pairs; spherical functions of type* δ; *generalised Abel transform; Gegenbauer transform pair; unitarizability*

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*) This report will be submitted for publication elsewhere.

1. INTRODUCTION

In 1947 two papers appeared on the representation theory of the two prototypes of noncompact semisimple Lie groups, namely by BARGMANN [1] on SL(2,ℝ) and by GELFAND & NAIMARK [24] on SL(2,ℂ). The methods in the two papers are surprisingly different. Bargmann uses the infinitesimal (i.e. Lie algebraic) approach*), while Gelfand & Naimark prefer non-infinitesimal (global) methods. In subsequent work to generalise these results for arbitrary noncompact semisimple Lie groups, the Bargmann approach has proved to be most successful, in particular by the work of Harish-Chandra. However, it is interesting to note MAUTNER'S [43] review of HARISH-CHANDRA'S paper [28], where Harish-Chandra's approach is compared with the approach of Gelfand and Naimark. Mautner states: "If one is mainly interested in the representations of the group itself, methods which use the group in the large should be considered as an alternative to the present author's algebraic analysis of the universal enveloping algebra".

Without denying the success of the infinitesimal approach, I want to add some motivation for a paper which favours the global approach:

(a) *The didactic argument.* The global approach is a more natural and direct one and it does not require so much sophisticated functional analysis as the infinitesimal approach.

(b) *Spin off to the theory of special functions and related harmonic analysis.* The global approach requires explicit knowledge of canonical matrix elements of representations as special functions. This provides new group theoretic interpretations of well-known special functions and it also yields new interesting special functions.

(c) *The philosophical argument.* The representation theory of semisimple Lie groups is one of the great topics in mathematics at the moment. It is good to have several distinct philosophies existing beside each other for the development of this theory, where each philosophy provides other insights.

In this paper a global approach to the representation theory of SL(2,ℝ) is presented. It is based on an explicit knowledge of the matrix elements of the principal series representations with respect to a basis which behaves nicely under the action of a maximal compact subgroup K.

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*) See SUGIURA [54, Ch.V] and van DIJK [11] for a modern account of this approach.

The theory required for such an approach is developed in the more general situation of Hilbert representations of a locally compact group G which are multiplicity free with respect to a compact subgroup K. In particular, this would apply to the noncompact semisimple Lie groups $SO_0(1,n)$ and SU(1,n). Furthermore, many things which are usually considered as standard material in Lie group papers, are explained here, and sometimes proved. Extensive notes after each section provide a guide to the literature. Thus the paper can also be used as a survey paper.

Our program consists of four parts:

(i) Determine all irreducible subquotient representations of the principal series representations of $SL(2,\mathbb{R})$.

(ii) Determine which equivalences do exist between the representations in (i).

(iii) Prove that each irreducible representation of $SL(2,\mathbb{R})$ is equivalent to some representation in (i).

(iv) Which of the representations in (i) are unitarisable?.

We will not only consider unitary representations, but, more generally, strongly continuous representations on a Hilbert space which are K-unitary and K-finite (cf.§2.1). Accordingly, we need a more general (but still non-infinitesimal) notion of equivalence than the notion of unitary equivalence, namely Naimark equivalence (cf.§4.1).

The four parts of the above program will be treated in sections 3,4,5 and 6, respectively. We start in section 2 with the computation of the canonical matrix elements of the principal series representations. They can be expressed in terms of hypergeometric or, more elegantly, Jacobi functions. These explicit expressions will be used throughout the paper.

This paper is a completely rewritten version of an earlier paper [37]. In the present paper more detailed proofs are given and, in contrast with [37], part (iii) of the program is also done by global methods. Also many bibliographic notes have been added here.

The results of this paper may be generalised rather easily to the universal covering group of $SL(2,\mathbb{R})$, cf. PUKANSKY [49] and SALLY [52] for the infinitesimal approach. The extension to $SL(2,\mathbb{C})$ was done by KOSTERS [39], see also NAIMARK [47, ch.3, §9]. Hopefully, an extension to $SO_0(1,n)$ and SU(1,n) is feasible.

Very recently, after completion of this manuscript, there appeared notes by TAKAHASHI [72] on the global approach to the representation theory of $SL(2,\mathbb{R})$.

ACKNOWLEDGEMENT. I am grateful to G. van Dijk and M. Flensted-Jensen, who read the earlier version [37] of this paper and gave useful comments.

2. THE CANONICAL MATRIX ELEMENTS OF THE PRINCIPAL SERIES

2.1 Preliminaries

Let G be a locally compact group satisfying the second axiom of countability (lcsc. group). A Hilbert representation τ of G on a separable Hilbert space $H(\tau)$ is a mapping τ from G into the space $L(H(\tau))$ of bounded linear operators on $H(\tau)$ such that (i) $\tau(g_1g_2) = \tau(g_1)\tau(g_2)$, $g_1,g_2 \in G$; (ii) $\tau(e) = I$; (iii) for each $v \in H(\tau)$ the mapping $g \to \tau(g)v$ is continuous from G to $H(\tau)$. The representation τ is called *irreducible* if $\{0\}$ and $H(\tau)$ are the only G-invariant closed subspaces of $H(\tau)$. The representation τ is called *unitary* if $\tau(g)$ is a unitary operator for all $g \in G$.

Let K be a compact subgroup of G. Each irreducible unitary representation of K is finite-dimensional. Let $\hat{K}$ be the set of equivalence classes of irreducible unitary representations of K. For $\delta \in \hat{K}$ let d_δ denote its degree and χ_δ its character. A Hilbert representation τ of G is called K-*unitary* if the restriction $\tau|_K$ of τ to K is a unitary representation of K. For a K-unitary representation τ of G let

$$(2.1)\qquad P_{\tau,\delta}v := d_\delta \int_K \chi_\delta(k^{-1})\ \tau(k)v\, dk, \qquad v \in H(\tau), \quad \delta \in \hat{K}.$$

Then $P_{\tau,\delta}$ is an orthogonal projection. If $H_\delta(\tau)$ denotes the range of $P_{\tau,\delta}$ then $\tau|_K$ acts on $H_\delta(\tau)$ as a multiple of δ and we have the orthogonal direct sum

$$(2.2)\qquad H(\tau) = \sum_{\delta\in\hat{K}}^{\oplus} H_\delta(\tau).$$

A K-unitary representation τ of G is called K-*finite* if each $\delta \in \hat{K}$ has finite multiplicity in $\tau|_K$. A K-unitary representation τ of G is called

K-*multiplicity free* if each $\delta \in \hat{K}$ has multiplicity 1 or 0 in $\tau|_K$. Throughout, when we use the terms K-finite or K-multiplicity free, it will be assumed that the representation is K-unitary. For a K-multiplicity free representation τ of G, the K-*content* $M(\tau)$ is the subset of $\hat{K}$ consisting of all δ which have multiplicity 1 in $\tau|_K$. If $\delta \in M(\tau)$ then write

$$(2.3)\qquad \tau_\delta(k) := \tau(k)|_{H_\delta(\tau)}, \qquad k \in K.$$

Thus τ_δ is a unitary representation of K belonging to the equivalence class δ and we have the decompositions

$$(2.4)\qquad \tau|_K = \sum_{\delta \in M(\tau)}^{\oplus} H_\delta(\tau),$$

$$(2.5)\qquad H(\tau) = \sum_{\delta \in M(\tau)}^{\oplus} H_\delta(\tau).$$

Let τ be a K-multiplicity free representation of G. For $\gamma, \delta \in M(\tau)$ and $g \in G$ we define a linear mapping $\tau_{\gamma,\delta}: H_\delta(\tau) \to H_\gamma(\tau)$ by

$$(2.6)\qquad \tau_{\gamma,\delta}(g)v := P_{\tau,\gamma}\tau(g)v, \qquad v \in H_\delta(\tau).$$

The operator-valued functions $\tau_{\gamma,\delta}$ $(\gamma,\delta \in M(\tau))$ on G are called the *canonical matrix elements* of τ (with respect to K).

2.2. The definition of the principal series.

Let G be a noncompact connected real semisimple Lie group with finite center. The prototype of such a group is SL(2,$\mathbb{R}$), which we treat in §2.3. [The reader who is not experienced with the general theory of semisimple Lie groups, may skip the present subsection.] Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ be an Iwasawa decomposition for the Lie algebra $\mathfrak{g}$ of G and let G = KAN be the corresponding global *Iwasawa decomposition* (cf. HELGASON [34, Ch.VI, §5]). K, A and N are closed subgroups of G, K is maximally compact in G, A is abelian and N is nilpotent. The mapping $(k,a,n) \to kan$ is a diffeomorphism from K × A × N onto G. For $g \in G$ write

$$(2.7)\qquad g = u(g)\, e^{H(g)}\, n(g),$$

where $u(g) \in K$, $H(g) \in \mathfrak{a}$ and $n(g) \in N$ are uniquely determined by g.

Let M be the centralizer of A in K. Let $\mathfrak{a}^*$ be the dual space of the linear space $\mathfrak{a}$ and let $\mathfrak{a}^*_{\mathbb{C}}$ be the complexification of $\mathfrak{a}^*$. Let $\rho \in \mathfrak{a}^*$ be defined by

$$\rho(H) := \tfrac{1}{2}\mathrm{tr}(\mathrm{ad}\,H\,|_{\mathfrak{n}}), \qquad H \in \mathfrak{a}. \tag{2.8}$$

We will define the so-called *principal series* of representations of G. Its members, denoted by $\pi_{\xi,\lambda}$, are labeled by all pairs (ξ,λ) $(\xi \in \hat{M},\ \lambda \in \mathfrak{a}^*_{\mathbb{C}})$. The representation $\pi_{\xi,\lambda}$ is obtained by inducing the (not necessarily unitary) finite-dimensional irreducible representation

$$man \to e^{\lambda(\log a)}\xi(m)$$

of the subgroup MAN.

Let us describe the representation $\pi_{\xi,\lambda}$ $(\xi \in \hat{M},\ \lambda \in \mathfrak{a}^*_{\mathbb{C}})$ in the so-called *compact picture* (cf. WALLACH [61, §8.3], see also KOORNWINDER & van der MEER [38, §6.2]). In this picture $\pi_{\xi,\lambda}$ is realized on the Hilbert space $L^2_\xi(K,\mathcal{H}(\xi))$ consisting of all $\mathcal{H}(\xi)$-valued L^2-functions f on K such that

$$f(km) = \xi(m^{-1})f(k), \quad k \in K,\ m \in M. \tag{2.9}$$

Then, using the notation of (2.7) and (2.8), we have

$$(\pi_{\xi,\lambda}(g)f)(k) := e^{-(\rho+\lambda)(H(g^{-1}k))}f(u(g^{-1}k)), \qquad f \in L^2_\xi(K,\mathcal{H}(\xi)),\ k \in K,\ g \in G. \tag{2.10}$$

$\pi_{\xi,\lambda}$ satisfies the properties of a Hilbert representation of G on $L^2_\xi(K,\mathcal{H}(\xi))$ (cf. §2.1). We have

$$(\pi_{\xi,\lambda}(g)f_1, \pi_{\xi,-\bar{\lambda}}(g)f_2) = (f_1,f_2), \quad f_1,f_2 \in L^2_\xi(K,\mathcal{H}(\xi)), \quad g \in G, \tag{2.11}$$

(cf. WALLACH [61, Lemma 8.3.11]). Hence, the representation $\pi_{\xi,i\mu}$ $(\xi \in \hat{M},\ \mu \in \mathfrak{a}^*)$ of G are unitary. They form the so-called *unitary principal* *series* of representations of G.

Restriction of $\pi_{\xi,\lambda}$ to K (take $g \in K$ in (2.10)) yields

$$(2.12) \qquad (\pi_{\xi,\lambda}(k_1)f)(k) = f(k_1^{-1}k), \quad f \in L^2_\xi(K,\mathcal{H}(\xi)), \quad k_1, k \in K.$$

Thus $\pi_{\xi,\lambda}|_K$ is unitary, so $\pi_{\xi,\lambda}$ is a K-unitary representation of G. It is also evident from (2.12) that $\pi_{\xi,\lambda}|_K$ is the representation of K on $L^2_\xi(K)$ which is induced by the irreducible unitary representation ξ of M (cf. KOORNWINDER & van der MEER [38, §4]). By the *Frobenius reciprocity theorem* in the case of a compact group (cf. WEIL [65, §23]), the multiplicity of $\delta \in \hat{K}$ in $\pi_{\xi,\lambda}|_K$ just equals the multiplicity of ξ in $\delta|_M$. Thus $\pi_{\xi,\lambda}$ is a K-finite representation of G. In particular, if, for each $\delta \in \hat{K}$, each $\eta \in \hat{M}$ occurs at most once in δ, then each principal series representation $\pi_{\xi,\lambda}$ of G is K-multiplicity free. In the following cases, this property of the pair (K,M) is true (cf. §5.3).

G	K	M
$SL(2,\mathbb{R})$	$SO(2)$	$O(1)$
$SL(2,\mathbb{C})$	$SU(2)$	$U(1)$
$SO_0(1,n)$	$SO(n)$	$SO(n-1)$
$SU(1,n)$	$U(n)$	$U(n-1)$

Assume $\pi_{\xi,\lambda}$ is K-multiplicity free. For $\gamma,\delta \in \mathcal{M}(\pi_{\xi,\lambda})$ let $\pi_{\xi,\lambda;\gamma,\delta}$ be the canonical matrix element of $\pi_{\xi,\lambda}$ (cf. 2.6)). There is the so-called *Cartan decomposition* G = KAK of G, i.e., each $g \in G$ can be written as $g = k_1ak_2$ for some $a \in A$ and $k_1,k_2 \in K$(cf. HELGASON [34, Ch.V, Theorem 6.7]). We have (using the notation (2.3))

$$(2.13) \qquad \pi_{\xi,\lambda;\gamma,\delta}(k_1ak_2) = \pi_{\xi,\lambda;\gamma}(k_1)\pi_{\xi,\lambda;\gamma,\delta}(a)\pi_{\xi,\lambda;\delta}(k_2),$$

$$a \in A,\ k_1,k_2 \in K,\ \gamma,\delta \in \mathcal{M}(\pi_{\xi,\lambda}).$$

If we assume $\pi_{\xi,\lambda;\delta}$ $(\delta \in \mathcal{M}(\pi_{\xi,\lambda}))$ to be known then $\pi_{\xi,\lambda}$ is completely determined by its canonical matrix elements restricted to A.

Let M' be the normalizer of A in K. Then M is a normal subgroup of M'.

The factor group W: = M'/M turns out to be finite (cf. HELGASON [34, Ch.VII, Prop. 2.1]). It is called the *Weyl group* of the pair (G,K). The group W acts on A by putting $w.a := m' a m'^{-1}$, $a \in A$, if $w = m'M$ for some $m' \in M'$. Thus

$$\pi_{\xi,\lambda;\gamma,\delta}(w.a) = \pi_{\xi,\lambda;\gamma}(m')\pi_{\xi,\lambda;\gamma,\delta}(a)\pi_{\xi,\lambda;\delta}(m'^{-1}), \tag{2.14}$$

$$\gamma,\delta \in M(\pi_{\xi,\lambda}),\ a \in A,\ m' \in M',\ w = m'M.$$

Finally, it follows from (2.11) and (2.6) that

$$(\pi_{\xi,\lambda;\gamma,\delta}(g))^* = \pi_{\xi,-\bar{\lambda};\delta,\gamma}(g^{-1}),\ \gamma,\delta \in M(\pi_{\xi,\lambda}), \qquad g \in G. \tag{2.15}$$

2.3. The principal series for SU(1,1).

Let us now specialize the above results to G = SL(2,ℝ). It is convenient to work with the isomorphic group G = SU(1,1):

$$G: = \left\{g_{\alpha,\beta} = \begin{pmatrix}\alpha & \beta\\ \bar{\beta} & \bar{\alpha}\end{pmatrix};\ \alpha,\beta \in \mathbb{C},\ |\alpha|^2 - |\beta|^2 = 1\right\}. \tag{2.16}$$

Let

$$K: = \left\{u_\theta = \begin{pmatrix}e^{\frac{1}{2}i\theta} & 0\\ 0 & e^{-\frac{1}{2}i\theta}\end{pmatrix};\qquad 0 \le \theta < 4\pi\right\}, \tag{2.17}$$

$$A: = \left\{a_t = \begin{pmatrix}\operatorname{ch}\tfrac{1}{2}t & \operatorname{sh}\tfrac{1}{2}t\\ \operatorname{sh}\tfrac{1}{2}t & \operatorname{ch}\tfrac{1}{2}t\end{pmatrix} = \exp t\begin{pmatrix}0 & \tfrac{1}{2}\\ \tfrac{1}{2} & 0\end{pmatrix};\ t \in \mathbb{R}\right\}, \tag{2.18}$$

$$N: = \left\{n_z = \begin{pmatrix}1+\tfrac{1}{2}iz & -\tfrac{1}{2}iz\\ \tfrac{1}{2}iz & 1-\tfrac{1}{2}iz\end{pmatrix};\ z \in \mathbb{R}\right\}. \tag{2.19}$$

Then G = KAN is an Iwasawa decomposition for G = SU(1,1) (cf. TAKAHASHI [55, §1]).

If $g_{\alpha,\beta} = u_\theta\, a_t\, n_z$ then

$$\alpha = e^{\frac{1}{2}i\theta}(\operatorname{ch}\tfrac{1}{2}t + ize^{\frac{1}{2}t}),$$

$$\beta = e^{\frac{1}{2}i\theta}(\operatorname{sh}\tfrac{1}{2}t - ize^{\frac{1}{2}t}),$$

so (2.7) takes the form

$$(2.20)\qquad g_{\alpha,\beta} = u_{\theta(g_{\alpha,\beta})} \exp\left(t(g_{\alpha,\beta})\begin{pmatrix}0 & \frac{1}{2}\\ \frac{1}{2} & 0\end{pmatrix}\right) n_{z(g_{\alpha,\beta})}$$

with

$$e^{\frac{1}{2}i\theta(g_{\alpha,\beta})} = \frac{\alpha+\beta}{|\alpha+\beta|}, \qquad e^{\frac{1}{2}t(g_{\alpha,\beta})} = |\alpha+\beta|.$$

It follows that

$$(2.21)\qquad \theta(g_{\alpha,\beta}^{-1}\, u_\psi) = 2\arg(\bar{\alpha}\, e^{\frac{1}{2}i\psi} - \beta e^{-\frac{1}{2}i\psi}),$$

$$(2.22)\qquad \exp(\tfrac{1}{2}t(g_{\alpha,\beta}^{-1}\, u_\psi)) = |\bar{\alpha}\, e^{\frac{1}{2}i\psi} - \beta e^{-\frac{1}{2}i\psi}|,$$

$$g_{\alpha,\beta} \in G, \qquad u_\psi \in K.$$

For ρ (cf. (2.8)) we find.

$$(2.23)\qquad \rho(\log a_t) = \tfrac{1}{2}t.$$

The centralizer of A in K is

$$M: = \{u_0, u_{2\pi}\}.$$

$\hat{M}$ consists of the two one-dimensional representations

$$(2.24)\qquad u_\theta \to e^{i\xi\theta}, \qquad u_\theta \in M, \quad \xi = 0 \text{ or } \tfrac{1}{2}.$$

The principal series representation $\pi_{\xi,\lambda}$ will be realized on the Hilbert space $L^2_\xi(K)$, where $f \in L^2_\xi(K)$ if and only if $f \in L^2(K)$ and $f(u_{\psi+2\pi}) = f(u_\psi)$ or $-f(u_\psi)$ according to whether $\xi = 0$ or $\frac{1}{2}$, respectively.

Let $\xi = 0$ or $\frac{1}{2}$ and $\lambda \in \mathbb{C}$. Using (2.10), (2.20), (2.22), (2.23) we realize the principal series representation $\pi_{\xi,\lambda}$ of $G = SU(1,1)$ as

$$(2.25)\qquad (\pi_{\xi,\lambda}(g_{\alpha,\beta})f)(u_\psi) := |\bar{\alpha}\, e^{\frac{1}{2}i\psi} - \beta e^{-\frac{1}{2}i\psi}|^{-2\lambda-1} f(u_{\psi'}),$$

$$\psi' := 2\arg(\bar{\alpha}\, e^{\frac{1}{2}i\psi} - \beta e^{-\frac{1}{2}i\psi}), \quad g_{\alpha,\beta} \in G,\ u_\psi \in K,\ f \in L^2_\xi(K).$$

This defines a class of Hilbert representations of G. By (2.11) the representation $\pi_{\xi,\lambda}$ is unitary if λ is imaginary.

On putting $g_{\alpha,\beta} := u_\theta \in K$ in (2.25) we get

$$(2.26)\qquad (\pi_{\xi,\lambda}(u_\theta)f)(u_\psi) = f(u_{\psi-\theta}),\quad f \in L^2_\xi(K),\ u_\theta,\ u_\psi \in K,$$

so $\pi_{\xi,\lambda}$ is K-unitary. $\hat{K}$ consists of the representations

$$(2.27)\qquad \delta_n(u_\theta) := e^{in\theta},\ u_\theta \in K,$$

where n runs through the set $\frac{1}{2}\mathbb{Z}$, i.e., $2n \in \mathbb{Z}$. An orthonormal basis for $L^2_\xi(K)$ is given by the functions

$$(2.28)\qquad \phi_n(u_\psi) := e^{-in\psi},\ u_\psi \in K,$$

where n runs through the set $\mathbb{Z} + \xi := \{m + \xi | m \in \mathbb{Z}\}$. This basis behaves nicely with respect to K:

$$(2.29)\qquad \pi_{\xi,\lambda}(u_\theta)\phi_n = \delta_n(u_\theta)\phi_n,\ u_\theta \in K,\ n \in \mathbb{Z} + \xi.$$

Thus $\pi_{\xi,\lambda}$ is K-multiplicity free and

$$(2.30)\qquad M(\pi_{\xi,\lambda}) = \{\delta_n \in \hat{K} | n \in \mathbb{Z} + \xi\}.$$

Identify $H_{\delta_n}(\pi_{\xi,\lambda})$ $(n \in \mathbb{Z} + \xi)$ with C by identifying ϕ_n with 1. Then the canonical matrix elements of $\pi_{\xi,\lambda}$ become scalar functions on G:

$$(2.31)\qquad \pi_{\xi,\lambda;m,n}(g) := (\pi_{\xi,\lambda}(g)\phi_n,\phi_m),\quad g \in G,\quad m,n \in \mathbb{Z} + \xi,$$

where (.,.) denotes the scalar product in $L^2_\xi(K)$.

It follows from (2.25) and (2.28) that

$$(\pi_{\xi,\lambda}(a_t)\phi_n)(u_\psi) = |\mathrm{ch}\tfrac{1}{2}t\ e^{\frac{1}{2}i\psi} - \mathrm{sh}\tfrac{1}{2}t\ e^{-\frac{1}{2}i\psi}|^{-2\lambda+2n-1}.$$

$$.(\mathrm{ch}\tfrac{1}{2}t\ e^{\frac{1}{2}i\psi} - \mathrm{sh}\tfrac{1}{2}t\ e^{-\frac{1}{2}i\psi})^{-2n}.$$

Hence

$$(2.32)\qquad \pi_{\xi,\lambda;m,n}(a_t) = (\mathrm{ch}\tfrac{1}{2}t)^{-2\lambda-1}\cdot$$

$$\cdot\ \frac{1}{4\pi}\int_0^{4\pi} (1-\mathrm{th}\tfrac{1}{2}t\ e^{i\psi})^{-\lambda+n-\frac{1}{2}}(1-\mathrm{th}\tfrac{1}{2}t\ e^{-i\psi})^{-\lambda-n-\frac{1}{2}}e^{i(m-n)\psi}d\psi,$$

$$t \in \mathbb{R},\quad m,\ n \in \mathbb{Z} + \xi.$$

The following two symmetries are evident from (2.32):

$$(2.33)\qquad \pi_{\xi,\lambda;-m,-n}(a_t) = \pi_{\xi,\lambda;m,n}(a_t),$$

$$(2.34)\qquad \pi_{\xi,\lambda;m,n}(a_{-t}) = (-1)^{m-n}\pi_{\xi,\lambda;m,n}(a_t).$$

The latter symmetry also follows from (2.14): The normalizer M' of A in K consists of the elements u_0, u_π, $u_{2\pi}$, $u_{3\pi}$. Let $w := u_\pi M$. Then $w.a_t = a_{-t}$ and

$$\pi_{\xi,\lambda;m,n}(a_{-t}) = \delta_m(u_\pi)\pi_{\xi,\lambda;m,n}(a_t)\delta_n(u_\pi^{-1}).$$

2.4. Calculation of the canonical matrix elements.

Let us calculate the integral (2.32). In view of (2.33) we can suppose $m \geq n$. The binomial expansion

$$(2.35)\qquad (1-z)^{-a} = \sum_{k=0}^{\infty} \frac{(a)_k}{k!} z^k, \qquad |z| < 1,\quad a \in \mathbb{C},$$

where

$$(2.36)\qquad (a)_k := a(a+1)\ \ldots\ (a+k-1) = \frac{\Gamma(a+k)}{\Gamma(a)},$$

can be substituted for the first two factors in the integrand of (2.32). Now interchange the order of summation and integration and perform the integration in each term. Then we obtain ($m \geq n$)

$$(2.37)\qquad \pi_{\xi,\lambda;m,n}(a_t) = \frac{(\lambda+n+\frac{1}{2})_{m-n}}{(m-n)!}\,(\mathrm{sh}\tfrac{1}{2}t)^{m-n}\,(\mathrm{ch}\tfrac{1}{2}t)^{n-m-2\lambda-1}\cdot$$

$$\cdot\,{}_2F_1(\lambda+m+\tfrac{1}{2},\lambda-n+\tfrac{1}{2};m-n+1;(\mathrm{th}\tfrac{1}{2}t)^2),$$

where the ${}_2F_1$ denotes a *hypergeometric series*, defined by

$$(2.38)\qquad {}_2F_1(a,b;c;z) := \sum_{k=0}^{\infty}\frac{(a)_k\,(b)_k}{(c)_k\,k!}\,z^k,\quad |z|<1,\ a,\ b,\ c\in\mathbb{C},$$

cf. [14, Vol. I, ch. 2].

The expression (2.38) is clearly symmetric in a and b. As a function of z, the ${}_2F_1$ has an analytic continuation to a one-valued function on $\mathbb{C}\setminus[1,\infty)$. Application of the transformation formulas

$$(2.39)\qquad {}_2F_1(a,b;c;z) = (1-z)^{-b}\,{}_2F_1(c-a,b;c;\tfrac{z}{z-1}) = $$

$$= (1-z)^{-a}\,{}_2F_1(a,c-b;c;\tfrac{z}{z-1})$$

(cf. [14,Vol.I, §2.1(22)]) to (2.37) yields ($m\geq n$):

$$(2.40)\qquad \pi_{\xi,\lambda;m,n}(a_t) =$$

$$= \frac{(\lambda+n+\frac{1}{2})_{m-n}}{(m-n)!}\,(\mathrm{sh}\tfrac{1}{2}t)^{m-n}(\mathrm{ch}\tfrac{1}{2}t)^{-m-n}\,{}_2F_1(\lambda-n+\tfrac{1}{2},-\lambda-n+\tfrac{1}{2};m-n+1;-(\mathrm{sh}\tfrac{1}{2}t)^2) =$$

$$= \frac{(\lambda+n+\frac{1}{2})_{m-n}}{(m-n)!}\,(\mathrm{sh}\tfrac{1}{2}t)^{m-n}(\mathrm{ch}\tfrac{1}{2}t)^{m+n}\,{}_2F_1(\lambda+m+\tfrac{1}{2},-\lambda+m+\tfrac{1}{2};m-n+1;-(\mathrm{sh}\tfrac{1}{2}t)^2).$$

It is more elegant to express the hypergeometric functions in (2.40) in terms of *Jacobi functions* $\phi_\mu^{(\alpha,\beta)}$ ($\mu,\alpha,\beta\in\mathbb{C}$, $\alpha\notin\{-1,-2,\ldots\}$), which are defined on $\mathbb{R}$ by

$$(2.41)\qquad \phi_\mu^{(\alpha,\beta)}(t) := {}_2F_1(\tfrac{1}{2}(\alpha+\beta+1+i\mu),\ \tfrac{1}{2}(\alpha+\beta+1-i\mu);\alpha+1;-(\mathrm{sh}t)^2)$$

(cf. KOORNWINDER [36,§2]). Clearly

$$(2.42)\qquad \phi_\mu^{(\alpha,\beta)}(0) = 1,$$

$$(2.43)\qquad \phi_\mu^{(\alpha,\beta)}(t) = \phi_\mu^{(\alpha,\beta)}(-t),$$

$$(2.44)\qquad \phi_\mu^{(\alpha,\beta)}(t) = \phi_{-\mu}^{(\alpha,\beta)}(t).$$

The function $\phi_\mu^{(\alpha,\beta)}$ satisfies the differential equation

$$(2.45)\qquad (\Delta_{\alpha,\beta}(t))^{-1}\frac{d}{dt}\left(\Delta_{\alpha,\beta}(t)\frac{du(t)}{dt}\right) = -(\mu^2+(\alpha+\beta+1)^2)u(t),$$

where

$$\Delta_{\alpha,\beta}(t) := (\text{sh } t)^{2\alpha+1}(\text{ch } t)^{2\beta+1},$$

and $u := \phi_\mu^{\alpha,\beta}$ is the unique solution of (2.45) which is regular at $t = 0$ and satisfies $u(0) = 1$. For fixed $\alpha > -1$, $\beta \in \mathbb{R}$, Jacobi functions $\phi_\mu^{(\alpha,\beta)}$ form a continuous orthogonal system with respect to the measure $\Delta_{\alpha,\beta}(t)dt$, $t > 0$.

Substitution of (2.41) into (2.40) yields ($m \geq n$):

$$(2.46)\qquad \pi_{\xi,\lambda;m,n}(a_t) =$$

$$= \frac{(\lambda+n+\frac{1}{2})_{m-n}}{(m-n)!}(\text{sh}\tfrac{1}{2}t)^{m-n}(\text{ch}\tfrac{1}{2}t)^{-m-n}\phi_{2i\lambda}^{(m-n,-m-n)}(\tfrac{1}{2}t) =$$

$$= \frac{(\lambda+n+\frac{1}{2})_{m-n}}{(m-n)!}(\text{sh}\tfrac{1}{2}t)^{m-n}(\text{ch}\tfrac{1}{2}t)^{m+n}\phi_{2i\lambda}^{(m-n,m+n)}(\tfrac{1}{2}t).$$

Application of (2.33) gives a similar result in the case $m < n$. Finally we conclude:

<u>THEOREM 2.1</u>. *The canonical matrix elements* $\pi_{\xi,\lambda;m,n}(a_t)$ ($\lambda \in \mathbb{C}$; $\xi = 0$ or $\frac{1}{2}$; $m,n \in \mathbb{Z} + \xi$; $t \in \mathbb{R}$) *of* SU(1,1) *can be expressed in terms of Jacobi functions by*

$$(2.47)\qquad \pi_{\xi,\lambda;m,n}(a_t) =$$

$$\frac{c_{\xi,\lambda;m,n}}{(|m-n|)!}\cdot(\text{sh}\tfrac{1}{2}t)^{|m-n|}(\text{ch}\tfrac{1}{2}t)^{m+n}\phi_{2i\lambda}^{(|m-n|,m+n)}(\tfrac{1}{2}t),$$

where

(2.48) $$c_{\xi,\lambda;m,n} := \begin{cases} (\lambda+n+\frac{1}{2})_{m-n} & \text{if } m \ge n, \\ (\lambda-n+\frac{1}{2})_{n-m} & \text{if } n \ge m. \end{cases}$$

In view of (2.42), formulas (2.47 and (2.48) describe the asymptotics of $\pi_{\xi,\lambda;m,n}$ near $t = 0$.

2.5. Notes

2.5.1. The principal series of representations was first written down for SL(2,ℝ) by BARGMANN [1], for SL(2,ℂ) by GELFAND & NAIMARK [24], and for a general noncompact semisimple Lie group by HARISH-CHANDRA [27, §12].

2.5.2. BARGMANN [1, §10] already obtained explicit expressions in terms of hypergeometric functions for the canonical matrix elements of the irreducible unitary representations of SL(2,ℝ). He solved the differential equation satisfied by these matrix elements, which is obtained from the Casimir operator. RÜHL [51, Ch.5] gives a derivation of these expressions which is similar to our derivation in §2.4, starting from the integral representation (2.32).

2.5.3. It follows from the present paper that the spherical functions for SL(2,ℝ) can be expressed as Jacobi functions of order $(\alpha,\beta) = (0,0)$. More generally, the spherical functions on any noncompact real semisimple Lie group of rank 1 (i.e., dim (A) = 1) can be written as Jacobi functions of certain order (cf. HARISH-CHANDRA [29, §13]). This motivated FLENSTED-JENSEN [17] to study harmonic analysis for Jacobi function expansions of quite general order (α,β), $\alpha \ge \beta \ge -\frac{1}{2}$. This research was continued in several papers by Flensted-Jensen and the author (see for instance [19]).

3. THE IRREDUCIBLE SUBQUOTIENT REPRESENTATIONS OF THE PRINCIPAL SERIES

3.1. Subquotient representations.

Let G be a lcsc. group and let τ be a Hilbert representation of G. Let

H_0 be a closed subspace of $H(\tau)$ and let P_0 be the orthogonal projection from $H(\tau)$ onto H_0. Define

$$(3.1) \qquad \tau_0(g)v := P_0\ \tau(g)v, \qquad g \in G, \qquad v \in H_0.$$

Then $\tau(g) \in L(H_0)$ for each $g \in G$, $\tau_0(e) = \text{id.}$, and $g \to \tau_0(g)v : G \to H_0$ is continuous for each $v \in H_0$. If also

$$(3.2) \qquad \tau_0(g_1g_2) = \tau_0(g_1)\tau_0(g_2), \qquad g_1,g_2 \in G,$$

then τ_0 is a Hilbert representation of G on H_0 and it is called a *subquotient representation* of τ. Formula (3.2) is clearly valid if H_0 is an *invariant subspace* of $H(\tau)$, i.e., if $\tau(g)v \in H_0$ for all $g \in G$, $v \in H_0$. In that case, τ_0 is called a *subrepresentation* of τ.

Let H_0 and H_1 be closed subspaces of $H(\tau)$ and assume $H_1 \subset H_0$. Let $P_0 : H(\tau) \to H_0$, $P_1 : H(\tau) \to H_1$ and $P_{0,1} : H_0 \to H_1$ be the corresponding orthogonal projections and let τ_0 be defined by (3.1). Then, for $g \in G$, $v \in H_1(\tau)$ we have:

$$\tau_1(g)v := P_1\tau(g)v = P_{0,1}P_0\tau(g)v = P_{0,1}\tau_0(g)v =: \tau_{0,1}(g)v.$$

Hence, if τ_0 is a subquotient representation of τ and if $H_1 \subset H(\tau_0)$ then there is a subquotient representation of τ on H_1 if and only if there is a subquotient representation of τ_0 on H_1, in which case these two representations coincide.

LEMMA 3.1. *Let H_0 be a closed subspace of $H(\tau)$, let H_2 be the closed G-invariant subspace of $H(\tau)$ which is generated by H_0 and let $H_1 := H_2 \cap H_0^{\perp}$. Then τ_0 is a subquotient representation if and only if H_1 is G-ivariant.*

PROOF. Let P_0 and P_1 denote the orthogonal projections on H_0 and H_1, respectively. It follows from (3.1) that

$$\tau_0(g_1g_2)v-\tau_0(g_1)\tau_0(g_2)v = P_0\tau(g_1)P_1\tau(g_2)v, \ g_1,g_2 \in G, \ v \in H_0.$$

H_1 is the closed linear span of all elements $P_1\tau(g_2)v$, $g_2 \in G$, $v \in H_0$. So (3.2) holds iff $P_0\tau(g_1)w = 0$ for all $g_1 \in G$, $w \in H_1$. □

Let K be a compact subgroup of G and suppose that τ is K-unitary. Let τ_0 be a subquotient representation of τ on H_0 and let H_1 and H_2 be as in Lemma 3.1. Then H_2 and H_1 are G-invariant subspaces, so $H_0 = H_2 \cap H_1^{\perp}$ is K-invariant. It follows that τ_0 is K-unitary and that

$$\tau_0(k)v = \tau(k)v, \qquad k \in K, \qquad v \in H_0.$$

If τ is K-multiplicity free then τ_0 is also K-multiplicity free, $M(\tau_0) \subset M(\tau)$, $H_\delta(\tau_0) = H_\delta(\tau)$ for $\delta \in M(\tau_0)$ and $\tau_{0;\gamma,\delta}(g) = \tau_{\gamma,\delta}(g)$, $\gamma,\delta \in M(\tau_0)$, $g \in G$.

3.2 Irreducible subquotient representations of K-multiplicity free representations.

Let τ be a K-multiplicity free representation of G. We will show that there is a canonical orthogonal direct sum decomposition of $H(\tau)$ such that there is an irreducible subquotient representation of τ on each of the summands and each irreducible subquotient representation of τ has this form.

Given $\delta \in M(\tau)$ we define two G-invariant subspaces of $H(\tau)$:

$$\text{(3.3)} \qquad \mathrm{Cycl}(\delta) := \mathrm{Cl\ Span}\{\tau(g)v \mid v \in H_\delta(\tau), \quad g \in G\},$$

$$\text{(3.4)} \qquad \mathrm{Anticycl}(\delta) := \{v \in H(\tau) \mid \tau(g)v \perp H_\delta(\tau) \text{ for all } g \in G\}.$$

Here Cl means closure. Note that Cycl(δ) is the smallest closed G-invariant subspace of $H(\tau)$ which includes $H_\delta(\tau)$ and that Anticycl (δ) is the biggest closed G-invariant subspace of $H(\tau)$ which does not include $H_\delta(\tau)$ (since τ is K-unitary). The subrepresentation of τ on Cycl(δ) is cyclic with any nonzero $v \in H_\delta(\tau)$ as a cyclic vector.

Both Cycl(δ) and Anticycl(δ) are a direct sum of certain subspaces $H_\gamma(\tau)$, $\gamma \in M(\tau)$. It is clear from (3.3) and (3.4) that

$$\text{(3.5)} \qquad H_\gamma(\tau) \subset \mathrm{Cycl}(\delta) \Longleftrightarrow \text{not } H_\delta(\tau) \subset \mathrm{Anticycl}(\gamma).$$

For $\delta \in M(\tau)$ define:

(3.6) $$\mathrm{Irr}(\delta) := \mathrm{Cycl}(\delta) \cap \mathrm{Anticycl}(\delta)^{\perp}.$$

It is convenient to define two relations $\prec$ and $\sim$ on $M(\tau)$ as follows:

(3.7) $$\gamma \prec \delta \overset{\text{def}}{\iff} H_{\gamma}(\tau) \subset \mathrm{Cycl}(\delta),$$

(3.8) $$\gamma \sim \delta \overset{\text{def}}{\iff} H_{\gamma}(\tau) \subset \mathrm{Irr}(\delta).$$

Then it follows from (3.5) and (3.6) that

(3.9) $$H_{\gamma}(\tau) \subset \mathrm{Anticycl}(\delta) \iff \text{not } \delta \prec \gamma,$$

(3.10) $$\gamma \sim \delta \iff \gamma \prec \delta \text{ and } \delta \prec \gamma.$$

It is clear from (3.3) and (3.7) that

(3.11) $$\beta \prec \gamma \text{ and } \gamma \prec \delta \Rightarrow \beta \prec \delta.$$

Hence, by (3.10) and (3.11), $\sim$ is an equivalence relation on $M(\tau)$ and $\mathrm{Irr}(\delta)$ is the closed K-invariant subspace of $H(\tau)$ corresponding to the equivalence class of δ with respect to $\sim$. It also follows that

(3.12) $$\gamma \sim \delta \Rightarrow \mathrm{Cycl}(\gamma) = \mathrm{Cycl}(\delta),$$

(3.13) $$\gamma \sim \delta \Rightarrow \mathrm{Anticycl}(\gamma) = \mathrm{Anticycl}(\delta).$$

LEMMA 3.2. *There is an irreducible subquotient representation of* τ *on each subspace* $\mathrm{Irr}(\delta)$, $\delta \in M(\tau)$. *All irreducible subquotient representations of* τ *have this form.*

PROOF. Let $\delta \in M(\tau)$ and $H_0 := \mathrm{Irr}(\delta)$. Let H_1 and H_2 be as in Lemma 3.1. Then $H_2 = \mathrm{Cycl}(\delta)$ by (3.12), so $H_1 = H_2 \cap \mathrm{Anticycl}(\delta)$ by (3.6). Since H_1 is G-invariant, τ_0 defined by (3.1) is a subquotient representation of

τ on H_0 (cf. Lemma 3.1). For the irreducibility proof of τ_0 suppose that τ_0' is a subrepresentation of τ_0 on $H_0' \subset H_0$, $H_0' \neq \{0\}$. Then τ_0' is a subquotient representation of τ, so it is a direct sum of subspaces $H_\gamma(\tau)$ for certain $\gamma \sim \delta$. Pick such a γ and let H_1' and H_2' be as in Lemma 3.1. Then, by (3.12), $H_2' = \mathrm{Cycl}(\gamma) = \mathrm{Cycl}(\delta) = H_2$. Since H_1' is τ-invariant and orthogonal to $H_\gamma(\tau)$, we have

$$H_1' \subset H_2 \cap \mathrm{Anticycl}(\gamma) = H_2 \cap \mathrm{Anticycl}(\delta) = H_1$$

(use (3.13)). Hence $H_0' \supset H_0$, so $H_0' = H_0$ and τ_0 is irreducible.

Next let τ_0 be an irreducible subquotient representation of τ on some subspace H_0. Then $H_\delta(\tau) \subset H_0$ for some $\delta \in M(\tau)$. Let H_1 and H_2 be as in Lemma 3.1. Then $\mathrm{Cycl}(\delta) \subset H_1$ and $H_2 \subset \mathrm{Anticycl}(\delta)$, so $\mathrm{Irr}(\delta) \subset H_0$. Hence the subquotient representation τ_0' of τ on $\mathrm{Irr}(\delta)$ is also a subquotient representation of τ_0. The irreducibility of τ_0 together with Lemma 3.1 implies that $\mathrm{Irr}(\delta) = H_0$. □

Next we derive a criterium for the relations $\prec$ and $\sim$ in terms of the canonical matrix elements of τ. Let $\gamma,\delta \in M(\tau)$. For $v \in H_\gamma(\tau)$, $w \in H_\delta(\tau)$, $g \in G$ we have:

$$(\tau(g)v,w) = (\tau_{\delta,\gamma}(g)v,w).$$

Hence we get from (3.9) and (3.4):

(3.14) not $\delta \prec \gamma \iff H_\gamma(\tau) \subset \mathrm{Anticycl}(\delta) \iff \tau_{\delta,\gamma} = 0.$

It follows from (3.14), (3.7), (3.8) and (3.10) that

(3.15) $\qquad \gamma \prec \delta \iff H_\gamma(\tau) \subset \mathrm{Cycl}(\delta) \iff \tau_{\gamma,\delta} \neq 0,$

(3.16) $\qquad \gamma \sim \delta \iff H_\gamma(\tau) \subset \mathrm{Irr}(\delta) \iff \tau_{\gamma,\delta} \neq 0$ and $\tau_{\delta,\gamma} \neq 0.$

Now we can formulate our main result.

THEOREM 3.3. *Let* G *be a lcsc. group with compact subgroup* K *and let* τ *be a K-multiplicity free representation of* G. *Let* $\mathcal{H}_0$ *be a closed subspace of* $\mathcal{H}(\tau)$ *and let* τ_0 *be defined by (3.1).*

(a) *If* τ_0 *is an irreducible subquotient representation of* τ *then*

$$(3.17) \qquad \mathcal{H}_0 = \sum_{\gamma \in M(\tau_0)}^{\oplus} \mathcal{H}_\gamma(\tau)$$

for some subset $M(\tau_0)$ *of* $M(\tau)$.

(b) *Suppose* $\mathcal{H}_0$ *has the form (3.17). Then the following three statements are equivalent:*

(i) τ_0 *is an irreducible subquotient representation of* τ.

(ii) $M(\tau_0) = \{\gamma \in M(\tau) \mid \tau_{\gamma,\delta} \neq 0$ and $\tau_{\delta,\gamma} \neq 0\}$ *for some* $\delta \in M(\tau_0)$.

(iii) $M(\tau_0) = \{\gamma \in M(\tau) \mid \tau_{\gamma,\delta} \neq 0$ and $\tau_{\delta,\gamma} \neq 0\}$ *for all* $\delta \in M(\tau_0)$.

(c) *Suppose* τ_0 *is an irreducible subquotient representation of* τ. *Then the following three statements are equivalent:*

(i) τ_0 *is a subrepresentation of* τ.

(ii) $M(\tau_0) = \{\gamma \in M(\tau) \mid \tau_{\gamma,\delta} \neq 0\}$ *for some* $\delta \in M(\tau_0)$.

(iii) $M(\tau_0) = \{\gamma \in M(\tau) \mid \tau_{\gamma,\delta} \neq 0\}$ *for all* $\delta \in M(\tau_0)$.

If G = KAK for some subset A of K and if τ is a K-multiplicity free representation of G then, for $\gamma, \delta \in M(\tau)$, we have:

$$\tau_{\gamma,\delta} \neq 0 \iff \tau_{\gamma,\delta}|_A \neq 0.$$

3.3. Irreducible subquotient representations for the principal series of SU(1,1).

For $\lambda \in \mathbb{C}$, $\xi = 0$ or $\frac{1}{2}$, the representation $\pi_{\xi,\lambda}$ of G = SU(1,1) on $L^2_\xi(K)$ (cf. (2.25)) is K-multiplicity free with K-content given by (2.30). By inspecting (2.47) for small but nonzero t and by using (2.42) it follows that

$$(3.18) \qquad \pi_{\xi,\lambda;m,n} \neq 0 \iff c_{\xi,\lambda;m,n} \neq 0,$$

where $c_{\xi,\lambda;m,n}$ is given by (2.48). Combination of (3.18) with Theorem 3.3 yields the following classification of irreducible subquotient representations of $\pi_{\xi,\lambda}$.

THEOREM 3.4. *Depending on* ξ *and* λ, *the representation* $\pi_{\xi,\lambda}$ *of* SU(1,1) *has the following irreducible subquotient representations:*

(a) $\lambda + \xi \notin \mathbb{Z} + \frac{1}{2}$:

$\pi_{\xi,\lambda}$ *is irreducible itself.*

(b) $\lambda = 0,\ \xi = \frac{1}{2}$:

$\pi^{+}_{\frac{1}{2},0}$ *on* Cl Span $\{\phi_{1/2}, \phi_{3/2}, \ldots\}$,

$\pi^{-}_{\frac{1}{2},0}$ *on* Cl Span $\{\ldots, \phi_{-3/2}, \phi_{-1/2}\}$.

$\pi^{+}_{\frac{1}{2},0}$ *and* $\pi^{-}_{\frac{1}{2},0}$ *are subrepresentations.*

(c) $\lambda + \xi \in \mathbb{Z} + \frac{1}{2},\ \lambda > 0$:

$\pi^{+}_{\xi,\lambda}$ *on* Cl Span $\{\phi_{\lambda+\frac{1}{2}}, \phi_{\lambda+3/2}, \ldots\}$,

$\pi^{-}_{\xi,\lambda}$ *on* Cl Span $\{\ldots, \phi_{-\lambda-3/2}, \phi_{-\lambda-\frac{1}{2}}\}$,

$\pi^{0}_{\xi,\lambda}$ *on* Span $\{\phi_{-\lambda+\frac{1}{2}}, \phi_{-\lambda+3/2}, \ldots, \phi_{\lambda-\frac{1}{2}}\}$.

$\pi^{+}_{\xi,\lambda}$ *and* $\pi^{-}_{\xi,\lambda}$ *but not* $\pi^{0}_{\xi,\lambda}$ *are subrepresentations.*

(d) $\lambda + \xi \in \mathbb{Z} + \frac{1}{2},\ \lambda < 0$:

$\pi^{+}_{\xi,\lambda}$ *on* Cl Span $\{\phi_{-\lambda+\frac{1}{2}}, \phi_{-\lambda+3/2}, \ldots\}$,

$\pi^{-}_{\xi,\lambda}$ *on* Cl Span $\{\ldots, \phi_{\lambda-3/2}, \phi_{\lambda-\frac{1}{2}}\}$,

$\pi^{0}_{\xi,\lambda}$ *on* Span $\{\phi_{\lambda+\frac{1}{2}}, \phi_{\lambda+3/2}, \ldots, \phi_{-\lambda-\frac{1}{2}}\}$.

$\pi^{0}_{\xi,\lambda}$ *but not* $\pi^{+}_{\xi,\lambda}, \pi^{-}_{\xi,\lambda}$ *is a subrepresentation.*

PROOF.

(a) $c_{\xi,\lambda;m,n} \neq 0$.

(b) $c_{\frac{1}{2},0;m,n} \neq 0 \Longleftrightarrow m,n \leq -\frac{1}{2}$ or $m,n \geq \frac{1}{2}$.

(c) $c_{\xi,\lambda;m,n} \neq 0 \Longleftrightarrow -\lambda + \frac{1}{2} \leq n \leq \lambda - \frac{1}{2}$ or $m,n \leq -\lambda - \frac{1}{2}$ or $m,n \geq \lambda + \frac{1}{2}$.

(d) $c_{\xi,\lambda;m,n} \neq 0 \Longleftrightarrow \lambda + \frac{1}{2} \leq m \leq -\lambda - \frac{1}{2}$ or $m,n \leq \lambda - \frac{1}{2}$ or $m,n \geq -\lambda + \frac{1}{2}$. □

The finite-dimensional representations occurring in the above classification are the representations $\pi^{0}_{\xi,\lambda}$ $(\lambda + \xi \in \mathbb{Z} + \frac{1}{2},\ \lambda \neq 0)$.

3.4. Notes.

3.4.1. In the case of the unitary principal series (λ imaginary), Theorem 3.4 was first proved by BARGMANN [1, sections 6 and 7]. See van DIJK [11, Theorem 4.1] for the statement and (infinitesimal) proof of our Theorem 3.4 in the general case. A proof of Theorem 3.4 similar to our proof was earlier given by BARUT & PHILLIPS [2, § II(4)].

3.4.2. Theorem 3.4 in the case of imaginary and nonzero λ is contained in a general theorem by BRUHAT [6, Théorème 7;2]: For $\xi \in \hat{M}$, $\lambda \in i\mathfrak{a}^*$, the principal series representation $\pi_{\xi,\lambda}$ of G(cf. (2.10)) is irreducible if $s.\lambda \neq \lambda$ for all $s \neq e$ in the Weyl group for (G,K).

3.4.3. GELFAND & NAIMARK [24, §5.4, Theorem 1] proved the irreducibility of the unitary principal series for SL(2,$\mathbb{C}$) by a global method different from ours. They wrote down the principal series in a noncompact realization related to the Bruhat decomposition (cf. WALLACH [61, §8.4]) and they calculated the "matrix elements" of the representation with respect to a (continuous) $\bar{N}$ -basis. ($\bar{N}$ is the image of N under the Cartan involution.) Then the irreducibility follows from the nonvanishing of these matrix elements. Although this method is analogous to ours, it requires more care, since the $\bar{N}$-basis is continuous rather than discrete.

3.4.4. Our technique of proving irreducibility (cf. Theorems 3.3 and 3.4) was probably first used by NAIMARK [47, Ch.3, §9, no. 15] in the case of the nonunitary principal series for SL(2,$\mathbb{C}$), see also KOSTERS [39]. Other applications of this technique, besides BARUT & PHILLIPS [2] (cf. 3.4.1 above), can be found in MILLER [44, Theorem 2], [45, Lemma 4.5] for the harmonic oscillator group and MILLER [45, Lemma 3.2] for the Euclidean motion group of $\mathbb{R}^2$.

3.4.5. Essentially, our technique is also used by TAKAHASHI [55, §3.4] for proving the irreducibility of the discrete series for SL(2,$\mathbb{R}$). In fact, consider a unitary representation τ of G which contains a certain one-dimensional representation δ of a compact subgroup K precisely once, and suppose that, for $0 \neq v \in \mathcal{H}_\delta(\tau)$, the function $g \to (\tau(g)w,v)$ is not identically zero for each nonzero w in $\mathcal{H}(\tau)$. Then τ is irreducible. The question whether this global argument, which works for the discrete series of $SL(2,\mathbb{R})$, could also be applied to the principal series, was the original motivation for writing this paper. The above argument was also used by TAKAHASHI [55, p.560, Cor.2] in order to decide when the principal series for $F_{4(-20)}$ is irreducible.

3.4.6. The method of this section does not show in an a priori way that a K-multiplicity free principal series representation has only finitely many irreducible subquotient representations. Actually, this property holds quite generally, cf. LEPOWSKY [70, Theorem 9.7], WALLACH [61, Theorem 8.13.3], KRALJEVIĆ [69].

4. EQUIVALENCES BETWEEN IRREDUCIBLE SUBQUOTIENT REPRESENTATIONS OF THE PRINCIPAL SERIES

4.1. The definition of Naimark equivalence.

Let G be a lcsc. group.

DEFINITION 4.1. Let σ and τ be Hilbert representations of G. The representation σ is called *Naimark* related to τ if there is a closed (possibly unbounded) injective linear operator A from $\mathcal{H}(\sigma)$ to $\mathcal{H}(\tau)$ with domain $\mathcal{D}(A)$ dense in $\mathcal{H}(\sigma)$ and range $\mathcal{R}(A)$ dense in $\mathcal{H}(\tau)$ such that $\mathcal{D}(A)$ is σ-invariant and $A\sigma(g)v = \tau(G)Av$ for all $v \in \mathcal{D}(A)$, $g \in G$. Then we use the notation $\sigma \simeq \tau$ or $\sigma \overset{A}{\simeq} \tau$.

Naimark relatedness is not necessarily a transitive relation (cf. WARNER [63, p.242]). However, we will see that it becomes an equivalence relation (called *Naimark equivalence*) when restricted to the class of unitary representations or of K-finite representations.

Two unitary representations σ and τ of G are called *unitarily equivalent* if there is an isometry A from $\mathcal{H}(\sigma)$ onto $\mathcal{H}(\tau)$ such that $A\sigma(g)v = \tau(g)Av$ for all $v \in \mathcal{H}(\sigma)$, $g \in G$. Clearly unitary equivalence is an equivalence relation.

THEOREM 4.2. *Two unitary representations of a lcsc. group* G *are Naimark related if and only if they are unitarily equivalent.*

PROOF. Clearly unitary equivalence implies Naimark relatedness. Conversely assume Naimark relatedness $\sigma \overset{A}{\simeq} \tau$ for two unitary representations σ and τ

of G. Then we have the polar decomposition $A = U|A|$ (cf. DUNFORD & SCHWARTZ [10, Ch. XII, Theorem 7.6]), where $|A|$ is the positive square root of the positive self-adjoint operator A^*A and where U is an isometry from $\mathcal{H}(\sigma)$ onto $\mathcal{H}(\tau)$, since A is injective and has dense range. Let $g \in G$. Then A $A\sigma(g)v = \tau(g)Av$, $v \in \mathcal{D}(A)$, so $A^*\tau(g^{-1})w = \sigma(g^{-1})A^*w$, $w \in \mathcal{D}(A^*)$, since σ and τ are unitary. Hence $\sigma(g)$ commutes with A^*A, so $\sigma(g)$ commutes with all projection operators in the spectral resolution of A^*A, and thus $\sigma(g)$ commutes with $|A| = (A^*A)^{\frac{1}{2}}$, cf. RUDIN [50, §13.22 - 13.33]. By using $A = U|A|$, it follows that $U\sigma(g)|A| = \tau(g)U|A|$, $g \in G$, so $U\sigma(g) = \tau(g)U$ because $|A|$ has dense range. □

Suppose that K is a compact subgroup of G. Use the notation of §2.1. Let τ be a K-finite representation of G. Write

$$(4.1) \qquad v_\delta := P_{\tau,\delta}v, \qquad v \in \mathcal{H}(\tau), \qquad \delta \in \hat{K}.$$

Then

$$(4.2) \qquad v = \sum_{\delta\in\hat{K}} v_\delta$$

and

$$\sum_{\delta\in\hat{K}} \|v_\delta\|^2 < \infty.$$

On the other hand, if, for each $\delta \in \hat{K}$, v_δ is an arbitrary element of $\mathcal{H}_\delta(\tau)$ then (4.2) defines an element of v if and only if (4.3) holds.

<u>LEMMA 4.3.</u> *Let* σ *and* τ *be K-finite representations of G. Let* $A:\mathcal{H}(\sigma) \to \mathcal{H}(\tau)$ *be an injective (possibly unbounded, not necessarily closed) linear operator with dense domain and range such that* $\mathcal{D}(A)$ *is* σ*-invariant and* $A\sigma(g)v = \tau(g)Av$ *for* $v \in \mathcal{D}(A)$, $g \in G$. *Consider the five statements:*

(i) *A is closed.*

(ii) $P_{\sigma,\delta}v \in \mathcal{D}(A)$ *and* $AP_{\sigma,\delta}v = P_{\tau,\delta}Av$ *for all* $v \in \mathcal{D}(A)$, $\delta \in \hat{K}$.

(iii) *For all* $\delta \in \hat{K}$: $\mathcal{H}_\delta(\sigma) \subset \mathcal{D}(A)$, $\mathcal{H}_\delta(\tau) \subset \mathcal{R}(A)$ *and the mapping* A_δ *defined by*

(4.4) $$A_\delta := A|_{H_\delta(\sigma)},$$

is a K-intertwining bijection from $H_\delta(\sigma)$ *onto* $H_\delta(\tau)$.

(iv) *The closure* $\bar{A}$ *of* A *is one-valued and injective.*

(v) $\mathcal{D}(\bar{A})$ *is* σ*-invariant and* $\bar{A}\sigma(g)v = \tau(g)\bar{A}v$ *for* $v \in \mathcal{D}(A)$, $g \in G$.

Among these statements there are the implications (i) ⇒ (ii), (ii) ⇒ (iii), (ii) ⇒ (iv) *and* (iv) ⇒ (v). *In particular, any of the three statements* (i), (ii) *or* (iv) *implies that* σ *and* τ *are Naimark related. Finally, if* (ii) *holds then*

(4.5) $$\mathcal{D}(\bar{A}) = \{v \in H(\sigma) \mid \sum_{\delta \in \hat{K}} \|A_\delta v_\delta\|^2 < \infty\},$$

so $\bar{A}$ *is completely determined by the restrictions* A_δ $(\delta \in \hat{K})$ *of* A.

PROOF.

(i) ⇒ (ii): Let A be closed. It follows from (2.1) and the intertwining property of A that

$$P_{\sigma,\delta}\, v = d_\delta \int_K \chi_\delta\, (k^{-1})\sigma(k)v\, dk,$$

$$P_{\tau,\delta}Av = d_\delta \int_K \chi_\delta\, (k^{-1})\tau(k)Av\, dx =$$

$$= d_\delta \int_K \chi_\delta\, (k^{-1})A\sigma(k)v\, dx.$$

Since A is closed, we get (ii).

(ii) ⇒ (iii): Assume (ii). Since $P_{\sigma,\delta}\mathcal{D}(A)$ is dense in $H_\delta(\sigma)$ and $H_\delta(\sigma)$ is finite-dimensional, we have $H_\delta(\sigma) = P_{\sigma,\delta}\mathcal{D}(A)$ and, similarly, $H_\delta(\tau) = P_{\tau,\delta}\mathcal{R}(A)$. Thus, by (ii), $H_\delta(\sigma) \subset \mathcal{D}(A)$ and

$$AH_\delta(\sigma) = AP_{\sigma,\delta}\mathcal{D}(A) = P_{\tau,\delta}A\mathcal{D}(A) = P_{\tau,\delta}\mathcal{R}(A) = H_\delta(\tau).$$

Hence, by the intertwining property and by the injectivity of A, A_δ is a K-intertwining bijection from $H_\delta(\sigma)$ onto $H_\delta(\tau)$.

(ii) ⇒ (iv): Assume (ii). Then (iii) also holds. Let $\{v_n\}$ be a sequence in $\mathcal{D}(A)$ such that $v_n \to v$ in $H(\sigma)$ and $Av_n \to w$ in $H(\tau)$. Then

$$w_\delta = \lim_{n\to\infty} P_{\tau,\delta} A v_n = \lim_{n\to\infty} A_\delta P_{\sigma,\delta} v_n = A_\delta v_\delta.$$

Hence, $v = 0$ iff $w = 0$, so $\bar{A}$ is one-valued and injective.

(iv) ⇒ (v): Assume (iv). Let $v \in \mathcal{D}(\bar{A})$, so $v_n \to v$, $Av_n \to \bar{A}v$ for some sequence $\{v_n\}$ in $\mathcal{D}(A)$. Let $g \in G$. Then $\sigma(g)v_n \to \sigma(g)v$ and $A\sigma(g)v_n = \tau(g)Av_n \to \tau(g)\bar{A}v$, so $\sigma(g)v \in \mathcal{D}(\bar{A})$ and $\bar{A}\sigma(g)v = \tau(g)\bar{A}v$.

PROOF OF (4.5). Assume (ii). Then also (iii) and (iv) holds. First suppose $v \in \mathcal{H}(\sigma)$ and $\sum_{\delta\in\hat{K}} \|A_\delta v_\delta\|^2 < \infty$. Put $w := \sum_{\delta\in\hat{K}} Av_\delta$. On comparing this with (4.2) we get $w = \bar{A}v$. Conversely, suppose that $v \in \mathcal{D}(\bar{A})$. Then

$$\bar{A}v = \sum_{\delta\in\hat{K}} P_{\tau,\delta}\bar{A}v = \sum_{\delta\in\hat{K}} \bar{A}P_{\sigma,\delta}v = \sum_{\delta\in\hat{K}} A_\delta v_\delta,$$

where we used the implication (i) ⇒ (ii) for the operator $\bar{A}$. Hence

$$\sum_{\delta\in\hat{K}} \|A_\delta v_\delta\|^2 < \infty. \qquad \square$$

THEOREM 4.4. *The relation of Naimark relatedness defines an equivalence relation in the class of K-finite representations of* G.

PROOF. Let ρ, σ, τ be K-finite representations of G. Clearly, $\tau \overset{id}{\simeq} \tau$ and $\sigma \overset{A}{\simeq} \tau \Rightarrow \tau \overset{A^{-1}}{\simeq} \sigma$. Next suppose that $\rho \overset{B}{\simeq} \sigma$ and $\sigma \overset{A}{\simeq} \tau$. Define $Cv := ABv$ whenever $v \in \mathcal{D}(B)$, $Bv \in \mathcal{D}(A)$. It follows from the implication (i) ⇒ (iii) in Lemma 4.3 that $\mathcal{H}_\delta(\rho) \subset \mathcal{D}(C)$ and $\mathcal{H}_\delta(\tau) \subset \mathcal{R}(C)$, $\delta \in \hat{K}$, so $\mathcal{D}(C)$ is dense in $\mathcal{H}(\rho)$ and $\mathcal{R}(C)$ is dense in $\mathcal{H}(\tau)$. Clearly, C is injective, $\mathcal{D}(C)$ is ρ-invariant and C is an intertwining operator for ρ and τ. If $v \in \mathcal{D}(C)$ and $\delta \in \hat{K}$ then $P_{\rho,\delta}v \in \mathcal{H}_\delta(\rho) \subset \mathcal{D}(C)$ and

$$CP_{\rho,\delta}v = ABP_{\rho,\delta}v = AP_{\sigma,\delta}Bv = P_{\tau,\delta}ABv = P_{\tau,\delta}Cv,$$

so (ii) of Lemma 4.3 holds for C. (we used (i) ⇒ (ii) in Lemma 4.3 for A and B.) Thus (iv) and (v) of Lemma 4.3 hold for C, so $\rho \overset{C}{\simeq} \tau$. $\square$

4.2. A criterium for Naimark equivalence.

In this subsection σ and τ are supposed to be K-multiplicity free representations of G. For each $\delta \in \mathcal{M}(\sigma) \cap \mathcal{M}(\tau)$ choose a K-intertwining isometry $I_\delta : \mathcal{H}_\delta(\sigma) \to \mathcal{H}_\delta(\tau)$, unique up to a complex scalar factor of absolute value 1. If $\sigma \overset{A}{\cong} \tau$ then $M(\sigma) = M(\tau)$ and, for each $\delta \in \hat{K}$, $A_\delta = c_\delta I_\delta$ for some $c_\delta \in \mathbb{C}$, $c_\delta \neq 0$ (cf. (iii) of Lemma 4.3).

THEOREM 4.5. *Let σ and τ be K-multiplicity free representations and, for each $\delta \in M(\sigma) \cap M(\tau)$, let $0 \neq c_\delta \in M(\sigma) \cap M(\tau)$, Let $0 \neq c_\delta \in \mathbb{C}$. Then the following two statements are equivalent:*

(a) $\sigma \overset{A}{\cong} \tau$ *and* $A_\delta = c_\delta I_\delta$ *for each* $\delta \in M(\sigma)$.

(b) $M(\sigma) = M(\tau)$ *and, for all* $\gamma, \delta \in M(\sigma)$,

$$\tau_{\gamma,\delta}(g) = C_{\gamma,\delta} I_\gamma \sigma_{\gamma,\delta}(g) I_\delta^{-1}, \qquad g \in G, \tag{4.6}$$

with $C_{\gamma,\delta} = c_\gamma / c_\delta$.

PROOF. If $\sigma \overset{A}{\cong} \tau$ then the intertwining property of A implies that

$$A_\gamma \sigma_{\gamma,\delta}(g) = \tau_{\gamma,\delta}(g) A_\delta, \qquad \gamma, \delta \in M(\sigma).$$

Now substitute $A_\delta = c_\delta I_\delta$, $\quad \delta \in M(\sigma)$.

Conversely assume (b). Let $A_\delta := c_\delta I_\delta$. Define A on the domain

$$\{v \in \mathcal{H}(\sigma) \mid \sum_{\delta \in M(\sigma)} \|A_\delta v_\delta\|^2 < \infty\}$$

by

$$A(\sum_{\delta \in M(\sigma)} v_\delta) := \sum_{\delta \in M(\sigma)} A_\delta v_\delta.$$

Then A is injective with dense domain and range and A satisfies (ii) of Lemma 4.3. We will prove that $\mathcal{D}(A)$ is G-invariant and that A is an intertwining operator. Let $v \in \mathcal{D}(A)$, $g \in G$. Then, by (4.6):

$$A_\gamma(\sigma(g)v)_\gamma = c_\gamma I_\gamma \sum_{\delta \in M(\sigma)} \sigma_{\gamma,\delta}(g) v_\delta =$$

$$= \sum_{\delta \in M(\sigma)} c_\delta \tau_{\gamma,\delta}(g) I_\delta v_\delta =$$

$$\sum_{\delta\in M(\sigma)} \tau_{\gamma,\delta}(g) A_\delta v_\delta = (\tau(g)Av)_\gamma .$$

Hence

$$\sum_{\gamma\in M(\sigma)} \| A_\gamma(\sigma(g)v)_\gamma \|^2 = \| \tau(g)Av \|^2 < \infty,$$

so $\sigma(g)v \in \mathcal{D}(A)$ and $A\sigma(g)v = \tau(g)Av$. Now, by Lemma 4.2, $\mathcal{D}(\bar{A})$ is given by (4.5), so $\bar{A} = A$, A is closed and $\sigma \overset{A}{\simeq} \tau$. □

The above theorem would be sufficient in order to check which of the irreducible subquotient representations of the principal series of SU(1,1) are equivalent to each other. However, for irreducible representations σ and τ we need not compare all generalized matrix elements $\sigma_{\gamma,\delta}$ and $\tau_{\gamma,\delta}$ to each other but just one, as will be shown in Theorem 4.9 below. We need the following lemmas.

LEMMA 4.6. *Let τ be a K-multiplicity free representation of G. Then, for* $\gamma,\ \delta \in M(\tau)$:

$$(4.7)\qquad \tau_{\gamma,\delta}(k_1gk_2) = \tau_\gamma(k_1)\tau_{\gamma,\delta}(g)\tau_\delta(k_2),\quad g \in G,\ k_1,k_2 \in K,$$

$$(4.8)\qquad \tau_{\gamma,\delta}(g_1kg_2) = \sum_{\beta\in M(\tau)} \tau_{\gamma,\beta}(g_1)\tau_\beta(k)\tau_{\beta,\delta}(g_2),\quad g_1,g_2 \in G,\ k \in K,$$

where the right hand side of (4.8) is an absolutely convergent series in $L(H_\delta(\tau),H_\gamma(\tau))$, *uniformly for* $k \in K$.

PROOF. We only need to prove (4.8). First observe that, for $v \in H_\beta(\tau)$, $w \in H_\gamma(\tau)$, $g \in G$:

$$(v,(\tau(g))^*w) = (\tau(g)v,w) = (\tau_{\gamma,\beta}(g)v,w) = (v,(\tau_{\gamma,\beta}(g))^*w).$$

Next, choose $v \in H_\delta(\tau)$, $w \in H_\gamma(\tau)$, $g_1,g_2 \in G$, $k \in K$. Then

$$(\tau_{\gamma,\delta}(g_1kg_2)v,w) = (\tau(g_1kg_2)v,w) =$$

$$= (\tau(g_1)\tau(k)\tau(g_2)v,w) = (\tau(k)\tau(g_2)v,(\tau(g_1))^*w) =$$

$$= \sum_{\beta\in M(\tau)} (\tau_\beta(k)\tau_{\beta,\delta}(g_2)v,\ (\tau_{\gamma,\beta}(g_1))^*w) =$$

$$= \sum_{\beta\in M(\tau)} (\tau_{\gamma,\beta}(g_1)\tau_\beta(k)\tau_{\beta,\delta}(g_2)v,w).$$

This series converges absolutely, uniformly in k, since

$$\sum_{\beta\in M(\tau)} |(\tau_\beta(k)\tau_{\beta,\delta}(g_2)v,\ (\tau_{\gamma,\beta}(g_1))^*w)| \le$$

$$\le \sum_{\beta\in M(\tau)} \|\tau_\beta(k)\tau_{\beta,\delta}(g_2)v\|\ \|(\tau_{\gamma,\beta}(g_1))^*w\| =$$

$$= \sum_{\beta\in M(\tau)} \|(\tau(g_2)v)_\beta\|\ \|((\tau(g_1))^*w)_\beta\| \le$$

$$\le \|\tau(g_2)v\|\ \|(\tau(g_1))^*w\|.$$

Now use that $H_\gamma(\tau)$ and $H_\delta(\tau)$ are finite-dimensional. □

A straightforward application of Schur's lemma shows:

LEMMA 4.7. *Let* β *be an irreducible unitary representation of a compact group* K *on a finite-dimensional Hilbert space* H. *Let* $B \in L(H)$. *Then*

$$\int_K \beta(k)B\beta(k^{-1})dk = d_\beta^{-1}\ \mathrm{tr}(B)I. \tag{4.9}$$

LEMMA 4.8. *Let* σ *and* τ *be irreducible K-multiplicity free representations of* G. *Suppose that, for some* $\gamma,\delta \in M(\sigma) \cap M(\tau)$ *and for some nonzero* $c_{\gamma,\delta} \in \mathbb{C}$, *we have*

$$\tau_{\gamma,\delta}(g) = c_{\gamma,\delta} I_\gamma \sigma_{\gamma,\delta}(g) I_\delta^{-1},\quad g \in G. \tag{4.10}$$

Then $M(\sigma) = M(\tau)$ *and, for each* $\beta \in M(\sigma)$, *there are nonzero complex constants* $c_{\gamma,\beta}$ *and* $c_{\beta,\delta}$ *such that*

$$\tau_{\gamma,\beta}(g) = c_{\gamma,\beta} I_\gamma \sigma_{\gamma,\beta}(g) I_\beta^{-1},\qquad g \in G, \tag{4.11}$$

(4.12) $$\tau_{\beta,\delta}(g) = c_{\beta,\delta} I_\beta \sigma_{\beta,\delta}(g) I_\delta^{-1}, \quad g \in G,$$

and

(4.13) $$c_{\gamma,\delta} = c_{\gamma,\beta} c_{\beta,\delta}.$$

PROOF. It follows from (4.10) and (4.8) that

(4.14) $$\sum_{\beta \in M(\tau)} \tau_{\gamma,\beta}(g_1)\tau_\beta(k)\tau_{\beta,\delta}(g_2) =$$

$$= c_{\gamma,\delta} \sum_{\beta \in M(\sigma)} I_\gamma \sigma_{\gamma,\beta}(g_1)\sigma_\beta(k)\sigma_{\beta,\delta}(g_2) I_\delta^{-1},$$

$$g_1, g_2 \in G, \quad k \in K.$$

Both sides are absolutely and uniformly convergent Fourier series in $k \in K$ (cf. Lemma 4.6). By using the irreducibility of τ, formula (3.17) and the irreducibility of τ_β, we conclude that each term at the left hand side of (4.14) is not identically zero in g_1, k, g_2. A similar statement is valid for the terms at the right hand side. For each $\beta \in \hat{K}$ the corresponding terms at both sides of (4.14) must be equal. Hence, $M(\sigma) = M(\tau)$ and

(4.15) $$\tau_{\gamma,\beta}(g_1)\tau_\beta(k)\tau_{\beta,\delta}(g_2) =$$

$$= c_{\gamma,\delta} I_\gamma \sigma_{\gamma,\beta}(g_1)\sigma_\beta(k)\sigma_{\beta,\delta}(g_2) I_\delta^{-1},$$

$$g_1, g_2 \in G, \ k \in K, \ \beta \in M(\sigma).$$

Substitution of $\tau_\beta(k) = I_\beta \sigma_\beta(k) I_\beta^{-1}$ and multiplication to the right and to the left of both sides of (4.15) with suitable operators yields:

$$I_\gamma^{-1} \tau_{\gamma,\beta}(g_1) I_\beta \sigma_\beta(k) I_\beta^{-1} \tau_{\beta,\delta}(g_2) (\tau_{\beta,\delta}(g_2))^* I_\beta \sigma_\beta(k^{-1}) =$$

$$= c_{\gamma,\delta} \sigma_{\gamma,\beta}(g_1)\sigma_\beta(k)\sigma_{\beta,\delta}(g_2) I_\delta^{-1} \ (\tau_{\beta,\delta}(g_2))^* I_\beta \sigma_\beta(k^{-1}).$$

Now integrate with respect to k and apply (4.9):

$$I_\gamma^{-1}\tau_{\gamma,\beta}(g_1)I_\beta \operatorname{tr}(\tau_{\beta,\delta}(g_2)(\tau_{\beta,\delta}(g_2))^*) =$$

$$= C_{\gamma,\delta}\sigma_{\gamma,\beta}(g_1)\operatorname{tr}(\sigma_{\beta,\delta}(g_2)I_\delta^{-1}(\tau_{\beta,\delta}(g_2))^* I_\beta).$$

$\tau_{\beta,\delta}$ is not identically zero (cf. formula (3.17)), so $\operatorname{tr}(\tau_{\beta,\delta}(g_2)(\tau_{\beta,\delta}(g_2))^*)$ $\neq 0$ for some $g_2 \in G$ and we obtain (4.11). Substitution of (4.11) into (4.15) gives:

$$C_{\gamma,\beta}I_\gamma\sigma_{\gamma,\beta}(g_1)\sigma_\beta(k)I_\beta^{-1}\tau_{\beta,\delta}(g_2) =$$

$$= C_{\gamma,\delta}I_\delta\sigma_{\gamma,\beta}(g_1)\sigma_\beta(k)I_\beta^{-1}I_\beta\sigma_{\beta,\delta}(g_2)I_\delta^{-1}.$$

Since $\sigma_{\gamma,\beta} \neq 0$ and σ_β is irreducible, it follows that

$$C_{\gamma,\beta}\tau_{\beta,\delta}(g_2) = C_{\gamma,\delta}I_\beta\sigma_{\beta,\delta}(g_2)I_\delta^{-1}.$$

Since $\sigma_{\beta,\delta} \neq 0$ and $C_{\gamma,\delta} \neq 0$, this implies $C_{\gamma,\beta} \neq 0$ and also the identities (4.12) and (4.13). Finally, from (4.13) we conclude that $C_{\beta,\delta} \neq 0$. □

THEOREM 4.9. *Let* σ *and* τ *be irreducible K-multiplicity free representations of* G *and, for each* $\delta \in M(\sigma) \cap M(\tau)$, *let* $0 \neq c_\delta \in \mathbb{C}$. *Then the statements* (a) *and* (b) *of Theorem 4.5 and statement* (c) *below are all equivalent:*

(c) *For some* $\gamma,\delta \in M(\sigma) \cap M(\tau)$

$$\tau_{\gamma,\delta}(g) = C_{\gamma,\delta}I_\gamma\sigma_{\gamma,\delta}(g)I_\delta^{-1},\quad g \in G, \tag{4.6}$$

holds for some complex $C_{\gamma,\delta} \neq 0$.

PROOF. Assume (c). It follows by a twofold application of Lemma 4.8 that $M(\sigma) = M(\tau)$ and, for all $\alpha,\beta \in M(\sigma)$,

$$\tau_{\alpha,\beta}(g) = C_{\alpha,\beta}I_\alpha\sigma_{\alpha,\beta}(g)I_\beta^{-1},\qquad g \in G,$$

with $C_{\alpha,\beta} \neq 0$ and $C_{\alpha_1,\alpha_3} = C_{\alpha_1,\alpha_2}C_{\alpha_2,\alpha_3}$ $(\alpha_1,\alpha_2,\alpha_3 \in M(\sigma))$. Thus (b) holds with $C_{\alpha,\beta} = C_{\alpha,\delta}/C_{\beta,\delta}$. □

Note that, under the conditions of Theorem 4.9, identity (4.6) uniquely determines $C_{\gamma,\delta}$. Hence, if $\sigma \overset{A}{\simeq} \tau$ then A is determined up to a constant factor. Note that (c) of Theorem 4.9 can be replaced by

(c)' *For some* $\delta \in M(\sigma) \cap M(\tau)$ *we have* $\tau_{\delta,\delta}(g) = I_\delta \sigma_{\delta,\delta}(g) I_\delta^{-1}$, $g \in G$.

If $G = KAK$ for some subset A of K then (c) of Theorem 4.9 can be replaced by

(c)" *For some* $\delta \in M(\sigma) \cap M(\tau)$ *we have* $\tau_{\delta,\delta}(a) = I_\delta \sigma_{\delta,\delta}(a) I_\delta^{-1}$, $a \in A$.

4.3. The case SU(1,1).

Consider irreducible subquotient representations of $\pi_{\xi,\lambda}$ as classified in Theorem 3.4. By comparing K-contents it follows that the only possible nontrivial Naimark equivalences are:

$$\pi_{\xi,\lambda} \simeq \pi_{\xi,\mu} (\lambda+\xi, \mu+\xi \notin \mathbb{Z} + \tfrac{1}{2},\ \lambda \neq \mu)$$

and

$$\pi^+_{\xi,\lambda} \simeq \pi^+_{\xi,-\lambda},\ \pi^0_{\xi,\lambda} \simeq \pi^0_{\xi,-\lambda},\ \pi^-_{\xi,\lambda} \simeq \pi^-_{\xi,-\lambda}$$

$$(\lambda+\xi \in \mathbb{Z} + \tfrac{1}{2},\ \lambda \neq 0).$$

Suppose that σ and τ are irreducible subquotient representations of $\pi_{\xi,\mu}$ and that $\phi_m \in H(\sigma) \cap H(\tau)$ for some $m \in \mathbb{Z} + \xi$. It follows from Theorem 4.9 (in particular, condition (c)") that $\sigma \simeq \tau$ if and only if

(4.16) $$\pi_{\xi,\lambda;m,m}(a_t) = \pi_{\xi,\mu;m,n}(a_t), \quad t \in \mathbb{R}.$$

In view of (2.47) and (2.48), formula (4.16) is equivalent to

(4.17) $$\phi^{(0,2m)}_{2i\lambda}(\tfrac{1}{2}t) = \phi^{(0,2m)}_{2i\mu}(\tfrac{1}{2}t), \quad t \in \mathbb{R}.$$

Formula (4.17) holds if $\lambda = \pm\mu$ (cf. (2.44)). Conversely, assume (4.17) and expand both sides of (4.17) as a power series in $-(\operatorname{sh}\tfrac{1}{2}t)^2$ by using (2.41) and (2.38). The coefficients of $-(\mathrm{sh}\frac{1}{2}t)^2$ yield the equality

$$(m+1+\lambda)(m+1-\lambda) = (m+1+\mu)(m+1-\mu).$$

Hence $\lambda = \pm\mu$. We have proved:

THEOREM 4.10. *Let σ and τ ($\sigma \neq \tau$) be irreducible subquotient representations of principal series. Then σ is Naimark equivalent to τ in precisely the following situations (cf. the notation of Theorem 3.4):*

(a) $\pi_{\xi,\lambda} \simeq \pi_{\xi,-\lambda}$ $\quad \lambda+\xi \notin \mathbb{Z}+\frac{1}{2},\ \lambda \neq 0)$

(b) $\pi^{+}_{\xi\,\lambda} \simeq \pi^{+}_{\xi,-\lambda},\ \pi^{0}_{\xi,\lambda} \simeq \pi^{0}_{\xi,-\lambda},\ \pi^{-}_{\xi,\lambda} \simeq \pi^{-}_{\xi,-\lambda}$ $\quad (\lambda+\xi \in \mathbb{Z}+\frac{1}{2},\ \lambda \neq 0).$

REMARK 4.11. It follows from Theorem 3.4 and Theorem 4.10 that each irreducible subquotient representation of some $\pi_{\xi,\lambda}$ is Naimark equivalent to some irreducible subrepresentation of some $\pi_{\xi,\lambda}$.

It follows from Theorem 4.10 and Theorem 4.9 that condition (b) of Theorem 4.5 applies to the equivalences of Theorem 4.10. This means that for each $\xi \in \{0,\frac{1}{2}\}$ and $\lambda \in \mathbb{C}\backslash\{0\}$ we have identities

$$\pi_{\xi,-\lambda;m,n}(a_t) = C_{\xi;\lambda;m,n}\pi_{\xi,\lambda;m,n}(a_t), \qquad t \in \mathbb{R}, \tag{4.18}$$

for certain nonzero complex constants $C_{\xi,\lambda;m,n}$, where $m,n \in \mathbb{Z}+\xi$ and, if $\lambda+\xi \in \mathbb{Z}+\frac{1}{2}$, we have the further restriction that $m,n \in (-\infty,-|\lambda|-\frac{1}{2}]$ or $m,n \in [-|\lambda|+\frac{1}{2},|\lambda|-\frac{1}{2}]$ or $m,n \in [|\lambda|+\frac{1}{2},\infty)$. Indeed, it follows from (2.47) and (2.44) that (4.18) holds with

$$C_{\xi,\lambda;m,n} = \frac{c_{\xi,-\lambda;m,n}}{c_{\xi,\lambda;m,n}}. \tag{4.19}$$

A calculation using (4.19) and (2.48) shows that

$$C_{\xi,\lambda;m,n} = c_{\xi,\lambda;m}/c_{\xi,\lambda;n} \tag{4.20}$$

with

$$(4.21)\qquad c_{\xi,\lambda;m} = \text{const.}\ \frac{\Gamma(-\lambda+m+\frac{1}{2})}{\Gamma(\lambda+m+\frac{1}{2})} = \text{const.}\ \frac{\Gamma(-\lambda-m+\frac{1}{2})}{\Gamma(\lambda-m+\frac{1}{2})} = \text{const.}\ (-1)^{m-\xi}\Gamma(-\lambda+m+\tfrac{1}{2})\Gamma(-\lambda-m+\tfrac{1}{2}) = \text{const.}\ \frac{(-1)^{m-\xi}}{\Gamma(\lambda+m+\frac{1}{2})\Gamma(\lambda-m+\frac{1}{2})}\ .$$

If $\lambda + \xi \notin \mathbb{Z} + \frac{1}{2}$ then we can use all alternatives for $c_{\xi,\lambda;m}$, but if $\lambda + \xi \in \mathbb{Z} + \frac{1}{2}$ then we can use precisely one alternative. Now, by Theorem 4.5, we obtain:

THEOREM 4.12. *Let* $\sigma \overset{A}{\simeq} \tau$ *be one of the equivalences of Theorem 4.10, with* σ *being a subquotient representation of* $\pi_{\xi,\lambda}$. *Then*

$$(4.22)\qquad A\phi_m = c_{\xi,\lambda;m}\ \phi_m,$$

where $m \in \mathbb{Z} + \xi$ *such that* $\delta_m \in M(\sigma)$ *and* $c_{\xi,\lambda;m}$ *is given by (4.21).*

4.4. Notes.

4.4.1. Ddfinition 4.1 of Naimark relatedness goes back to NAIMARK [46]. He introduced this concept in the context of representations of the Lorentz group on a reflexive Banach space. Next he gave a much more involved definition in his book [47, Ch.3, §9, no.3]. Afterwards, many different versions of this definition appeared in literature, which all refer to [47]. We mention ZELOBENKO & NAIMARK [67, Def.2] ("weak equivalence" for representations on locally convex spaces), FELL [16, §6] (Naimark relatedness for "linear system representations") and WARNER [63, p.232 and p.242]. Warner starts with the definition of Naimark relatedness for Banach representations of an associative algebra over $\mathbb{C}$ (this definition is similar to our Definition 4.1) and next he defines Naimark relatedness for Banach representations of a lcsc. group G in terms of Naimark relatedness for the corresponding representations of $M_c(G)$ or (equivalently) $C_c(G)$. Warner's definition seems to be standard now. POULSEN [48, Def. 3.3] gives Naimark's original definition [46] and he calls it weak equivalence. FELL [16] (see also WARNER [63, Theorem 4.5.5.2]) proved that, for K-finite Banach representations of a connected unimodular Lie group, two representations are Naimark related iff they are infinitesimally equivalent.

4.4.2. In the case of a connected semisimple Lie group G with finite center our implication

(c)' of Theorem 4.9 ⇒ (a) of Theorem 4.5

is contained in WALLACH [60, Cor.2.1]. He also shows [60, Theorem 2.1] that two irreducible K-finite representations of G for which the characters are the same must be infinitesimally equivalent. By a theorem of Harish-Chandra (cf. [60, Theorem 3.1]) the characters of two principal series representations $\pi_{\xi,\lambda}$ and $\pi_{\xi',\lambda'}$ (cf. (2.10)) are the same iff $\xi^s = \xi'$, $\lambda' = s.\lambda$ for some $s \in W$. Thus, if $\pi_{\xi,\lambda}$ is irreducible and if $s \in W$ then $\pi_{\xi,\lambda}$ is irreducible and if $s \in W$ then $\pi_{\xi^s,s.\lambda}$ is Naimark equivalent to $\pi_{\xi,\lambda}$. For $\lambda \in i\mathfrak{a}^*$ this was already proved by BRUHAT [6, Theorême 7.2]. Wallach's results cover our Theorem 4.10(a). Note that Theorem 4.10(b) (equivalence in the case of non-irreducible $\pi_{\xi,\lambda}$) can be obtained from (a) by a limit argument. Indeed, by continuity in λ, $\pi_{\xi,\lambda;m,n}(g) = \pi_{\xi,-\lambda;m,n}(g)$ for all (ξ,λ) and m. Now use Theorem 3.3. Theorem (4.10(b) is also contained in LEPOWSKY'S [70, Theorem 9.8] result for general semisimple G with finite center that $\pi_{\xi,\lambda}$ and $\pi_{\xi^s,s.\lambda}$ have equivalent composition series for $s \in W$.

4.4.3. Theorem 4.10 was first proved in the unitarizable cases by BARGMANN [1]. He used infinitesimal methods. TAKAHASHI [55] proved Theorem 4.10 (again in the unitarizable cases) by calculating the diagonal matrix elements $\pi_{\xi,\lambda;m,n}(a_t)$ and by observing that they are even in λ. GELFAND, GRAEV & VILENKIN [23, Ch.VII, §4] obtained Theorem 4.10 by working in the noncompact realization of the principal series and by explicitly constructing all possible intertwining operators.

5. EQUIVALENCE OF IRREDUCIBLE REPRESENTATIONS OF SU(1,1) TO SUBREPRESENTATIONS OF THE PRINCIPAL SERIES

The first five subsections of this section contain generalities about Gelfand pairs and spherical functions. By using the concepts developed there we can next, in §5.6, translate the problem of classifying the irreducible representations of SU(1,1) in such a way that the problem can be solved by global methods. This is done in §5.8, with the generalized Abel transform (§5.7) and the Gegenbauer transform pair of Deans (§5.9) being the main tools. The problem is finally reduced to finding the continuous characters on the convolution algebra $\mathcal{D}_{\mathrm{even}}(\mathbb{R})$ (Proposition 5.16). In the earlier subsections Theorems 5.7 and 5.8 may be particularly noteworthy. They give new global proofs that, for instance, the restrictions of the irreducible representations of SO(n+1) to SO(n) are multiplicity free.

5.1. The connection between representations of G and spherical representations of G × K.

Let G be a lcsc. group with compact subgroup K. Let $K^* := \{(k,k) \in G \times K \mid k \in K\}$. Then K^* is a compact subgroup of $G \times K$, isomorphic to K. If $\delta \in \hat{K}$ then let $\check{\delta}$ denote the representation of K which is contragredient to δ. If $\delta \in \hat{K}$ and π is a K-unitary representation of G then $\pi \otimes \check{\delta}$ is a K^*-unitary representation of $G \times K$ on $\mathcal{H}(\pi) \otimes \mathcal{H}(\check{\delta})$.

LEMMA 5.1. *Let $\delta \in \hat{K}$ and let π be a K-unitary representation of* G. *The multiplicity of δ in $\pi|_K$ is equal to the multiplicity of the representation* 1 *of K^* in $\pi \otimes \check{\delta}|_{K^*}$. π is irreducible iff $\pi \otimes \check{\delta}$ is irreducible. π is unitary iff $\pi \otimes \check{\delta}$ is unitary.*

PROOF. Let $\pi|_K = \sum_{\gamma\in\hat{K}} n_\gamma \gamma$ $(n_\gamma \in \{0,1,2,\ldots,\infty\})$. Then $\pi \otimes \check{\delta}|_{K\times K} = \sum^{\oplus}_{\gamma\in\hat{K}} n_\gamma\, \gamma \otimes \check{\delta}$. The representation 1 of K^* has multiplicity 0 in $\gamma \otimes \check{\delta}$ for $\gamma \neq \delta$ and 1 for $\gamma = \delta$. This proves the first statement. We omit the easy proofs of the other two statements. □

5.2. Gelfand pairs.

Let G be a lcsc. group with compact subgroup K. The space $\mathcal{K}(K\backslash G/K)$ of all continuous K-biinvariant functions on G with compact support becomes an associative algebra with respect to the convolution product

$$(5.1)\qquad (f_1 * f_2)(x) := \int_G f_1(y) f_2(y^{-1}x)\,dy,$$

where dy is a left Haar measure on G.

DEFINITION 5.2. The pair (G,K) is called a *Gelfand pair* if the algebra $\mathcal{K}(K\backslash G(K)$ is commutative.

It can be shown by an easy argument that G is unimodular if (G,K) is a Gelfand pair (cf. BERG [3, p.136]).

THEOREM 5.3. *Let* G *be a lcsc. group with compact subgroup* K. *Suppose that there is a continuous involutive automorphism* α *on* G *such that* $\alpha(KxK) = Kx^{-1}K$ *for all* $x \in G$. *Then* (G,K) *is a Gelfand pair.*

PROOF. For $f \in \mathcal{K}(K\backslash G/K)$ we have $f(\alpha(x)) = f(x^{-1})$, $x \in G$. Also $d\alpha(x) = dx$, since the automorphism α is involutive. Let $f_1, f_2 \in \mathcal{K}(K\backslash G/K)$. Then

$$(f_1 * f_2)(x) = (f_1 * f_2)(\alpha(x^{-1})) = \int_G f_1(y) f_2(y^{-1}\alpha(x^{-1}))dy =$$

$$= \int_G f_1((\alpha(y))^{-1}) f_2(x\alpha(y))dy = \int_G f_1(y^{-1}) f_2(xy)dy =$$

$$= \int_G f_1(y^{-1}x) f_2(y)dy = (f_2 * f_1)(x). \qquad \square$$

THEOREM 5.4. *Let* (G,K) *be a Gelfand pair. Let* π *be a* K*-unitary irreducible representation of* G *and let, in addition,* π *be unitary or* K*-finite. Then the representation* 1 *of* K *has multiplicity* 0 *or* 1 *in* $\pi|_K$.

PROOF. Suppose that $H_1(\pi)$ has nonzero dimension. The formula

$$f^{\#}(x) := \int_{K_1}\int_{K_2} f(k_1 x k_2)\,dk_1\,dk_2 \tag{5.2}$$

defines a projection from $\mathcal{K}(G)$ (the space of continuous functions on G with compact support) onto $\mathcal{K}(K\backslash G/K)$. Let P be the orthogonal projection from $H(\pi)$ onto $H_1(\pi)$:

$$Pv := \int_K \pi(k)v\,dk, \qquad v \in H(\pi). \tag{5.3}$$

For $f \in \mathcal{K}(G)$ define

$$\pi(f)v := \int_G f(x)\pi(x)v\,dx, \qquad v \in H(\pi). \tag{5.4}$$

Then π is a homomorphism from the algebra $K(G)$ into the algebra $L(H(\pi))$. Since π is an irreducible representation of G, the family of operators $\pi(K(G))$ also acts irreducibly on $H(\pi)$. It follows from (5.2) and (5.3) that

$$\pi(f^{\#})v = P\pi(f)Pv, \qquad f \in K(G),\ v \in H(\pi).$$

If $v \in H_1(\pi)$ then

$$\pi(f^{\#})v = P\pi(f)v, \qquad f \in K(G).$$

So $\pi(K(K\backslash G/K))v = P\pi(K(G))v$, $v \in H_1(\pi)$. Let $v \in H_1(\pi)$, $v \neq 0$. By the irreducibility of $\pi(K(G))$, $\pi(K(G))v$ is dense in $H(\pi)$, so $P\pi(K(G))v$ is dense in $H_1(\pi)$. We conclude that the algebra $\pi(K(K\backslash G/K))$ acts irreducibly on $H_1(\pi)$. Now $\pi(K(K\backslash G/K))$ is a commutative algebra. Therefore, if $\dim H_1(\pi) < \infty$, the finite-dimensional version of Schur's lemma yields that $\pi(f)|_{H_1(\pi)}$ is a multiple of the identity for each $f \in K(K\backslash G/K)$. Then $\dim H_1(\pi) = 1$. On the other hand, if π is unitary and $\dim H_1(\pi) = \infty$ is admitted, then $\pi(K(K\backslash G/K))$ is a commutative $*$-algebra and the generalization of Schur's lemma again yields the same conclusion. □

5.3. Gelfand pairs of the form $(G \times K, K^*)$.

Let G be a lcsc. group with compact subgroup K. Use the notation of §5.1. We conclude from Lemma 5.1 and Theorems 5.3, 5.4:

COROLLARY 5.5. *Suppose that there exists a continuous involutive automorphism* α *on* G *such that for each* $(g,k) \in G \times K$ *there exist* $k_1, k_2 \in K$ *with the property that* $\alpha(g) = k_1 g^{-1} k_2$, $\alpha(k) = k_1 k^{-1} k_2$.
Then:
(a) $(G \times K, K^*)$ *is a Gelfand-pair.*
(b) *If* π *is an irreducible* K*-unitary representation of* G *which is* K*-finite or unitary then* π *is* K*-multiplicity free.*

THEOREM 5.6. *The conclusions of Corollary* 5.5 *apply to the case* $G = SU(1,1)$.

PROOF. For $g = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \in SU(1,1)$ define

$$\alpha(g) := (g^{-1})^t = \begin{pmatrix} \bar{a} & -\bar{b} \\ -b & a \end{pmatrix}.$$

Then α is a continuous involutive automorphism on G and $\alpha(a_t) = a_{-t}$ on A, $\alpha(u_\Theta) = u_{-\Theta}$ on K. By using (2.1) we conclude that α satisfies the property required in Corollary 5.5. □

Let $SO_0(1,n)$ be the group of all real $(n+1) \times (n+1)$ matrices g of determinant 1 such that the first diagonal matrix entry is positive and $g^tJg = J$, where $J = \mathrm{diag}(-1,1,1,\ldots,1)$.

THEOREM 5.7. *The conclusions of Corollary 5.5 apply to the cases where* $G = SO_0(1,n)$ *or* $SO(n+1)$ *and* $K = SO(n)$, $n = 1,2,\ldots$.

PROOF. For $m = 3,4,\ldots,n+1$ let B_m consist of all matrices

$$\left(\begin{array}{c|cc|c} I_{n-m+1} & 0 & & 0 \\ \hline 0 & \cos\theta & -\sin\theta & 0 \\ & \sin\theta & \cos\theta & \\ \hline 0 & 0 & & I_{m-2} \end{array}\right).$$

Let $A := B_{n+1}$ if $G = SO(n+1)$ and let A consist of all matrices

$$\left(\begin{array}{cc|c} \mathrm{ch}\ t & \mathrm{sh}\ t & 0 \\ \mathrm{sh}\ t & \mathrm{ch}\ t & \\ \hline & 0 & I_{n-1} \end{array}\right)$$

if $G = SO_0(1,n)$. For $m = 2,3,\ldots,n$ let

$$K_m := \begin{pmatrix} I_{n-m+1} & 0 \\ 0 & SO(m) \end{pmatrix} .$$

Then $K = K_n$. We have the decompositions $G = KAK$ and $K_m = K_{m-1}B_mK_{m-1}$. Note that B_m commutes with $B_{m-2}, B_{m-3}, \ldots$ and with K_{m-2}. In the following,

g, k, a, b_i, k_i will denote some element of G, K, A, B_i, K_i, respectively, and $(g,k) \sim (g',k')$ will mean that $(g,k) \in K^*(g',k')K^*$.

Let $(g,k) \in G \times K$. We will prove that

$$(5.5) \qquad \begin{cases} (g,k) \sim (b_3 b_5 \ldots b_{n-1} a, k_2 b_4 \ldots b_n), & n \text{ even}, \\ (g,k) \sim (k_2 b_4 \ldots b_{n-1} a, b_3 b_5 \ldots b_n), & n \text{ odd}, \end{cases} \quad \text{and}$$

for certain b_i's, a and k_2. More generally hold:

$$(5.6) \qquad (g,k) \sim (k_{n-2i+1} b_{n-2i+3} b_{n-2i+5} \ldots b_{n-1} a, b_{n-2i+2} b_{n-2i+4} \ldots b_n),$$

$$i = 1, \ldots, [\tfrac{1}{2}(n-1)],$$

and

$$(5.7) \qquad (g,k) \sim (b_{n-2i+1} b_{n-2i+3} \ldots b_{n-1} a, k_{n-2i} b_{n-2i+2} b_{n-2i+4} \ldots b_{n-2} b_n),$$

$$i = 0, \ldots, [\tfrac{1}{2}(n-3)].$$

We will prove (5.6) and (5.7) by complete induction with respect to i. Indeed,

$$(g,k) \sim (k'ak'',k) \sim (a,k'''),$$

so (5.7) is true for i = 0. Next, (5.6) implies (5.7) and (5.7) implies (5.6) with i replaced by i+1. For instance, to prove (5.6) ⇒ (5.7) substitute into (5.6)

$$k_{n-2i+1} = k_{n-2i} b_{n-2i} k'_{n-2i}.$$

Then

$$(g,k) \sim (k_{n-2i} b_{n-2i+1} \ldots b_{n-1} a k'_{n-2i}, b_{n-2i+2} \ldots b_n) \sim$$

$$\sim (b_{n-2i+1} \ldots b_{n-1} a, k''_{n-2i} b_{n-2i+2} \ldots b_n k'''_{n-2i}) \sim$$

$$\sim (b_{n-2i+1} \cdots b_{n-1} a, k''''_{n-2i} b_{n-2i+2} \cdots b_n).$$

Now let $J_1 := \mathrm{diag}(1,-1,1,-1,\ldots,1,-1,(1))$ and put $\alpha(g) := J_1 g J_1$, $g \in G$. Then $\alpha(K) = K$, $\alpha(a) = a^{-1}$, $\alpha(b_m) = b_m^{-1}$ and $\alpha(k_2) = k_2^{-1}$. Thus, in view of (5.5), α satisfies the property required in Corollary 5.5. □

Let $U(1,n)$ be the group of all complex $(n+1) \times (n+1)$ matrices g with $g^* J g = J$.

<u>THEOREM 5.8</u>. *The conclusions of Corollary 5.5 apply to the cases:*

G	SU(1,n)	SU(n+1)	U(1,n)	U(n+1)
K	S(U(1)×U(n))	S(U(1)×U(n))	U(n)	U(n)

where $n = 1,2,\ldots$.

<u>PROOF</u>. This is analogous to the proof of Theorem 5.7 with the following modifications. B_m $(m = 2,\ldots,n+1)$ consists of all matrices

$$\begin{pmatrix} I_{n-m+1} & 0 & 0 & 0 \\ 0 & \cos\theta\, e^{i\phi} & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta\, e^{-i\phi} & 0 \\ 0 & 0 & 0 & I_{m-2} \end{pmatrix}.$$

A is as in Theorem 5.7 in the first two cases, $A := B_{n+1}$ in the fourth case and A consists of all matrices

$$\begin{pmatrix} \mathrm{ch}\, t\, e^{i\phi} & \mathrm{sh}\, t & 0 \\ \mathrm{sh}\, t & \mathrm{ch}\, t\, e^{i\phi} & \\ 0 & & I_n \end{pmatrix}$$

in the third case. In the first two cases let K_m $(m=1,\ldots,n)$ consist of all matrices of determinant 1 of the form

$$\begin{pmatrix} e^{i\psi} I_{n-m+1} & 0 \\ 0 & T \end{pmatrix}, \qquad T \in U(m),$$

and in the last two cases let

$$K_m := \begin{pmatrix} I_{n-m+1} & 0 \\ 0 & U(m) \end{pmatrix}.$$

Again we have the decompositions $G = KAK$ and $K_m = K_{m-1} B_m K_{m-1}$. Instead of (5.5) it can now be proved that

$$(g,k) \sim (k_1 b_3 b_5 \ldots b_{n-1} a, b_2 b_4 \ldots b_n), \text{ n even,}$$

$$(g,k) \sim (b_2 b_4 \ldots b_{n-1} a, k_1 b_3 b_5 \ldots b_n), \text{ n odd.}$$

Define the involutive automorphism α by $\alpha(g) := J_1 \bar{g} J_1$, where $\bar{g}$ is the complex conjugate of the matrix g. □

5.4. Spherical functions.

Let (G,K) be a Gelfand pair. A function $\phi \in C(G)$ is called a *spherical function* on G (with respect to K) if $\phi \neq 0$ and

(5.8) $$\phi(x)\phi(y) = \int_K \phi(xky)\,dk, \qquad x,y \in G.$$

It follows from this definition that ϕ is biinvariant with respect to K and that

(5.9) $$\phi(e) = 1.$$

If ϕ is a spherical function then the linear functional α on $\mathcal{K}(K\backslash G/K)$ defined by

(5.10) $$\alpha(f) := \int_G f(x)\phi(x^{-1})\,dx$$

(dx Haar measure on G; the same as in (5.1)) is an algebra homomorphism from

$K(K\backslash G/K)$ onto $\mathbb{C}$. This is easily seen from (5.8). If, in addition, G is a Lie group, let $\mathcal{D}(K\backslash G/K)$ be the space of K-biinvariant C^∞-functions with compact support on G. Then $\mathcal{D}(K\backslash G/K)$ is a subalgebra of $K(K\backslash G/K)$ under convolution and α, defined by (5.10), is an homomorphism from $\mathcal{D}(K\backslash G/K)$ onto $\mathbb{C}$ as well. Actually, the converse implications also hold. Provide $K(G)$ and $\mathcal{D}(G)$ with the usual inductive limit topologies. $K(K\backslash G/K)$ and $\mathcal{D}(K\backslash G/K)$, being closed subspaces of $K(G)$ respectively $\mathcal{D}(G)$, inherit these topologies. The dual spaces of $K(K\backslash G/K)$ and $\mathcal{D}(K\backslash G/K)$ can be identified with the spaces of K-biinvariant Radon measures and distributions, respectively.

THEOREM 5.9. *Let* (G,K) *be a Gelfand pair. There is a one-to-one correspondence (5.10) between the spherical functions* ϕ *and the nonzero continuous homomorphisms* $\alpha: K(K\backslash G/K) \to \mathbb{C}$. *If, in addition,* G *is a Lie group then (5.10) also establishes a one-to-one correspondence between the spherical functions* ϕ *and the nonzero continuous homomorphisms* $\mathcal{D}(K\backslash G/K) \to \mathbb{C}$. *In the Lie group case all spherical functions are analytic.*

A proof of this theorem is sketched by GODEMENT [26], see also HELGASON [33, Ch. X, §3 and 4].

Let again (G,K) be a Gelfand pair. Let π be a K-unitary representation of G such that $\dim H_1(\pi) = 1$. Choose $v \in H_1(\pi)$ such that $\|v\| = 1$ and define

(5.11) $$\phi(x) := (\pi(x)v,v), \quad x \in G.$$

PROPOSITION 5.10. *The function* ϕ *given by (5.11) is a spherical function.*

PROOF. ϕ is continuous and $\phi(e) = 1$. We will show that ϕ satisfies (5.8):

$$\int_K \phi(xky)\,dk = \int_K (\pi(xky)v,v)\,dk = \left(\int_K \pi(ky)v\,dk,\ \pi(x)^*v\right) =$$

$$= (P_{\pi,1}\pi(y)v,\ \pi(x)^*v) = (c(y)v,\ \pi(x)^*v) =$$

$$= c(y)(\pi(x)v,\ v) = c(y)\phi(x)$$

for some constant c(y) depending on y. Substitution of x = e in the identity

$$\int_K \phi(xky)\,dk = c(y)\phi(x)$$

yields $\phi(y) = c(y)$. □

5.5. Spherical functions of type δ.

Let G be a lcsc. group with compact subgroup K. Then the mapping $F \to F|_{G\times\{e\}}$ is a topological and algebraic isomorphism from the algebra $\mathcal{K}(K^*\backslash G \times K/K^*)$ onto the algebra

$$(5.12)\qquad I_c(G) := \{f \in \mathcal{K}(G) \mid f(kgk^{-1}) = f(g), \quad g \in G,\ k \in K\},$$

and, if G is a Lie group, also from $\mathcal{D}(K^*\backslash G \times K/K^*)$ onto

$$(5.13)\qquad I_c^\infty(G) := \{f \in \mathcal{D}(G) \mid f(kgk^{-1}) = f(g), \quad g \in G,\ k \in K\}.$$

Thus $(G \times K, K^*)$ is a Gelfand pair if and only if the algebra $I_c(G)$ is commutative.

Suppose that $(G \times K, K^*)$ is a Gelfand pair. Then the mapping $\Phi \to \Phi|_{G\times\{e\}}$ is a bijection from the class of spherical functions Φ on $G \times K$ onto the class of functions $\phi \in C(G)$ such that $\phi \neq 0$ and

$$(5.14)\qquad \phi(x)\phi(y) = \int_K \phi(xkyk^{-1})\,dk, \quad x,y \in G.$$

For such functions ϕ we have $\phi(e) = 1$ and $\phi(kgk^{-1}) = \phi(g)$, $g \in G$, $k \in K$. The mapping $\phi \to \alpha$ (cf. (5.10)) identifies the class of nonzero functions $\phi \in C(G)$ satisfying (5.14) with the class of nonzero continuous homomorphisms from $I_c(G)$ (or $I_c^\infty(G)$) to $\mathbb{C}$.

LEMMA 5.11. *Let $\phi \in C(G)$ satisfy (5.14), $\phi \neq 0$. Then there is a unique $\delta \in \hat{K}$ such that*

$$(5.15)\qquad d_\delta \int_K \phi(xk)\chi_\delta(k^{-1})\,dk = \phi(x), \quad x \in G.$$

PROOF. Let $\gamma, \delta \in \hat{K}$, $\gamma \neq \delta$. Then

$$d_\gamma \int_K \phi(x\ell)\chi_\gamma(\ell^{-1})d\ell . d_\delta \int_K \phi(ym)\chi_\delta(m^{-1})dm =$$

$$= d_\gamma d_\delta \int_K\int_K\int_K \phi(x\ell k y m k^{-1})\chi_\gamma(\ell^{-1})\chi_\delta(m^{-1})dk\,d\ell\,m =$$

$$= d_\gamma d_\delta \int_K\int_K \phi(x\ell y m)\left(\int_K \chi_\gamma(k\ell^{-1})\chi_\delta(k^{-1}m^{-1})dk\right)d\ell dm = 0.$$

Hence the left hand side of (5.15) is identically zero in x for all but one $\delta \in \hat{K}$. □

If a nonzero $\phi \in C(G)$ satisfies (5.14) and (5.15) then it is called a *spherical function of type* δ on G (with respect to K). Note that the spherical functions of type 1 are precisely the ordinary spherical functions.

For $\delta \in \hat{K}$ let $I_{c,\delta}(G)$ (or $I^\infty_{c,\delta}$ (G) if G is a Lie group) be the space consisting of all $f \in I_c(G)$ (or I^∞_c (G)) such that

$$(5.16) \qquad d_\delta \int_K f(xk)\chi_\delta(k^{-1})dk = f(x), \qquad x \in G.$$

$I^{(\infty)}_{c,\delta}(G)$ is a closed subalgebra of $I^{(\infty)}_c(G)$ (being commutative, since $I^{(\infty)}_c(G)$ is commutative) and for distinct $\gamma,\delta \in \hat{K}$ we have $f_1 * f_2 = 0$ if $f_1 \in I^{(\infty)}_{c,\gamma}(G)$, $f_2 \in I^{(\infty)}_{c,\delta}(G)$. It follows from the preceding results and from Theorem 5.9 that:

<u>THEOREM 5.12</u>. *Let* $(G \times K, K^*)$ *be a Gelfand pair. Let* $\delta \in \hat{K}$. *There is a one-to-one correspondence (5.10) between the spherical functions* ϕ *of type* δ *and the nonzero continuous homomorphisms* $\alpha\colon I_{c,\delta}(G) \to \mathbb{C}$ *(also* $\alpha\colon I^\infty_{c,\delta}(G) \to \mathbb{C}$ *if* G *is a Lie group).*

Let π be a K-multiplicity free representation of G and let $\delta \in M(\pi)$. Then, by Lemma 5.1, the representation 1 of K^* has multiplicity 1 in the representation $\pi \otimes \check{\delta}$ of $G \times K$. Let Φ be the spherical function associated with $\pi \otimes \check{\delta}$ according to (5.11) and let $\phi := \Phi|_{G\times\{e\}}$. Then

$$\phi(g) = d_\delta^{-1} \operatorname{tr} \pi_{\delta,\delta}(g).$$

Indeed, let $e_1,\dots,e_{d_\delta}$ be an orthonormal basis for $H_\delta(\pi)$ and let $f_1,\dots,f_{d_\delta}$ be a dual basis for $H(\check{\delta})$. Then $d_\delta^{-\frac{1}{2}} \sum_{i=1}^{d_\delta} e_i \otimes f_i$ is a normalized K^*-invariant vector in $H(\pi) \otimes H(\check{\delta})$ and

$$\phi(g) = d_\delta^{-1}\Big(\sum_{i=1}^{d_\delta} \pi(g)e_i \otimes f_i,\ \sum_{j=1}^{d_\delta} e_j \otimes f_j\Big) =$$

$$= d_\delta^{-1} \sum_{i,j=1}^{d_\delta} (\pi(g)e_i,e_j)(f_i,f_j) =$$

$$= d_\delta^{-1} \sum_{i=1}^{d_\delta} (\pi(g)e_i,e_i) = d_\delta^{-1} \mathrm{tr}\ \pi_{\delta,\delta}(g).$$

Clearly, ϕ is a spherical function of type δ.

5.6. Formulation of the main theorem.

It is the purpose of this section to prove:

THEOREM 5.13. *Let τ be an irreducible K-unitary representation of* SU(1,1) *which is K-finite or unitary. Then τ is Naimark equivalent to an irreducible subrepresentation of some principal series representation $\pi_{\xi,\lambda}$.*

By Theorem 5.6 τ is K-multiplicity free. If $\delta_n \in M(\tau)$ then write $\tau_{n,n}$ for the corresponding diagonal matrix element τ_{δ_n,δ_n}. In view of Remark 4.11 and Theorem 4.9 it is sufficient for the proof of Theorem 5.13 to show that for some $\delta_n \in M(\tau)$, for some $\lambda \in \mathbb{C}$ and for $\xi \in \{0,\frac{1}{2}\}$ with $n \in \mathbb{Z} + \xi$ we have

$$(5.17)\qquad \tau_{n,n} = \pi_{\xi,\lambda;n,n}.$$

Both sides of (5.17) are spherical functions of type δ_n. Write $I_{c,n}^\infty(G)$ for $I_{c,\delta_n}^\infty(\mathrm{SU}(1,1))$. Then (5.17) holds if the corresponding characters (cf. (5.10)) on $I_{c,n}^\infty(G)$ are equal. Hence Theorem 5.13 will follow from:

PROPOSITION 5.14. *Let $G = \mathrm{SU}(1,1)$, $n \in \frac{1}{2}\mathbb{Z}$. Let α be a nonzero continuous homomorphism from $I_{c,n}^\infty(G)$ to $\mathbb{C}$. Then*

$$(5.18) \qquad \alpha(f) = \int_G f(x)\, \pi_{\xi,\lambda;n,n}(g^{-1})\,dg, \qquad f \in I^{\infty}_{c,n}(g),$$

for some $\lambda \in \mathbb{C}$ *and for* $\xi \in \{0,\frac{1}{2}\}$ *such that* $n \in \mathbb{Z} + \xi$.

Let us now summarize the ingredients for the proof of Proposition 5.14. By (5.13) and (5.16) the space $I^{\infty}_{c,n}(G)$ $(G = SU(1,1))$ consists of all $f \in \mathcal{D}(G)$ such that $f(kgk^{-1}) = f(g)$ $(g \in G,\ k \in K)$ and $f(gu_\Theta) = e^{in\Theta}f(g)$ $(g \in G,\ u_\Theta \in K)$. For $f \in I^{\infty}_{c,n}(G)$ define

$$(5.19) \qquad F^n_f(t) = e^{\frac{1}{2}t} \int_{-\infty}^{\infty} f(a_t n_z)\,dz.$$

Then F^n_f is in the space $\mathcal{D}_{\text{even}}(\mathbb{R})$ of even C^{∞}-functions on $\mathbb{R}$ with compact support. Both $I^{\infty}_{c,n}(G)$ and $\mathcal{D}_{\text{even}}(\mathbb{R})$ are convolution algebras and topological vector spaces. We will prove:

THEOREM 5.15. *Let* $n \in \frac{1}{2}\mathbb{Z}$. *The mapping* $f \to F^n_f$ *is an homeomorphic and isomorphic mapping from* $I^{\infty}_{c,n}(G)$ *onto* $\mathcal{D}_{\text{even}}(\mathbb{R})$. *Furthermore, if* $\xi = 0$ *or* $\frac{1}{2}$ *such that* $n \in \mathbb{Z} + \xi$ *then*

$$(5.20) \qquad \int_{-\infty}^{\infty} F^n_f(t)\,e^{-\lambda t}\,dt = \int_G f(g)\,\pi_{\xi,\lambda;n,n}(g^{-1})\,dg, \qquad f \in I^{\infty}_{c,n}(G),$$

where $dg = (2\pi)^{-1}e^{t}\,d\Theta\,dt\,dz$ *if* $g = u_\Theta a_t n_z$.

We will also prove:

PROPOSITION 5.16. *Let* α *be a nonzero continuous homomorphism from* $\mathcal{D}_{\text{even}}(\mathbb{R})$ *to* $\mathbb{C}$. *Then, for some* $\lambda \in \mathbb{C}$,

$$(5.21) \qquad \alpha(f) = \int_{-\infty}^{\infty} f(t)\,e^{-\lambda t}\,dt, \qquad f \in \mathcal{D}_{\text{even}}(\mathbb{R}).$$

Now Proposition 5.14 (and hence Theorem 5.13) follows from Theorem 5.15 and Proposition 5.16.

5.7. The generalized Abel transform.

In this subsection we discuss the analogue of (5.19) for a general noncompact connected semisimple Lie group G with finite center. Let

G = KAN be an Iwasawa decomposition. For given Haar measures dk, da, dn on K,A,N, respectively, normalize the Haar measure on G such that

$$(5.22)\qquad \int_G f(g)\,dg = \int_{K\times A\times N} f(kan)\,e^{2\rho(\log a)}\,dk\,da\,dn, \qquad f \in K(G)$$

(cf. HELGASON [33, Ch.X, Prop. 1.11]). Note the property

$$(5.23)\qquad \int_N f(n)\,dn = e^{2\rho(\log a)} \int_N f(ana^{-1})\,dn, \qquad f \in K(N),\ a \in A$$

(cf. [33, Ch.X, proof of Prop. 1.11]).

For $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ let U^λ be the representation of G induced by the one-dimensional representation $an \to e^{\lambda(\log a)}$ of the subgroup AN:

$$(5.24)\qquad (U^\lambda(g)f)(k) := e^{-(\rho+\lambda)H(g^{-1}k)} f(u(g^{-1}k)), \quad f \in L^2(K),\ g \in G,\ k \in K,$$

where $u(g^{-1}k)$ and $H(g^{-1}k)$ are defined by (2.7). The representation U^λ is easily seen to split as a direct sum of principal series representations $\pi_{\xi,\lambda}$ ($\xi \in \hat{M}$), cf. §2.2. U^λ restricted to K is the left regular representation of K:

$$(5.25)\qquad (U^\lambda(k_1)f)(k) = f(k_1^{-1}k), \qquad f \in L^2(K),\ k,k_1 \in K.$$

For each $\gamma \in \hat{K}$ choose an orthonormal basis $e_1^\gamma,\ldots,e_{d_\gamma}^\gamma$ of $H(\gamma)$ and let

$$(5.26)\qquad \gamma_{ij}(k) := (\gamma(k)e_j^\gamma,\ e_i^\gamma), \qquad k \in K,\quad i,j = 1,\ldots,d_\gamma.$$

Then, by (5.25):

$$(5.27)\qquad U^\lambda(k)\gamma_{ij} = \sum_{\ell=1}^{d_\gamma} \overline{\gamma_{\ell i}(k)}\ \gamma_{\ell j}, \quad k \in K.$$

For the contragredient representation $\check{\gamma}$ we have

$$\check{\gamma}_{ij}(k) = \overline{\gamma_{ij}(k)} = \gamma_{ji}(k^{-1}).$$

For $\delta \in \hat{K}$ and $f \in I_{c,\delta}(G)$ define

$$(5.28) \qquad F_f^\delta(a) := e^{\rho(\log a)} \int_K \int_N f(kan)\, \delta(k^{-1})\, dn\, dk, \quad a \in A.$$

Then F_f^δ belongs to the space $K(A;H(\delta))$ of $H(\delta)$-valued continuous functions on A with compact support. If $f \in I_{c,\delta}^\infty(G)$ then $F_f^\delta \in \mathcal{D}(A;H(\delta))$. For reasons which will become clear in §5.8, the mapping $f \to F_f^\delta$ is called a *generalized Abel transform*.

<u>THEOREM 5.17</u>. *Let* G *be a noncompact connected semisimple Lie group with finite center and let* G *have a faithful finite-dimensional representation. Then, for each* $\delta \in \hat{K}$, *the mapping* $f \to F_f^\delta : I_{c,\delta}(G) \to K(A;H(\delta))$ *has the following properties:*

(a) *It is continuous (also from* $I_{c,\delta}^\infty(G)$ *to* $\mathcal{D}(A;H(\delta))$.

(b) *It is an homomorphism, i.e.,*

$$(5.29) \qquad F_{f_1 * f_2}^\delta(a) = (F_{f_1}^\delta * F_{f_2}^\delta)(a) := \int_A F_{f_1}^\delta(a_1) F_{f_2}^\delta(a_1^{-1}a)\, da,$$

$$f_1, f_2 \in I_{c,\delta}, \quad a \in A.$$

(c) *It satisfies*

$$(5.30) \qquad d_\delta^{-1} \int_A (F_f^\delta(a))_{jq}\, e^{-\lambda(\log a)}\, da =$$

$$= \int_G f(g)\, (U^\lambda(g^{-1})\delta_{ij}, \delta_{iq})\, dg, \quad f \in I_{c,\delta}(G),\ i,j,q = 1,\ldots,d_\delta,\ \lambda \in \mathfrak{a}_{\mathbb{C}}^*,$$

where $(.,.)$ *denotes the inner product on* $L^2(K)$.

(d) *It is injective.*

For the proofs of (a), (b) and (c) we will not use the property that G has a faithful finite-dimensional representation.

The proof of (a) is immediate. (5.29) is also easily proved from (5.28), (5.22) and (5.23) (cf. WARNER [64, pp.34,35]).

It follows from (5.13) and (5.16) that

(5.31) $$\int_G f(g)(U^\lambda(g^{-1})\gamma_{ij},\ \beta_{pq})dg = 0, \qquad f \in I_{c,\delta}(G),$$

if not $\beta = \gamma = \check{\delta}$ and $i = p$, and

(5.32) $$\int_G f(g)(U^\lambda(g^{-1})\check{\delta}_{ij},\ \check{\delta}_{iq})dg = d_\delta^{-1} \sum_{\ell=1}^{d_\delta} \int_G f(g)(U^\lambda(g^{-1})\check{\delta}_{\ell j},\check{\delta}_{\ell q})dg.$$

Now, for the proof of (5.30), substitute (5.24) into the right hand side of (5.32). Then

$$\int_G f(g)(U^\lambda(g^{-1})\check{\delta}_{ij},\check{\delta}_{iq})dg =$$

$$= d_\delta^{-1} \sum_{\ell=1}^{d_\delta} \int_G\int_K f(g)e^{-(\rho+\lambda)H(gk)}\overline{\delta_{\ell j}(u(gk))}\ \delta_{\ell q}(k)dk\ dg =$$

$$= d_\delta^{-1} \int_G\int_K f(g)\delta_{jq}((u(gk))^{-1}k)e^{-(\rho+\lambda)H(gk)}dk\ dg =$$

$$= d_\delta^{-1}\int_G\int_K f(gk^{-1})\delta_{jq}((u(g))^{-1}k)e^{-(\rho+\lambda)H(g)}dk\ dg =$$

$$= d_\delta^{-1} \int_{K\times A\times N}\int_K f(k_1ank^{-1})\delta_{jq}(k_1^{-1}k)e^{(\rho-\lambda)\log a}\, dk\ dk_1\ da\ dn =$$

$$= d_\delta^{-1} \int_{K\times A\times N} f(kan)\delta_{jq}(k^{-1}e^{(\rho-\lambda)\log a}dk\ da\ dn =$$

$$= d_\delta^{-1} \int_A (F_f^\delta(a))_{jq}\ e^{-\lambda(\log a)}da.$$

This settles (5.30).

For the proof of the injectivity suppose that $f \in I_{c,\delta}(G)$ and $F_f^\delta = 0$. Then, in view of (5.30), (5.31) is valid for all $\gamma,\beta \in \hat{K}$,

$i,j = 1,\dots,d_\gamma$, $p,q = 1,\dots,d_\beta$, $\lambda \in \mathbb{C}$. Now use the following lemma:

LEMMA 5.18. *Let* τ *be a finite-dimensional irreducible representation of* G. *Then, for some* $\lambda \in \mathbb{C}$, τ *is a subrepresentation of* U^λ.

The proof of this lemma follows from GODEMENT [25, Lemma 7] by observing that the group AN is solvable and that its one-dimensional representations are precisely the representations $an \to e^{\lambda(\log a)}$, $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$. Thus, by Lemma 5.18, $F^\delta_f = 0$ implies that

$$\int_G f(g)\ \tau(g^{-1})\,dg = 0$$

for all finite-dimensional irreducible representations τ of G. Now, since G has a faithful finite-dimensional representation, the conclusion $f = 0$ follows by the Stone-Weierstrass theorem (cf. GODEMENT [25, Lemma 5]).

5.8. Completion of the proof of the main theorem.

In §5.6 we pointed out that Theorem 5.13 will be implied by Theorem 5.15 and Proposition 5.16. A part of Theorem 5.15 already follows from Theorem 5.17: The mapping $f \to F^n_f$ is a continuous injective homomorphism from $I^\infty_{c,n}(G)$ into $\mathcal{D}(\mathbb{R})$ and (5.20) holds. The property that F^n_f is even follows from HELGASON [33, Ch. X, Theorem 1.15], but it will also be clear from the way we will rewrite (5.19). So, regarding Theorem 5.15, it is left to prove that $f \to F^n_f$ maps $I^\infty_{c,n}(G)$ onto $\mathcal{D}_{even}(\mathbb{R})$ and that the inverse mapping is continuous. In order to establish this we identify both $I^\infty_{c,n}(G)$ and $\mathcal{D}_{even}(\mathbb{R})$, considered as topological vector spaces, with $\mathcal{D}([1,\infty))$ and we rewrite (5.19) as a mapping from $\mathcal{D}([1,\infty))$ into itself. This mapping turns out to be a known integral transformation, for which an inverse transformation can be explicitly given. First note

LEMMA 5.19. *The formula*

$$f(x) = g(x^2) \tag{5.33}$$

defines an homeomorphic linear bijection $f \to g$ *from* $\mathcal{D}_{even}(\mathbb{R})$ *onto* $\mathcal{D}([0,\infty))$.

PROOF. Clearly, if $g \in \mathcal{D}([0,\infty))$ then $f \in \mathcal{D}_{\text{even}}(\mathbb{R})$ and the mapping $g \to f$ is continuous. Conversely, let $f \in \mathcal{D}_{\text{even}}(\mathbb{R})$ and let g be defined by (5.33). By complete induction with respect to n we prove: $g^{(n)}(0)$ exists and there is a function $f_n \in \mathcal{D}_{\text{even}}(\mathbb{R})$ such that

$$f_n(x) = g^{(n)}(x^2), \qquad x \in \mathbb{R},$$

and $f \to f_n : \mathcal{D}_{\text{even}}(\mathbb{R}) \to \mathcal{D}_{\text{even}}(\mathbb{R})$ is continuous. Indeed, suppose this is proved up to n-1. Then

$$2x(g^{(n-1)})'(x^2) = f'_{n-1}(x) = \int_0^x f''_{n-1}(y)\,dy,$$

so

$$g^{(n)}(x^2) = \tfrac{1}{2}\int_0^1 f''_{n-1}(tx)\,dt =: f_n(x). \qquad \square$$

For $f \in I^{\infty}_{c,n}(G)$ define

$$(5.34) \qquad \tilde{f}(x) := f\left(\begin{pmatrix} x & (x^2-1)^{\frac{1}{2}} \\ (x^2-1)^{\frac{1}{2}} & x \end{pmatrix}\right), \qquad x \in [1,\infty).$$

For $h \in \mathcal{D}_{\text{even}}(\mathbb{R})$ define

$$(5.35) \qquad \tilde{h}(\operatorname{ch}\tfrac{1}{2}t) := h(t), \qquad t \in \mathbb{R}.$$

LEMMA 5.20. *The mapping* $f \to \tilde{f}$ *defined by (5.34) is an homeomorphic linear bijection from* $I^{\infty}_{c,n}(G)$ *onto* $\mathcal{D}([1,\infty))$. *The mapping* $h \to \tilde{h}$ *defined by (5.35) is an homeomorphic linear bijection from* $\mathcal{D}_{\text{even}}(\mathbb{R})$ *onto* $\mathcal{D}([1,\infty))$.

PROOF. The second statement follows from Lemma 5.19. For the proof of the first statement introduce global real analytic coordinates on G by the mapping

$$(z,\phi) \to \begin{pmatrix} e^{\frac{1}{2}i\phi}(1+|z|^2)^{\frac{1}{2}} & z \\ \bar{z} & e^{-\frac{1}{2}i\phi}(1+|z|^2)^{\frac{1}{2}} \end{pmatrix}$$

from $\mathbb{C} \times (\mathbb{R}/4\pi\mathbb{Z})$ onto G. If $g \in \mathcal{D}([1,\infty))$ and

$$f\left(\begin{pmatrix} e^{\frac{1}{2}i\phi}(1+|z|^2)^{\frac{1}{2}} & z \\ \bar{z} & e^{-\frac{1}{2}i\phi}(1+|z|^2)^{\frac{1}{2}} \end{pmatrix}\right) := e^{in\phi} g((1+|z|^2)^{\frac{1}{2}})$$

then $f \in I^{\infty}_{c,n}(G)$, $\tilde{f} = g$ and the mapping $g \to f$ if continuous. Conversely, if $f \in I^{\infty}_{c,n}(G)$ then f, as a function of z and ϕ, is radial in z, so the function

$$z \to f\begin{pmatrix} (1+z^2)^{\frac{1}{2}} & z \\ z & (1+z^2)^{\frac{1}{2}} \end{pmatrix}, \qquad z \in \mathbb{R},$$

belongs to $\mathcal{D}_{\text{even}}(\mathbb{R})$. Now make the transformation $z = (x^2-1)^{\frac{1}{2}}$ and apply Lemma 5.19. It follows that $\tilde{f} \in \mathcal{D}([1,\infty))$ and that the mapping $f \to \tilde{f}$ is continuous. □

Define the Chebyshev polynomial $T_n(x)$ by

$$T_n(\cos\theta) := \cos n\theta. \tag{5.36}$$

It follows from (5.19) that, for $f \in I^{\infty}_{c,n}(G)$:

$$F^n_f(t) = e^{\frac{1}{2}t}\int_{-\infty}^{\infty} f\left(\begin{pmatrix} \operatorname{ch}\tfrac{1}{2}t+\tfrac{1}{2}ize^{\frac{1}{2}t} & * \\ * & * \end{pmatrix}\right) dz$$

$$= e^{\frac{1}{2}t}\int_{-\infty}^{\infty} \tilde{f}(|\operatorname{ch}\tfrac{1}{2}t+\tfrac{1}{2}ize^{\frac{1}{2}t}|)\left(\frac{\operatorname{ch}\frac{1}{2}t+\frac{1}{2}iz\, e^{\frac{1}{2}t}}{|\operatorname{ch}\frac{1}{2}t+\frac{1}{2}iz\, e^{\frac{1}{2}t}|}\right)^{2n} dz =$$

$$= e^{\frac{1}{2}t}\int_{0}^{\infty} \tilde{f}(|\operatorname{ch}\tfrac{1}{2}t+\tfrac{1}{2}iz\, e^{\frac{1}{2}t}|)\; T_{2|n|}\left(\frac{\operatorname{ch}\frac{1}{2}t}{|\operatorname{ch}\frac{1}{2}t+\frac{1}{2}iz\, e^{\frac{1}{2}t}|}\right) dz,$$

so

$$F_f^n(t) = 2 \int_{\mathrm{ch}\frac{1}{2}t}^{\infty} \tilde{f}(y) T_{2|n|}(y^{-1}\mathrm{ch}\tfrac{1}{2}t) \quad (y^2 - \mathrm{ch}^2\tfrac{1}{2}t)^{-\frac{1}{2}} y\,dy.$$

This formula again shows that F_f^n is even on $\mathbb{R}$. Thus, by (5.35):

$$(5.37) \qquad \tilde{F}_f^n(x) = 2 \int_x^{\infty} \tilde{f}(y) T_{2|n|}(y^{-1}x)(y^2-x^2)^{-\frac{1}{2}} y\,dy, \qquad x \in [1,\infty).$$

For $n = 0$, (5.37) takes the form

$$\tilde{F}_f^0(x) = 2 \int_x^{\infty} \tilde{f}(y)(y^2-x^2)^{-\frac{1}{2}} y\,dy.$$

The problem of inverting this just means to solve the Abel integral equation. This explains the name "generalized Abel transform" for $f \to F_f^\delta$. We get

$$\tilde{f}(y) = -\pi^{-1} \int_y^{\infty} \frac{d}{dx} \tilde{F}_f^0(x) \quad (x^2-y^2)^{-\frac{1}{2}} dx.$$

For general n, the inversion formula is obtained by DEANS [8,(30)]. He uses the inversion formula for the Radon transform. His result is:

$$(5.38) \qquad \tilde{f}(y) = -\pi^{-1} \int_y^{\infty} \frac{d}{dx} \tilde{F}_f^n(x) T_{2|n|}(y^{-1}x)(x^2-y^2)^{-\frac{1}{2}} dx.$$

In §5.9 we will give another proof of this result. In order to prove the continuity of the mapping $\tilde{F}_f^n \to \tilde{f}\colon \mathcal{D}([1,\infty)) \to \mathcal{D}([1,\infty))$ defined by (5.38), expand $T_{2|n|}(y^{-1}x)$ as a polynomial and use that

$$\left(y^{-1} \frac{d}{dy}\right)^p \int_y^{\infty} h(x)(x^2-y^2)^{-\frac{1}{2}} x\,dx =$$

$$= \int_y^{\infty} \left(x^{-1} \frac{d}{dx}\right)^p h(x) \quad (x^2-y^2)^{-\frac{1}{2}} x\,dx$$

by the properties of the Weyl fractional integral transform (cf. §5.9). This completes the proof of Theorem 5.15.

PROOF OF PROPOSITION 5.16. Extend α to a continuous linear functional on $\mathcal{D}(\mathbb{R})$, for instance by putting $\alpha(f) = 0$ if f is odd. Choose $f_1 \in \mathcal{D}_{even}(\mathbb{R})$ such that $\alpha(f_1) \neq 0$. Let $(\lambda(y)f_1)(x) := f_1(x-y)$, $x,y \in \mathbb{R}$. By the continuity and homomorphism property of α we have, for $f \in \mathcal{D}_{even}(\mathbb{R})$:

$$\alpha(f_1)\alpha(f) = \alpha(f_1 * f) = \int_{-\infty}^{\infty} \alpha(\lambda(y)f_1)f(y)\,dy.$$

Hence

$$\alpha(f) = \int_{-\infty}^{\infty} f(y)\beta(y)\,dy, \qquad f \in \mathcal{D}_{even}(\mathbb{R}),$$

where

$$\beta(y) := \tfrac{1}{2}(\alpha(f_1))^{-1}(\alpha(\lambda(y)f_1) + \alpha(\lambda(-y)f_1)).$$

Then β is even and it is a C^∞-function by the continuity of α. It follows from the homomorphism property of α and from the fact that β is even, that

$$\beta(x)\beta(y) = \tfrac{1}{2}(\beta(x+y) + \beta(x-y)),$$

so $\beta(0) = 1$. On differentiating twice with respect to y and next putting $y = 0$ we obtain

$$\beta(x)\beta''(0) = \beta''(x).$$

Hence $\beta(x) = \mathrm{ch}(\sqrt{\beta''(0)}\,x)$. □

Note that we actually have given a proof of part of Theorem 5.9 in the case that G is the semidirect product of $K := \mathbb{Z}_2$ and $\mathbb{R}$. Unfortunately, we did not succeed in finding a "global" proof of Proposition 5.16.

5.9. The Gegenbauer transform pair of Deans.

Let the *Jacobi polynomial* $R_n^{(\alpha,\beta)}(x)$ be defined by

$$R_n^{(\alpha,\beta)}(x) := {}_2F_1(-n, n+\alpha+ +1; \alpha+1; \tfrac{1}{2}(1-x)). \tag{5.39}$$

Then

$$(5.40)\qquad T_n(x) = R_n^{(-\frac{1}{2},-\frac{1}{2})}(x).$$

If $\alpha = \beta$ then the Jacobi polynomial is called a *Gegenbauer polynomial.* The fact that (5.38) is the inversion formula to (5.37) is contained in the more general result:

<u>THEOREM 5.21</u>. *Let* $m = 0,1,2,\ldots,\nu = -\frac{1}{2},0,\frac{1}{2},1,\ldots$. *For* $g \in \mathcal{D}([1,\infty))$ *define*

$$(5.41)\qquad (J_1 g)(x) := \frac{2\pi^{\nu+1}}{\Gamma(\nu+1)} \int_x^\infty g(y) R_m^{(\nu,\nu)}(y^{-1}x)(y^2-x^2)^\nu y\,dy, \qquad x \in [1,\infty),$$

$$(5.42)\qquad (J_2 g)(x) := \frac{(-1)^{2\nu} x^{-2\nu-1}}{2^{2\nu+1}\pi^{\nu+1}\Gamma(\nu+1)} \cdot$$

$$\int_x^\infty g^{(2\nu+2)}(y) R_m^{(\nu,\nu)}(x^{-1}y)(y^2-x^2)^\nu dy, \qquad x \in [1,\infty).$$

Then J_1 *and* J_2 *are homeomorphic linear bijections from* $\mathcal{D}([1,\infty))$ *onto itself and* $J_1J_2 = J_2J_1 = \mathrm{id}$.

DEANS [7], [8] proved that J_1 and J_2 are inverse to each other by using the inversion formula for the Radon transform. Here we will present an alternative proof. We need some preliminaries.

For Re $\mu > 0$ define the *Weyl fractional integral transform* $\mathcal{W}_\mu$ by

$$(5.43)\qquad (\mathcal{W}_\mu g(x) := (\Gamma(\mu))^{-1} \int_x^\infty g(y)(y-x)^{\mu-1} dy, \quad g \in \mathcal{D}([1,\infty)),\ x \in [1,\infty),$$

cf. [15, Chap. 13]. Then $\mathcal{W}_\mu g \in \mathcal{D}([1,\infty))$,

$$(5.44)\qquad \mathcal{W}_{\mu_1}\mathcal{W}_{\mu_2} = \mathcal{W}_{\mu_1+\mu_2},$$

$$(5.45)\qquad (\mathcal{W}_{\mu+1}g)' = -\mathcal{W}_\mu g = \mathcal{W}_{\mu+1}g'.$$

By (5.45) $(\mathcal{W}_\mu g)(x)$ has an analytic continuation to all complex μ and (5.44) and (5.45) remain valid for all μ. Furthermore, (5.46) $\mathcal{W}_0 = \mathrm{id}.$

It follows that $\mathcal{W}_\mu$ is an homeomorphic linear bijection from $\mathcal{D}([1,\infty))$ onto itself. By the transformation of variables $x \to x^2$ in (5.43) we obtain transformations $\mathcal{V}_\mu$ defined by

$$(5.47) \qquad (\mathcal{V}_\mu g)(x) := 2(\Gamma(\mu))^{-1} \int_x^\infty g(y)(y^2-x^2)^{\mu-1} y\,dy, \quad g \in \mathcal{D}([1,\infty)),$$

$$g \in \mathcal{D}([1,\infty)),\ x \in [1,\infty),$$

which have similar properties as $\mathcal{W}_\mu$.

For $\nu,\mu,\lambda \in \mathbb{C}$, $\mathrm{Re}(\lambda+\mu) > 0$, SPRINKHUIZEN [53, (3.1)] introduced the transformations $I_\nu^{\mu,\lambda}$ defined by

$$(5.48) \qquad (I_\nu^{\mu,\lambda} g)(x) := \frac{1}{2^{\lambda+\mu-1}\Gamma(\lambda+\mu)} \cdot$$

$$\cdot \int_x^\infty g(y)\, {}_2F_1(\lambda+\tfrac{1}{2}(\mu+\nu-1), \tfrac{1}{2}\mu; \lambda+\mu; 1-y^{-2}x^2) \cdot$$

$$\cdot (y^2-x^2)^{\lambda+\mu-1} y^{1-\mu} dy, \quad g \in \mathcal{D}([1,\infty)),\ x \in [1,\infty).$$

Special cases are

$$(5.49) \qquad I_\nu^{0,\lambda} = 2^{-\lambda}\mathcal{V}_\lambda \quad \text{(independent of } \nu),$$

$$(5.50) \qquad I_0^{\mu,0} = \mathcal{W}_\mu,$$

where the latter formula is obtained by substituting [14, 2.8(6)] into (5.48). The transformations satisfy the composition formula

$$(5.51) \qquad I_\nu^{\mu_2,\lambda_2}\, I_{\nu+2\lambda_2}^{\mu_1,\lambda_1} = I_\nu^{\mu_1+\mu_2,\lambda_1+\lambda_2}$$

(cf. [53, 3.4]), which follows from an integral formula for hypergeometric functions due to ERDÉLYI [13], [14, 2.4(3)]. It follows from (5.51) that

$$I_\nu^{\mu,\lambda} = I_\nu^{0,-\frac12\nu}\, I_0^{\mu,\lambda+\frac12\nu} = I_\nu^{0,-\frac12\nu}\, I_0^{\mu,0}\, I_0^{0,\lambda+\frac12\nu}$$

for Re $\lambda > -\frac12$Re $\nu > 0$, Re$(\lambda+\mu+\frac12\nu) > 0$.

Hence

(5.52) $$I_\nu^{\mu,\lambda} = 2^{-\lambda} V_{-\frac12\nu} \mathcal{W}_\mu V_{\lambda+\frac12\nu},$$

and by analytic continuation this formula remains valid for Re$(\lambda+\mu) > 0$ and it gives meaning to $I_\nu^{\mu,\lambda}$ for all complex ν,μ,λ. It also follows that $I_\nu^{\mu,\lambda}$ is an homeomorphic linear bijection from $\mathcal{D}([1,\infty))$ onto itself.

Now we turn to the proof of Theorem 5.21. By (5.39) and [14, 2.11(2)] we have

(5.53) $$R_m^{(\nu,\nu)}(x) = {}_2F_1(-\tfrac12 m, \tfrac12 m+\nu+\tfrac12; \nu+1; 1-x^2),$$

so

(5.54) $$(J_1 g)(x) = (2\pi)^{\nu+1} (I_0^{m+2\nu+1,-m-\nu}(y \mapsto y^{m+2\nu+1} g(y)))(x).$$

By combining (5.53) and (2.39) we have

(5.55) $$R_m^{(\nu,\nu)}(x) = x^m {}_2F_1(-\tfrac12 m, -\tfrac12 m+\tfrac12; \nu+1; 1-x^{-2}),$$

so

(5.56) $$(J_2 g)(x) = (-1)^{2\nu} (2\pi)^{-\nu-1} x^{-m-2\nu-1} (I_{-2m-2\nu}^{1-m,m+\nu}\, g^{(2\nu+2)})(x).$$

Formulas (5.54) and (5.56) show that J_1 and J_2 are homeomorphic linear bijections from $\mathcal{D}([1,\infty))$ onto itself. Now, by (5.51), (5.50),

$$J_1 J_2 g = (-1)^{2\nu}\, I_0^{m+2\nu+1,-m-\nu}\, I_{-2m-2\nu}^{1-m,m+\nu} g^{(2\nu+2)}$$

$$= (-1)^{2\nu}\, I_0^{2\nu+2,0}\, g^{(2\nu+2)} = (-1)^{2\nu} \mathcal{W}_{2\nu+2} g^{(2\nu+2)} = g.$$

This completes the proof of Theorem 5.21.

Note that Theorem 5.21 remains valid for general $\nu > -1$ if in (5.42) we replace $(-1)^{2\nu} g^{(2\nu+2)}$ by $\mathcal{W}_{-2\nu-2}g$, a fractional derivative.

5.10. Notes.

5.10.1. GELFAND [22] first observed that the algebra $\mathcal{K}(K\backslash G/K)$ is commutative for symmetric pairs (G,K). The general theory of Gelfand pairs and spherical functions*) can be found in GODEMENT [26]. Spherical functions of type δ were introduced in GODEMENT [25]. The equivalence between pairs (G,K) with commutative algebra $I_c(G)$ and spherical functions of type δ on the one hand and Gelfand pairs $(G\times K,K^*)$ and ordinary spherical functions for such pairs on the other hand is widely known, but is is probably not in the literature.

5.10.2. The proof that the irreducible representations of SO(n+1) (or SU(n+1)) are SO(n) - (or S(U(1)×U(n))-) multiplicity free (cf. Theorems 5.7,5.8) is usually given by infinitesimal methods, cf. the branching theorems in BOERNER [4, Ch.VII, §12; Ch.V, §6]. Another global proof of Theorem 5.7 is given by DIXMIER [9], by the use of Lemma 5.18, Frobenius reciprocity and complete induction with respect to n. KRÄMER [40] shows that the cases (G,K) with G compact occuring in our Theorems 5.7, 5.8 are, together with the pair (SO(8), Spin(7)), the building blocks for general pairs (G,K) for which the compact connected Lie group G has all its irreducible representations K-multiplicity free, see also HECKMAN [32, §4.3].

5.10.3. Theorem 5.13 was first proved, for unitary representations, by BARGMANN [1]. More generally, there is the subquotient theorem of HARISH-CHANDRA [28, Theorem 4], [68, Theorem 4], see also LEPOWSKY [70, Theorem 1.1]. This theorem states (or rather implies) that each irreducible K-finite Hilbert representation of a noncompact connected semisimple Lie group G with finite center is Naimark equivalent to a subquotient of some principal series representation. CASSELMAN (cf. WALLACH [62, Cor.7.5]) showed that "subquotient" in the above theorem can be replaced by "subrepresentation", at least if G has a faithful finite-dimensional representation.

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*) See also the recent notes by FARAUT [71].

5.10.4. TAKAHASHI [55] also reduces the proof of Theorem 5.13 to a proof of Prop. 5.14. However, he proves Prop. 5.14 by considering eigenfunctions of the Casimir operator, like in the earlier version [37, §5.7] of the present paper. The idea of proving Prop. 5.14 from Theorem 5.15 and Prop. 5.16 is taken from TAKAHASHI [56, §4.1], where it is done in the spherical case, for $G = SO_0(1,n)$.

5.10.5. The generalized Abel transform $f \to F_f^\delta$ is a generalization of HARISH-CHANDRA'S [30, p.595] transform $f \to F_f$ in the spherical case. Later HARISH-CHANDRA [31, §21] proved the injectivity of the transform $f \to F_f$. The transform $f \to F_f^\delta$ was introduced by TAKAHASHI [56, §2] in the case $G = SO_0(n,1)$ and by WARNER [64, §6.2.2] in the general case. However, in Warner's definition of F_f^δ ([64, p.34]) $\mu_{\check{\delta}}$ should be replaced by μ_δ. Warner proves the injectivity of $f \to F_f^\delta$ without the assumption that G has a faithful finite-dimensional representation. However, his proof uses the subquotient theorem. The image of $I_{c,\delta}^\infty(G)$ under the mapping $f \to F_f^\delta$ is generally unknown (cf. WARNER [64, p.36]). As a consequence of the Paley-Wiener theorem for the spherical Fourier transform (cf. GANGOLLI [21]) this image is known if $\delta = 1$: it is the space of all Weyl group invariant functions in $\mathcal{D}(A)$.

5.10.6. Our formula (5.37) for $F_f^n(x)$ also occurs, essentially, in TAKAHASHI [55, (2.8)]. However, TAKAHASHI [55, p.66] mentions that he did not succeed in inverting this transformation, except in the spherical case. The transformation $f \to F_f^n$ occurs for general real n in MATSUSHITA [42, §2.3], in the context of the universal covering group of $SL(2,\mathbb{R})$, and the inversion formula is also derived there*). Further references containing some inversion formula to the Gegenbauer transform (5.41) are TA LI [58] (in the Chebyshev case), HIGGINS [35] and WIMP [66].

5.10.7. Our method of proving Theorem 5.13 is similar to the proof of the analogous theorem in the case $G = SL(2,\mathbb{C})$, as given in NAIMARK [47, Ch.3 §9].

5.10.8. DEANS [8] obtained his Gegenbauer transform from a study of the Radon transform. The Radon transform also occurs naturally in the context of $SL(2,\mathbb{R})$. Indeed, the formula

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*) with a proof due to T. Shintani (unpublished)

$$F_f(e^{\frac{1}{2}i\Theta},t) := e^{\frac{1}{2}t} \int_{-\infty}^{\infty} f(u_\Theta a_t n_z)\,dz$$

defines an isomorphism $f \to F_f$ from $I_c^\infty(G)$ into the space of functions belonging to $\mathcal{D}(\mathbb{T} \times \mathbb{R})$ which are even in the second argument. Furthermore,

$$F_f(e^{\frac{1}{2}i\Theta},t) = e^{in\,\Theta}\, F_f^n(t),\ f \in I_{c,n}^\infty(G),$$

and

$$(4\pi)^{-1} \int_0^{4\pi} \int_{-\infty}^{\infty} F_f(e^{\frac{1}{2}i\Theta},t)\, e^{-in\Theta} e^{-\lambda t} d\Theta\, dt =$$

$$= \int_G f(g)\, \pi_{\xi,\lambda,n,n}(g^{-1})\,dg,\quad f \in I_c^\infty(G).$$

The formula

$$\tilde{f}(w) := f\left(\begin{pmatrix} \frac{w}{(|w|^2-1)^{\frac{1}{2}}} & (|w|^2-1)^{\frac{1}{2}} \\ & \bar{w} \end{pmatrix}\right)$$

defines an homeomorphic linear bijection $f \to \tilde{f}$ from $I_c^\infty(G)$ onto $\mathcal{D}(\{w \in \mathbb{C} \mid |w| \geq 1\})$. It follows that

$$F_f(e^{\frac{1}{2}i\Theta},t) =$$

$$= 2\int_{-\infty}^{\infty} \tilde{f}(e^{\frac{1}{2}i\Theta}(\mathrm{ch}\tfrac{1}{2}t+iy))\,dy =$$

$$= 2(\mathcal{R}\tilde{f})(e^{\frac{1}{2}i\Theta},\mathrm{ch}\tfrac{1}{2}t),$$

where

$$(\mathcal{R}f)(\xi,t) := \int_{x.\xi = t} f(x)\,dS(x)$$

$(f \in \mathcal{D}(\mathbb{R}^n),\ t \in \mathbb{R},\ \xi \in S^{n-1},$

$dS(x)$ is Lebesgue measure on the hyperplane $x.\xi = t$)

defines the *Radon transform* (cf. LUDWIG [41]). In order to characterize the image of $I_c^\infty(G)$ under the mapping $f \to F_f$ one needs the Paley-Wiener theorem for the Fourier transform on $I_c^\infty(G)$, cf. EHRENPREIS & MAUTNER [12] or (in a more general case) FLENSTED-JENSEN [18].

6. UNITARIZABILITY OF IRREDUCIBLE SUBREPRESENTATIONS OF THE PRINCIPAL SERIES

In this section we deal with the last part of our program formulated in the introduction.

6.1. The conjugate contragredient to a representation of G.

Let G be a lcsc. group. In the definition of a Hilbert representation τ of G (cf. §2.1) we assumed that the mapping $g \to \tau(g)$ is strongly continuous, i.e., $g \to \tau(g)v\colon G \to \mathcal{H}(\tau)$ is continuous for all $v \in \mathcal{H}(\tau)$. This implies weak continuity, i.e., $g \to (\tau(g)v,w)\colon G \to \mathbb{C}$ is continuous for all $v,w \in \mathcal{H}(\tau)$. Conversely, we will show that weak continuity of τ implies strong continuity.

Assume that τ is a weakly continuous Hilbert representation of G. A twofold application of the Banach-Steinhaus theorem shows that τ is locally bounded, that is, $\sup_{g\in C}\|\tau(g)\| < \infty$ for compact subsets C of G. For each $f \in \mathcal{K}(G)$ we define

$$(6.1)\qquad \tau(f) := \int_G f(g)\tau(g)dg,$$

where dg is a left Haar measure and the operator-valued integral is considered in the weak sense. Let $\{V_n\}$ be a decreasing sequence of open neighbourhoods of e in G such that $\{V_n\}$ is a base for the neighbourhoods of e. Choose a sequence $\{f_n\}$ in $\mathcal{K}(G)$ such that $f_n \geq 0$, $\mathrm{supp}(f_n) \subset V_n$ and $\int_G f_n(g)\,dg = 1$. Then $(\tau(f_n)v,w) \to (v,w)$ for all $v,w \in \mathcal{H}(\tau)\}$. We conclude that the linear span of $\{\tau(f)v \mid f \in \mathcal{K}(G),\ v \in \mathcal{H}(\tau)\}$ is weakly dense in $\mathcal{H}(\tau)$. Also observe that

$$(6.2)\qquad \tau(x)\tau(f) = \tau(\lambda(x)f),\ f \in \mathcal{K}(\tau),\ x \in G,$$

where $(\lambda(x)f)(g) := f(x^{-1}g)$.

PROPOSITION 6.1 (cf. WARNER [63, Prop. 4.2.2.1]). *Let* $\tau : G \to \mathcal{L}(H(\tau))$ *satisfy* $\tau(g_1 g_2) = \tau(g_1)\tau(g_2)$ *and* $\tau(e) = I$. *Then* τ *is strongly continuous if and only if it is weakly continuous.*

PROOF. Assume that τ is weakly continuous. Let $\mathcal{H}_s(\tau)$ be the linear subspace of $\mathcal{H}(\tau)$ consisting of all $v \in \mathcal{H}(\tau)$ for which $g \to \tau(g)v$ is continuous from G to $\mathcal{H}(\tau)$. It is easily seen that $\mathcal{H}_s(\tau)$ is a closed subspace of $\mathcal{H}(\tau)$. Furthermore it follows from (6.1) and (6.2) that $\tau(f)v \in \mathcal{H}_s(\tau)$ for all $f \in \mathcal{K}(G)$, $v \in \mathcal{H}(\tau)$. Since weak closure and closure in norm coincide for linear subspaces of $\mathcal{H}(\tau)$, the weak closure $\mathcal{H}(\tau)$ of the linear span of $\{\tau(f)v \mid f \in \mathcal{K}(G), v \in \mathcal{H}(\tau)\}$ must be included in $\mathcal{H}_s(\tau)$. □

COROLLARY 6.2. *If* τ *is a (strongly continuous) Hilbert representation of* G *then* $\tilde{\tau}$ *defined by*

$$(6.3) \qquad \tilde{\tau}(g) := \tau(g^{-1})^*, \qquad g \in G,$$

is again a (strongly continuous) Hilbert representation of G *on* $\mathcal{H}(\tau)$.

The representation $\tilde{\tau}$ is called the *conjugate contragredient* to τ. The representation τ is unitary if and only if $\tilde{\tau} = \tau$.

6.2. A criterium for unitarizability

Let G be a lcsc. group with compact subgroup K. Let σ and τ be K-finite representations of G and let $\sigma \overset{A}{\simeq} \tau$ (cf. §4.1). Let $A_\delta := A|_{\mathcal{H}_\delta(\sigma)}$, $\delta \in \hat{K}$. Then it is easily seen that A^* is an injective closed linear operator from $\mathcal{H}(\tau)$ to $\mathcal{H}(\sigma)$ with dense domain and range and such that $A^*|_{\mathcal{H}_\delta(\tau)} = A_\delta{}^*$ maps $\mathcal{H}_\delta(\tau)$ onto $\mathcal{H}_\delta(\sigma)$. Furthermore, since

$$(\tilde{\tau}(g)v,\ Aw) = (v, \tau(g^{-1})Aw) = (v, A\sigma(g^{-1})w) = (A^*v,\ \sigma(g^{-1})w),$$
$$v \in \mathcal{D}(A^*),\ w \in \mathcal{D}(A),$$

we conclude that $\mathcal{D}(A^*)$ is $\tilde{\tau}$-invariant and that $A^*\tilde{\tau}(g)v = \tilde{\sigma}(g)A^*v$ if $v \in \mathcal{D}(A^*)$, $g \in G$. Thus we have:

LEMMA 6.3. *If* $\sigma \overset{A}{\simeq} \tau$ *then* $\tilde{\tau} \overset{A^*}{\simeq} \tilde{\sigma}$.

Now we can prove:

<u>THEOREM 6.4</u>. *Let* G *be a lcsc. group with compact subgroup* K. *Let* τ *be a* K-*finite representation of* G. *Then* τ *is equivalent to some unitary representation of* G *iff* $\tau \overset{A}{\simeq} \tilde{\tau}$ *with* A *self-adjoint and positive definite.*

<u>PROOF</u>. First suppose that $\tau \overset{B}{\simeq} \sigma$ with σ unitary. Then $\sigma = \tilde{\sigma}$ and $\tilde{\sigma} \overset{B^*}{\simeq} \tilde{\tau}$ (cf. Lemma 6.3), so $\tau \overset{A}{\simeq} \tilde{\tau}$, where A is the closure of B^*B (cf. proof of Theorem 4.4). Obviously, A is self-adjoint and positive definite.

Next suppose that $\tau \overset{A}{\simeq} \tilde{\tau}$ with A self-adjoint and positive definite. Let $(.\,,\,.)$ be the inner product on $\mathcal{H}(\tau)$. We define a new inner product $\langle .\,,\,. \rangle$ on $\mathcal{D}(A)$ by

$$(6.4) \qquad \langle v,w \rangle := (Av,w), \quad v,w \in \mathcal{D}(A).$$

This is indeed a positive definite sesquilinear form on $\mathcal{D}(A)$. For $v,w \in \mathcal{D}(A)$, $g \in G$, we have:

$$\langle \tau(g)v,\ \tau(g)w \rangle = (A\tau(g)v,\ \tau(g)w) = (\tilde{\tau}(g^{-1})A\tau(g)v,w) =$$
$$= (A\tau(g^{-1})\tau(g)v,w) = (Av,w) = \langle v,w \rangle,$$

i.e.,

$$(6.5) \qquad \langle \tau(g)v,\ \tau(g)w \rangle = \langle v,w \rangle.$$

Thus τ is a unitary representation of G on $\mathcal{D}(A)$ with respect to the inner product $\langle .\,,\,. \rangle$. (Weak continuity is obvious from (6.4) and the weak continuity of the original representation.) Let σ be the extension of this representation to a unitary representation in the Hilbert space completion $\mathcal{H}(\sigma)$ of $\mathcal{D}(A)$ with respect to $\langle .\,,\,. \rangle$. Then $\tau \overset{B}{\simeq} \sigma$, where B is the closure of the identity operator on $\mathcal{D}(A)$ (cf. Lemma 4.3). □

Next we restrict ourselves to the case that τ is a K-multiplicity free representation of G. Then the same holds for $\tilde{\tau}$. Furthermore, $\mathcal{M}(\tilde{\tau}) = \mathcal{M}(\tau)$ and

$$(6.6) \qquad \tilde{\tau}_{\gamma,\delta}(g) = \tau_{\delta,\gamma}(g^{-1})^*, \ \gamma,\delta \in M(\tau), \ g \in G.$$

It follows from Theorem 4.5 that $\tau \overset{A}{\cong} \tilde{\tau}$ with $A_\delta = c_\delta I_\delta$, $\delta \in M(\tau)$ (I_δ identity operator on $H_\delta(\tau)$) iff

$$\tilde{\tau}_{\gamma,\delta}(g) = \frac{c_\gamma}{c_\delta} \tau_{\gamma,\delta}(g), \ \gamma,\delta \in M(\tau), \ g \in G.$$

Now A is self-adjoint and positive definite iff $c_\delta > 0$ for all $\delta \in M(\tau)$. Thus Theorem 6.4 implies:

<u>THEOREM 6.5</u>. *Let τ be a K-multiplicity free representation of G. Then τ is equivalent to some unitary representation iff there are positive numbers c_δ, $\delta \in M(\tau)$, such that*

$$(6.7) \qquad \tau_{\gamma,\delta}(g^{-1})^* = \frac{c_\delta}{c_\gamma} \tau_{\delta,\gamma}(g), \ \gamma,\delta \in M(\tau), \quad g \in G.$$

In case of unitarizability of τ, the new inner product (6.4) becomes

$$(6.8) \qquad \langle v,w\rangle = \sum_{\delta \in M(\tau)} c_\delta (v_\delta, w_\delta).$$

<u>REMARK 6.6</u>. In the case that τ is an irreducible K-multiplicity free representation of G, unitarizability of τ is already implied if (6.7) holds for some δ and all $\gamma \in M(\tau)$ with $c_\gamma > 0$ (cf. Lemma 4.8).

6.3. <u>The case G = SU(1,1)</u>.

It follows from (2.15) and (6.6) that

$$(6.9) \qquad \tilde{\pi}_{\xi,\lambda} = \pi_{\xi,-\bar{\lambda}}, \ \xi \in \{0,\tfrac{1}{2}\}, \quad \lambda \in \mathbb{C}.$$

This can also be derived from (2.47), (2.48) and (6.6). Note that, essentially, (6.9) is equivalent to the identity

$$(6.10) \qquad \overline{c_{\xi,\lambda;n,m}} = (-1)^{m-n} c_{\xi,-\bar{\lambda};m,n}.$$

In §6.2 we showed that a necessary condition for unitarizability of an irreducible subquotient representation τ of $\pi_{\xi,\lambda}$ is the equivalence of τ and $\tilde{\tau}$. In view of (6.9) and Theorem 4.10 this is only possible if $\bar{\lambda} = \pm\lambda$, that is, if λ is real or imaginary. If λ is imaginary then $\tilde{\pi}_{\xi,\lambda} = \pi_{\xi,\lambda}$, so $\pi_{\xi,\lambda}$ is already unitary. Let us now examine the case that λ is real and non-zero. Then $\tilde{\pi}_{\xi,\lambda} = \pi_{\xi,-\lambda}$. If τ is an irreducible subquotient representation of $\pi_{\xi,\lambda}$ then $\tau \overset{A}{\cong} \tilde{\tau}$ with (cf. (4.22))

$$A\phi_m = c_{\xi,\lambda;m}\,\phi_m,\quad \phi_m \in H(\tau), \tag{6.11}$$

where $c_{\xi,\lambda;m}$ is given by (4.21). Now a sufficient condition for the unitarizability of τ is that the coefficients $c_{\xi,\lambda;m}$ are all positive or all negative for $\phi_m \in H(\tau)$. Referring to the classification in Theorem 3.4 we will examine these coefficients. (Because of equivalence, it is not necessary to treat the cases where $\lambda < 0$.)

(a) $\pi_{0,\lambda}(\lambda > 0,\ \lambda \notin \mathbb{Z} + \frac{1}{2})$.

$$c_{0,\lambda;m} = \frac{(-\lambda+\frac{1}{2})_{|m|}}{(\lambda+\frac{1}{2})_{|m|}},\quad m \in \mathbb{Z}.$$

$c_{0,\lambda;m}$ has fixed sign iff $0 < \lambda < \frac{1}{2}$.

(b) $\pi_{\frac{1}{2},\lambda}(\lambda > 0,\ \lambda \notin \mathbb{Z})$.

$$c_{\frac{1}{2},\lambda;m} = \frac{(-\lambda)_{m+\frac{1}{2}}}{(\lambda)_{m+\frac{1}{2}}},\quad m + \tfrac{1}{2} \in \{0,1,2,\dots\}.$$

No fixed sign.

(c) $\pi^{+}_{\xi,\lambda}$ and $\pi^{-}_{\xi,\lambda}$ $(\lambda+\xi \in \mathbb{Z} + \frac{1}{2},\ \lambda > 0)$.

$$c_{\xi,\lambda;m} = \frac{(|m|-(\lambda+\frac{1}{2}))!}{(2\lambda+1)_{|m|-(\lambda+\frac{1}{2})}},\quad m \in \mathbb{Z} + \xi,\ |m| \geq \lambda + \tfrac{1}{2}.$$

Fixed sign.

(d) $\pi^0_{\xi,\lambda}$ $(\lambda+\xi \in \mathbb{Z} + \frac{1}{2},\ \lambda > 0)$.

$$c_{\xi,\lambda;m} = \frac{(-1)^{m-\xi}}{(\lambda-\frac{1}{2}+m)!\,(\lambda+\frac{1}{2}-m)!}\ ,\quad m \in \{-\lambda+\tfrac{1}{2}, -\lambda+\tfrac{3}{2}, \ldots, \lambda-\tfrac{1}{2}\}.$$

No fixed sign except if $\lambda = \frac{1}{2}$, $\xi = 0$.

Combining these results with Theorems 3.4, 4.2, 4.10 and 5.13 we reobtain BARGMANN'S [1] classification of all irreducible unitary representations of SU(1,1):

THEOREM 6.7. *Any irreducible unitary representation of* SU(1,1) *is unitarily equivalent to one and only one of the following representations:*

1) $\pi_{\xi,i\nu}(\xi = 0,\frac{1}{2},\ \nu > 0)$, $\pi_{0,0}$, $\pi^+_{\frac{1}{2},0}$, $\pi^-_{\frac{1}{2},0}$.

2) $\pi_{0,\lambda}(0 < \lambda < \frac{1}{2})$ *on* Cl Span $\{\ldots,\phi_{-1},\phi_0,\phi_1,\ldots\}$

with respect to the inner product

$$\langle\phi_m,\phi_n\rangle := \frac{(-\lambda+\frac{1}{2})_{|m|}}{(\lambda+\frac{1}{2})_{|m|}}\ \delta_{m,n}.$$

3) $\pi^+_{\xi,\lambda}$ *and* $\pi^-_{\xi,\lambda}$ $(\xi = 0 \text{ or } \frac{1}{2},\ \lambda = \xi+\frac{1}{2},\ \xi+\frac{3}{2},\ldots)$

on

$$\text{Cl Span } \{\phi_{\lambda+\frac{1}{2}},\ \phi_{\lambda+\frac{3}{2}},\ldots\}$$

and

$$\text{Cl Span } \{\ldots.,\ \phi_{-\lambda-\frac{3}{2}},\ \phi_{-\lambda-\frac{1}{2}}\},$$

respectively, with respect to the inner product

$$\langle\phi_m,\phi_n\rangle := \frac{(|m|-(\lambda+\frac{1}{2}))!}{(2\lambda+1)_{|m|-(\lambda+\frac{1}{2})}}\ \delta_{m,n}.$$

4) $\pi^0_{0,\frac{1}{2}}$.

This is the identity representation of SU(1,1).

6.4. An addition formula approach.

Let G be a lcsc. group with compact subgroup K and let τ be a K-multiplicity free representation of G. In this paper we developed machinery to obtain irreducible sub(quotient) representations of τ and to decide about their unitarizability. This machinery works quite well if we have an explicit knowledge of the generalized matrix elements $\tau_{\gamma,\delta}(g)$, $\gamma,\delta \in M(\tau)$. This happened to be the case in our simple example $G = SU(1,1)$, $\tau = \pi_{\xi,\lambda}$, but for more general G these matrix elements are usually unknown. Now it may happen that we have an explicit expression for $\tau_{\delta,\delta}(g)$ for some $\delta \in M(\tau)$ (for instance, for $\delta = 1$) and that we are able to calculate explicitly the Fourier expansion

$$(6.12) \qquad \tau_{\delta,\delta}(g_1 k g_2^{-1}) = \sum_{\gamma\in\hat{K}} \tau_{\delta,\delta;\gamma}(g_1,g_2,k),$$

where

$$(6.13) \qquad \tau_{\delta,\delta;\gamma}(g_1,g_2,k) := d_\gamma \int_K \tau_{\delta,\delta}(g_1 k_1 g_2^{-1})\chi_\gamma(k_1^{-1}k)\,dk_1.$$

Formula (6.12) is called the *addition formula* for $\tau_{\delta,\delta}$.

THEOREM 6.8. *Let τ be a K-multiplicity free representation of G. Let $\delta \in M(\tau)$. Let τ_0 be the irreducible subquotient representation of τ on* Irr(δ) *(cf. Lemma* 3.2). *Then:*

(a) $\qquad M(\tau_0) = \{\gamma \in \hat{K} \mid \tau_{\delta,\delta;\gamma} \neq 0\}$

(b) $\qquad \tau_0$ *is unitarizable if and only if*

$$(6.14) \qquad \tau_{\delta,\delta}(g^{-1}) = \tau_{\delta,\delta}(g)^*, \quad g \in G,$$

and the matrices $\tau_{\delta,\delta;\gamma}(g,g,e)$ are positive (semi-) definite for all $\gamma \in \hat{K}$, $g \in G$.

PROOF.

(a) On comparing (6.12) with (4.8) we have

$$\tau_{\delta,\delta;\gamma}(g_1,g_2,k) = \begin{cases} \tau_{\delta,\gamma}(g_1)\gamma(k)\tau_{\gamma,\delta}(g_2^{-1}) & \text{if } \gamma \in M(\tau_0), \\ 0 & \text{if } \gamma \notin M(\tau_0). \end{cases} \tag{6.15}$$

By irreducibility of τ_0, $\tau_{\delta,\delta;\gamma}$ cannot be identically zero if $\gamma \in M(\tau_0)$.

(b) Suppose that τ_0 is unitarizable. Then Theorem 6.5 implies (6.14) and substitution of (6.7) into (6.15) yields

$$\tau_{\delta,\delta;\gamma}(g,g,e) = \frac{c_\delta}{c_\gamma}\,\tau_{\delta,\gamma}(g)\tau_{\delta,\gamma}(g^*),\ \gamma \in M(\tau_0), \tag{6.16}$$

where the c_γ's are positive numbers. Thus $\tau_{\delta,\delta;\gamma}(g,g,e)$ is positive semi-definite.

Conversely, assume (6.14) and the positive semi-definiteness of the matrices $\tau_{\delta,\delta;\gamma}(g,g,e)$. Formula (6.14) together with Theorem 4.9 implies that there are complex constants c_γ, $\gamma \in M(\tau_0)$, such that (6.7) holds. Substitution into (6.15) again gives (6.16). By irreducibility of τ_0, the function $g \to \tau_{\delta,\delta;\gamma}(g,g,e)$ is not identically zero for $\gamma \in M(\tau_0)$. Thus the positive semi-definiteness of $\tau_{\delta,\delta;\gamma}(g,g,e)$ implies that $c_\delta/c_\gamma > 0$. Now the unitarizability of τ_0 follows from Remark 6.6. □

6.5. Notes.

6.5.1. Following BARGMANN [1], most authors prove theorem 6.7 by infinitesimal methods. VILENKIN [59, Ch.VI] uses the method of the present paper. TAKAHASHI [55, §6] decides about unitarizability by considering whether $\pi_{\xi,\lambda;n,n}$ is a positive definite function on G.

6.5.2. The family of unitary representations in 2) of Theorem 6.7 is called the *complementary series*. See [37, §7.1] for a different characterization of these representations. The family of unitary representations in 3) of Theorem 6.7 is called the *discrete series*. By comparing K-contents these representations can be identified with more usual realizations of the discrete series (cf. VAN DIJK [11, §7]).

6.5.3. A unitary representation τ of a unimodular lcsc. group G is called *square integrable* if for each $v,w \in \mathcal{H}(\tau)$ the function $x \to (\tau(x)v,w)$ is in $L^2(G)$. If τ is irreducible and if $x \to (\tau(x)v,w)$ is in $L^2(G)$ for some nonzero $v,w \in \mathcal{H}(\tau)$ then it can be shown that τ is square integrable (cf. BOREL [5, Théorème 5.15]). Thus we can examine square integrability of the irreducible unitary representations of SU(1,1) listed in Theorem 6.7 by considering which of the functions $\pi_{\xi,\lambda n,n}$ are in $L^2(G)$. It can be shown that

$$\|\pi_{\xi,\lambda;n,n}\|^2_{L^2(G)} = \text{const.} \int_0^\infty |\phi_{2i\lambda}^{(0,2n)}(\tfrac{1}{2}t)|^2 \ \operatorname{sh}\tfrac{1}{2}t(\operatorname{ch}\tfrac{1}{2}t)^{4n+1}dt.$$

To decide whether this integral is finite is a purely analytic exercise in the theory of Jacobi functions. The solution is given by FLENSTED-JENSEN [18, Lemma A.3]. It turns out that precisely the discrete series representations are square integrable.

6.5.4. In FLENSTED-JENSEN & KOORNWINDER [20] Theorem 6.7(b) was used in order to find all irreducible unitary spherical representations of non-compact semisimple Lie groups G of rank one. Theorem 6.7(b) was proved there by using that a spherical function on G corresponds to a unitary representation iff it is a positive definite function on G.

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