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A super-balanced hypergraph has a nest point

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ABSTRACT

A super-balanced hypergraph is a hypergraph such that in any cycle of length at least three there is an edge containing at least three vertices of the cycle. A nest point is a vertex such that the edges containing it are totally ordered by inclusion. It is proved that a super-balanced hypergraph contains at least two nest points.

KEY WORDS & PHRASES: *hypergraphs, balanced hypergraphs, chordal graphs*

1. INTRODUCTION

Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite set, and let $\mathcal{E} = \{E_i | i = 1,2,\ldots,m\}$ be a family of subsets of X. The family $\mathcal{E}$ is said to be a *hypergraph on* X if

(1) $E_i \neq \emptyset \quad i = 1,2,\ldots,m,$

(2) $\bigcup_{i=1}^{m} E_i = X$

The couple $H = (X,\mathcal{E})$ is called a *hypergraph*. The elements $x_1, x_2, \ldots, x_n$ are called *vertices* and the sets $E_1, E_2, \ldots, E_m$ are called *edges*.

The *dual hypergraph* $H^* = (X^* \mathcal{E}^*)$ is defined by $X^* = \{e_1, \ldots, e_m\}$, $\mathcal{E}^* = \{X_1, X_2, \ldots, X_n\}$, where $X_j = \{e_i | x_j \in E_i, \ i = 1,\ldots,m\}$, $j = 1,2,\ldots,n$.

In a hypergraph $H = (X,\mathcal{E})$, *a chain of length* q is defined to be a sequence $(x_1, E_1, x_2, E_2, \ldots, E_q, X_{q+1})$ such that

(1) $x_1, x_2, \ldots, x_q$ are all distinct vertices of H,

(2) $E_1, E_2, \ldots, E_q$ are all distinct edges of H,

(3) $x_k, x_{k+1} \in E_k$ for $k = 1,2,\ldots,q$.

If $q > 1$ and $x_{q+1} = x_1$, then this chain is called a *cycle of length* q.

A hypergraph H is said to be *balanced* if every odd cycle has an edge that contains at least three vertices of the cycle. Balanced hypergraphs have been studied quite extensively (BERGE [1,2]). They have the following property.

<u>PROPERTY 1.1</u> Let $J \subseteq \{1,2,\ldots,m\}$ be such that if $E_i \cap E_j \neq \emptyset$ for all $i,j \in J$, then $\bigcap_{i \in J} E_i \neq \emptyset$, i.e., the edges of a balanced hypergraph have Helly's property.

We consider a more restrictive class of hypergraphs called super-balanced. A hypergraph is said to be *super-balanced* if every cycle of length at least three has an edge containing at least three vertices of the cycle.

<u>PROPERTY 1.2</u>. The dual hypergraph H^* of a super-balanced hypergraph H is super-balanced.

<u>PROOF</u>. Consider a cycle $\mu = (e_1, X_1, e_2, \ldots, X_p, e_1)$. It corresponds to a cycle $(x_1, E_2, e_2, \ldots, x_p, E_1, x_1)$ in H.

As H is super-balanced an edge E_i contains three of the x_j's and therefore in H^* an e_i belongs to three of the X_j's; or equivalently an X_k contains three of the e_i's. □

As an example of a super-balanced hypergraph we consider the following.

EXAMPLE. Let T be a tree with vertex set $V = \{v_1,\dots,v_p\}$. Each edge of the tree has a positive length. The distance $d(v_i,v_j)$ between two vertices $v_i,v_j \in V$ is defined to be the length of the shortest path between these two vertices. For each i, $1 \le i \le p$, let r_i be a nonnegative integer and define $E_i = \{v \in V | d(v,v_i) \le r_i\}$. It was shown by GILES [4] that the hypergraph $(V,\{E_1,\dots,E_p\})$ is super-balanced.

A *subhypergraph* of $(X,\mathcal{E})$ is a hypergraph $(A,\mathcal{E}^A)$, where $A \subset X$ and $\mathcal{E}^A = \{E_i \cap A \mid E_i \in \mathcal{E},\ E_i \cap A \neq \emptyset\}$.
A *partial hypergraph* of $(X,\mathcal{E})$ is a hypergraph $(X_{\mathcal{F}},\mathcal{F})$, where $\mathcal{F} \subset \mathcal{E}$ and $X_{\mathcal{F}} = \bigcup_{E_i \in \mathcal{F}} E_i$.
The following properties of super-balanced hypergraphs are trivial.

PROPERTY 1.3. If H is a super-balanced hypergraph, then every partial hypergraph H' is super-balanced.

PROPERTY 1.4. If H is a super-balanced hypergraph, then every subhypergraph H' is super-balanced.

A *nest point* of a hypergraph is a vertex with the property that the edges containing it are totally ordered by inclusion.
The *incidence matrix* $A = (a_{ij})$ of a hypergraph $(X,\mathcal{E})$ is defined by $a_{ij} = 1$ if $x_j \in E_i$, $a_{ij} = 0$ otherwise.
A (0,1)-matrix is called *super-balanced* if it does not contain a square submatrix of size at least three with row and column sums equal to two. It is clear that the definition of a super-balanced hypergraph that the incidence matrix of a super-balanced hypergraph is super-balanced. The converse is trivially true: every super-balanced matrix defines a super-balanced hypergraph. Our interest in proving that a super-balanced hypergraph has a nest point arises from the fact that this result enables us to solve the set covering problem on a super-balanced matrix in polynomial time; in a subsequent paper we will show how this is done.
The vertex *intersection graph* $G = (X,\Gamma)$ corresponding to a hypergraph $(X,\mathcal{E})$ has vertex set X and two vertices are adjacent if and only if they have an edge of $\mathcal{E}$ in common.

A *chordal graph* is a graph with the property that every cycle with more than three vertices has a chord, i.e., an edge incident to two vertices of the cycle which are not incident to an edge of the cycle.
The following porperty follows from the definition of a super-balanced hypergraph.

PROPERTY 1.5. The vertex intersection graph of a super-balanced hypergraph is a chordal graph.

Chordal graphs are sometimes called *rigid circuit graphs* or *triangulated graphs*.
A *simplicial vertex* of a graph is a vertex with the property that if two vertices are adjacent to this vertex, then they are also adjacent to each other, i.e., all vertices adjacent to a simplicial vertex form a clique.
The following property of a chordal graphs was first proved by DIRAC [3].

PROPERTY 1.6. A chordal graph has a simplicial vertex.

2. MAIN RESULT.

In this section we will prove our main result, namely that every super-balanced hypergraph $(X,\mathcal{E})$ with at least two vertices contains at least two nest points. To prove this result we will use induction on the number $|X| + |\mathcal{E}|$. It is clear that if $|X| = 2$, then both vertices are nest points.
Consider a super-balanced hypergraph $(X,\mathcal{E})$ and assume that all super-balanced hypergraphs $(\hat{X},\hat{\mathcal{E}})$ with $|\hat{X}| + |\hat{\mathcal{E}}| < |X| + |\mathcal{E}|$ have at least two nest points. In particular this is the case for all partial subhypergraphs of $(X,\mathcal{E})$ (by Property 1.3 and 1.4). Our result follows from the next two theorems.

THEOREM 2.1. *The super-balanced hypergraph* $(X,\mathcal{E})$ *has a nest point.*

THEOREM 2.2. *The super-balanced hypergraph* $(X,\mathcal{E})$ *does not contain exactly one nest point.*

Before proving these theorems we give some definitions and prove some useful lemmas.

DEFINITIONS.
- $\mathcal{E}_x = \{E \mid E \in \mathcal{E},\ x \in E\}$, for $x \in X$.
- $\mathcal{E}(A) = \{E \backslash A \mid E \in \mathcal{E}\}$, for $A \subset X$.
- Two edges E_1 and E_2 are *comparable* if $E_1 \subseteq E_2$ or $E_2 \subseteq E_1$
- Two edges are *incomparable* if they are not comparable.
- Let $\mathcal{E}_x$ $(x \in X)$ be totally ordered by inclusion.
  Then *min* $\mathcal{E}_x$ is an edge belonging to $\mathcal{E}_x$ with the property that it is included in all edges of $\mathcal{E}_x$, *max* $\mathcal{E}_x$ is the edge belonging to $\mathcal{E}_x$ which includes all other edges of $\mathcal{E}_x$.

LEMMA 2.3. *Let* $E, F_1, F_2 \in \mathcal{E}$ *such that* $E \subseteq F_1 \cap F_2$ *and* F_1 *and* F_2 *incomparable. Then* $(X,\mathcal{E})$ *contains two nest points.*

PROOF. Consider the partial hypergraph obtained from $(X,\mathcal{E})$ by deleting the edge E. By induction this hypergraph has two nest points y_1 and y_2. Since F_1 and F_2 are incomparable it follows that $y_i \notin E \subseteq F_1 \cap F_2$ $(i = 1,2)$. Hence y_1 and y_2 are also nest points of $(X,\mathcal{E})$. □

According to Lemma 2.3. we may assume without loss of generality that $(X,\mathcal{E})$ satisfies Property 2.4.

PROPERTY 2.4. If $E, F_1, F_2 \in \mathcal{E}$ and $E \subseteq F_i$ $(i=1,2)$, then F_1 and F_2 are comparable.

LEMMA 2.5. *Let* $(X,\mathcal{E})$ *be a super-balanced hypergraph. Then there is a vertex* x_1 *and an edge* E_1 *with* $x_1 \in E_1$ *such that* $\forall F \in \mathcal{E}_{x_1} [F \subseteq E_1]$.

PROOF. Consider the vertex intersection graph G of $(X,\mathcal{E})$, By Properties 1.5 and 1.6 there is a simplicial vertex x_1 of G. Let $x_2,\ldots,x_k$ be all vertices adjacent to x_1. Clearly all edges belonging to $\mathcal{E}_{x_1}$ are included in $\{x_1,x_2,\ldots,x_k\}$. Consider the dual hypergraph of $(X,\mathcal{E})$. Since $x_1,x_2,\ldots,x_k$ form a clique in G we know that for the corresponding edges $X_1,X_2,\ldots,X_k$ of the dual hypergraph $X_i \cap X_j \neq \emptyset$ for all $i,j = 1,\ldots,k$. By Property 1.1 we known that there is a vertex e_1 of the dual hypergraph such that $e_1 \in \bigcap_{i=1}^{k} X_i$. Therefore the corresponding edge E_1 of $(X,\mathcal{E})$ contains $x_1,x_2,\ldots,x_k$. Since all edges containing x_1 are contained in $\{x_1,x_2,\ldots,x_k\}$ we know that $E_1 = \{x_1,x_2,\ldots,x_k\}$. □

PROOF OF THEOREM 2.1. Let x_1, E_1 be the vertex and edge found in Lemma 2.5. Define the set I by $I = \{i \in E_1 \mid \forall F \in \mathcal{E}_i [F \subseteq E_1]\}$. Since $x_1 \in I$ we know that $I \neq \emptyset$. We consider two possibilities.

1. $I = E_1$.
   Consider the partial hypergraph obtained from $(X,\mathcal{E})$ by deleting E_1. By induction it contains two nest points y_1 and y_2. If $y_i \notin E_1$, then y_i is nest point of $(X,\mathcal{E})$ $(i = 1,2)$. If $y_i \in E_1 = I$, then by definition of I all edges of $\mathcal{E}_{y_i}$ are included in E_1, hence y_i is also a nest point of $(X,\mathcal{E})$ $(i = 1,2)$.

2. $I \subsetneq E_1$.
   Consider the subhypergraph of $(X,\mathcal{E})$ obtained by deleting the set I from X. Let $\hat{\mathcal{E}} = \mathcal{E}(I)$. Let y be a nest point of this hypergraph. If there is no edge belonging to $\mathcal{E}$ which contains y and a point $i \in I$, then y is nest point of $(X,\mathcal{E})$. So we may assume that there is an edge $F \in \mathcal{E}$ containing both y and a point $i \in I$. By definition of I we know that $F \subseteq E_1$ and therefore $y \in E_1 \backslash I$. Since $y \notin I$ we know that y is contained in an edge E which contains a point not belonging to E_1. Consider $E_2 = \max \hat{\mathcal{E}}_y$. E_2 includes both $E_1 \backslash I$ and E. Since E_2 contains a point not belonging to E_1 it follows from the definition of I that $E_2 \in \mathcal{E}$. We conclude that $(X,\mathcal{E})$ contains two incomparable edges E_1 and E_2 with $E_1 \cap E_2 = E_1 \backslash I$.

CASE 2.1. $I = \{i\}$.
Suppose i is not a nest point. Then there are two incomparable edges F_1 and F_2 containing i. Choose $a_1 \in F_1 \backslash F_2$ and $a_2 \in F_2 \backslash F_1$. Then $(i, F_1, a_1, E_2, a_2, F_2, i)$ is a cycle of length three not containing an edge which contains all three vertices, contradicting the assumption that $(X,\mathcal{E})$ is super-balanced. Hence i is a nest point.

CASE 2.2. $|I| > 1$.
Let $i \in I$. Consider the subhypergraph of $(X,\mathcal{E})$ obtained by deleting $I\backslash\{i\}$ from X. By induction this hypergraph has two nest points; at least one of them, say y, is different from i. We have $y \notin E_1 \backslash (I\backslash\{i\})$ since all points from $E_1 \backslash (I\backslash\{i\})$ except i are contained in two incomparable edges, namely $E_1 \backslash (I\backslash\{i\})$ and E_2. By definition of I it follows that y is not contained in an edge of $\mathcal{E}$ which also contains a point of I, hence y is also a nest point

of $(X,\mathcal{E})$. $\square$

LEMMA 2.6. *Let* $(X,\mathcal{E})$ *be a super-balanced hypergraph satisfying Property 2.4 and let* y *be a nest point of the subhypergraph obtained from* $(X,\mathcal{E})$ *by deleting a vertex* x*, but not a nest point of* $(X,\mathcal{E})$*. Then* $\min \hat{\mathcal{E}}_y \notin \mathcal{E}$ *and* $\min \hat{\mathcal{E}}_y \cup \{x\} \in \mathcal{E}$*, where* $\hat{\mathcal{E}} = \mathcal{E}(\{x\})$.

PROOF. Since y is not a nest point of $(X,\mathcal{E})$ there are edges $F_1, F_2 \in \mathcal{E}$ such that $F_1\backslash\{x\} \subsetneq F_2\backslash\{x\}$ and $x \in F_1$, $x \notin F_2$. Since $\min \hat{\mathcal{E}}_y \subseteq (F_1 \cap F_2)$ it follows from Property 2.4 that $\min \hat{\mathcal{E}}_y \notin \mathcal{E}$ and hence $\min \hat{\mathcal{E}}_y \cup \{x\} \in \mathcal{E}$. $\square$

PROOF OF THEOREM 2.2. Assume $(X,\mathcal{E})$ contains exactly one nest point x. Consider the subhypergraph obtained from $(X,\mathcal{E})$ by deleting x from X. By induction there are two nest points y_1 and y_2. Define $E_i = \min \hat{\mathcal{E}}_{y_i} \cup \{x\}$, where $\hat{\mathcal{E}} = \mathcal{E}(\{x\})$ $(i = 1,2)$. Since y_1 and y_2 are not nest points of $(X,\mathcal{E})$ it follows from Lemma 2.6 that $E_i \in \mathcal{E}$ $(i = 1,2)$. Since $x \in E_i$ $(i = 1,2)$ and x is a nest point it follows that E_1 and E_2 are comparable, say $E_1 \subseteq E_2$.

CASE 1. $E_1 = E_2$

Then y_1 and y_2 are contained in exactly the same edges. Consider the sub-hypergraph of $(X,\mathcal{E})$ obtained by deleting y_1 from X. It has two nest points z_1 and z_2. Clearly z_1 and z_2 are also nest points of $(X,\mathcal{E})$, contradicting our assumption that there is exactly one nest point of $(X,\mathcal{E})$.

CASE 2. $E_1 \subsetneq E_2$

Consider the partial hypergraph obtained from $(X,\mathcal{E})$ by deleting E_1 from $\mathcal{E}$. It has two nest points; at least one, say z, is different from x. If $z \notin E_1$, then z is also a nest point of $(X,\mathcal{E})$, contradicting the assumption that x was the only nest point of $(X,\mathcal{E})$. So assume $z \in E_1$. But then $\mathcal{E}_{y_2} \subset \hat{E}_z$, where $\hat{\mathcal{E}} = \mathcal{E}\backslash\{E_1\}$, and hence y_2 is also a nest point of $(X,\mathcal{E})$. Since $y_2 \neq x$ this again leads to a contradiction. $\square$

We conclude that $(X,\mathcal{E})$ contains at least two nest points.

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