stichting
mathematisch
centrum

AFDELING ZUIVERE WISKUNDE ZW 153/81 MAART
(DEPARTMENT OF PURE MATHEMATICS)

T.H. KOORNWINDER

INVARIANT DIFFERENTIAL OPERATORS ON NON-REDUCTIVE HOMOGENEOUS SPACES

kruislaan 413 1098 SJ amsterdam

Printed at the Mathematical Centre, 413 Kruislaan, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O.).

1980 Mathematics subject classification: 22E30, 43A85, 17B35

Invariant differential operators on non-reductive homogeneous spaces

by

T.H. Koornwinder

ABSTRACT

A systematic exposition is given of the theory of invariant differential operators on a not necessarily reductive homogeneous space. This exposition is modelled on Helgason's treatment of the general reductive case and the special non-reductive case of the space of horocycles. As a final application the differential operators on (not a priori reductive) isotropic pseudo-Riemannian spaces are characterized.

KEY WORDS & PHRASES: *invariant differential operators; non-reductive homogeneous spaces; space of horocycles; isotropic pseudo-Riemannian spaces*

1. INTRODUCTION

Let G be a Lie group and H a closed subgroup. Let $\mathfrak{g}$ and $\mathfrak{h}$ denote the corresponding Lie algebras. Suppose that the coset space G/H is *reductive*, i.e., there is a complementary subspace $\mathfrak{m}$ to $\mathfrak{h}$ in $\mathfrak{g}$ such that $\mathrm{Ad}_G(H)\mathfrak{m} \subset \mathfrak{m}$. Let $\mathbb{D}(G/H)$ denote the algebra of G-invariant differential operators on G/H. The main facts about $\mathbb{D}(G/H)$ are summarized below (cf. HELGASON [3, Ch. III], [4, Cor. X.2.6, Theor. X.2.7], [6, §2].

Let $\mathbb{D}(G)$ be the algebra of left invariant differential operators on G, $\mathbb{D}_H(G)$ the subalgebra of operators which are right invariant under H and $S(\mathfrak{g})$ the complexified symmetric algebra over $\mathfrak{g}$. Let $\lambda\colon S(\mathfrak{g}) \to \mathbb{D}(G)$ denote the symmetrization mapping. $I(\mathfrak{m})$ denotes the set of $\mathrm{Ad}_G(H)$-invariants in $S(\mathfrak{m})$. Then

$$(1.1) \qquad \mathbb{D}_H(G) = \mathbb{D}(G)\mathfrak{h} \cap \mathbb{D}_H(G) \oplus \lambda(I(\mathfrak{m})).$$

Let $\pi\colon G \to G/H$ be the natural mapping. Let $C_H^\infty(G)$ consist of the C^∞-functions on G which are right invariant under H. Write $\tilde{f} := f\circ\pi\,(f \in C^\infty(G/H))$ and $(D_u f)^\sim := u\tilde{f}$ $(f \in C^\infty(G/H),\ u \in \mathbb{D}_H(G))$. Then $D_u \in \mathbb{D}(G/H)$.

<u>THEOREM 1.1</u>. *The mapping* $u \to D_u$ *is an algebra homomorphism from* $\mathbb{D}_H(G)$ *onto* $\mathbb{D}(G/H)$ *with kernel* $\mathbb{D}(G)\mathfrak{h} \cap \mathbb{D}_H(G)$. *The mapping* $P \to D_{\lambda(P)}\colon I(\mathfrak{m}) \to \mathbb{D}(G/H)$ *is a linear bijection.*

Theorem 1.1. is of basic importance for the analysis on symmetric spaces. In particular, it can be shown that $\mathbb{D}(G/H)$ is commutative if G/H is a pseudo-Riemannian symmetric space which admits a relatively invariant measure. In its most general form this result was proved by DUFLO [1] in an algebraic way. G. van Dijk kindly communicated a short analytic proof of Duflo's result to me (unpublished). In [1] DUFLO used generalizations of (1.1) and Theorem 1.1 to the case of homogeneous line bundles over G/H. These can be proved by only minor changes of Helgason's original proofs.

There exist non-reductive coset spaces G/H for which $\mathbb{D}(G/H)$ is still commutative. For instance, let G be a connected real semisimple Lie group and let M and N be the usual subgroups of G. Then G/MN is the space of horocycles and $\mathbb{D}(G/MN)$ is commutative. In order to prove this, formula (1.1) and Theorem 1.1 have to be adapted to the non-reductive case. While HELGASON [5, § 4], [6, § 3] has done this in an ad hoc way for the special coset spaces under consideration, it is the purpose of the present note to give a more systematic exposition of the theory of $\mathbb{D}(G/H)$ for a not necessarily reductive coset space.

Furthermore, following Duflo, the theory will be developed for invariant differential operators on homogeneous line bundles over G/H. As a final application we will characterize $\mathbb{D}(G/H)$ for isotropic pseudo-Riemannian symmetric spaces G/H without a priori knowledge that G/H is reductive. Throughout HELGASON [4] will be our standard reference.

2. DEVELOPMENT OF THE GENERAL THEORY

Let G be a Lie group with Lie algebra $\mathfrak{g}$. For $X \in \mathfrak{g}$ define the vector field $\tilde{X}$ on G by

$$(2.1)\qquad (\tilde{X}f)(g) := \frac{d}{dt} f(g \exp tX)\Big|_{t=0}, \qquad f \in C^\infty(G),\ g \in G.$$

Then the mapping $X \to \tilde{X}$ is an isomorphism from $\mathfrak{g}$ onto the Lie algebra of left invariant vector fields on G. Throughout this section let $X_1,\ldots,X_n$ be a fixed basis of $\mathfrak{g}$.

For a finite-dimensional real vector space V the symmetric algebra S(V) is defined as the algebra of all polynomials with complex coefficients on V^*, the dual of V. Let $S^m(V)$ respectively $S_m(V)$ $(m = 0,1,2,\ldots)$ denote the space of homogeneous polynomials of degree m on V^*, respectively of polynomials of degree $\le m$ on V^*. Thus $S^m(\mathfrak{g})$ is spanned by the monomials $X_{i_1}X_{i_2}\ldots X_{i_m}$ $(i_1,\ldots,i_m \in \{1,\ldots,n\})$.

Let $\mathbb{D}(G)$ be the algebra of left invariant differential operators on G with complex coefficients. For $P \in S(\mathfrak{g})$ define an operator $\lambda(P)$ on $C^\infty(G)$ by

$$(2.2)\qquad (\lambda(P)f)(g) := P\left(\frac{\partial}{\partial t_1},\ldots,\frac{\partial}{\partial t_n}\right) f(g \exp(t_1X_1+\ldots+t_nX_n))\Big|_{t_1=\ldots=t_n=0},$$

where

$$P\left(\frac{\partial}{\partial t_1},\ldots,\frac{\partial}{\partial t_n}\right) := \frac{\partial^m}{\partial t_{i_1}\ldots\partial t_{i_m}} \qquad \text{for } P = X_{i_1}\ldots X_{i_m}.$$

It is proved in [4, Prop. II.1.9 and p. 392] that:

PROPOSITION 2.1. *The mapping* $P \to \lambda(P)$ *is a linear bijection from* $S(\mathfrak{g})$ *onto* $\mathbb{D}(G)$. *It satisfies:*

$$(2.3) \qquad \lambda(Y^m) = \tilde{Y}^m, \qquad Y \in \mathfrak{g};$$

$$(2.4) \qquad \lambda(Y_1 \ldots Y_m) = \frac{1}{m!} \sum_{\sigma \in S_m} \tilde{Y}_{\sigma(1)} \ldots \tilde{Y}_{\sigma(m)}, \qquad Y_1, \ldots, Y_m \in \mathfrak{g}.$$

The definition of λ *is independent of the choice of the basis of* $\mathfrak{g}$.

The mapping λ is called *symmetrization*. The Lie algebra $\mathfrak{g}$ is embedded as a subspace of $\mathbb{D}(G)$ under the mapping $X \to \tilde{X}$. Any homomorphism from $\mathfrak{g}$ to $\mathfrak{g}$ uniquely extends to a homomorphism from $\mathbb{D}(G)$ to $\mathbb{D}(G)$ and any linear mapping from $\mathfrak{g}$ to $\mathfrak{g}$ uniquely extends to a homorphism from $S(\mathfrak{g})$ to $S(\mathfrak{g})$. In particular, for $g \in G$, the automorphism $\mathrm{Ad}(g)$ of $\mathfrak{g}$ uniquely extends to automorphisms of both $S(\mathfrak{g})$ and $\mathbb{D}(G)$ and

$$(2.5) \qquad \lambda(\mathrm{Ad}(g)P) = \mathrm{Ad}(g)\lambda(P), \qquad P \in S(\mathfrak{g}),\ g \in G.$$

For $g, g_1 \in G$, $f \in C^\infty(G)$, $D \in \mathbb{D}(G)$ write

$$f^{R(g)}(g_1) := f(g_1 g); \qquad D^{R(g)}f := (Df^{R(g^{-1})})^{R(g)}.$$

Then

$$(2.6) \qquad \mathrm{Ad}(g)D = D^{R(g^{-1})}, \qquad D \in \mathbb{D}(G),\ g \in G.$$

Let H be a closed subgroup of G and let $\mathfrak{h}$ be the corresponding subalgebra. Let $\mathfrak{m}$ be a subspace of $\mathfrak{g}$ complementary to $\mathfrak{h}$. Let $X_1, \ldots, X_r$ be a basis of $\mathfrak{m}$ and $X_{r+1}, \ldots, X_n$ a basis of $\mathfrak{h}$. Let χ be a character of H, i.e. a continuous homomorphism from H to the multiplicative group $\mathbb{C}\backslash\{0\}$. Throughout this section, H, $\mathfrak{m}$, the basis and χ will be assumed fixed.

Let $\pi: G \to G/H$ be the canonical mapping. Write $0 := \pi(e)$. Let

(2.7) $$C^{\infty}_{H,\chi}(G) := \{f \in C^{\infty}(G) \mid f(gh) = f(g)\chi(h^{-1}),\ g \in G,\ h \in H\}.$$

Sometimes we will assume that χ has an extension to a character on G. This assumption clearly holds if $\chi \equiv 1$ on H, but it does not hold for general H and χ. For instance, if $G = SU(2)$ or $SL(2,\mathbb{R})$ and $H = SO(2)$ then nontrivial characters on H do not extend to characters on G.

If χ extends to a character on G then we define a linear bijection $f \to \tilde{f}: C^{\infty}(G/H) \to C^{\infty}_{H,\chi}(G)$ by

(2.8) $$\tilde{f}(g) := f(\pi(g))\chi(g^{-1}), \qquad g \in G.$$

<u>LEMMA 2.2</u>. *Let* $P \in S(m)$. *If* $\lambda(P)f = 0$ *for all* $f \in C^{\infty}_{H,\chi}(G)$ *then* $P = 0$.

<u>PROOF</u>. For each $f \in C^{\infty}(G/H)$ we can find $F \in C^{\infty}_{H,\chi}(G)$ such that

$$F(\exp(t_1X_1+\ldots+t_rX_r)) = f(\exp(t_1X_1+\ldots+t_rX_r).0)$$

for $(t_1,\ldots,t_r)$ in some neighbourhood of $(0,\ldots,0)$. Hence

$$0 = (\lambda(P)F)(e) =$$

$$= P\left(\frac{\partial}{\partial t_1},\ldots,\frac{\partial}{\partial t_r}\right) f(\exp(t_1X_1+\ldots+t_rX_r).0)\Big|_{t_1=\ldots=t_r=0}$$

for all $f \in C^{\infty}(G/H)$, so $P = 0$. □

Let the differential of χ also be denoted by χ. Let $h^{\mathbb{C}}$ be the complexification of h. Let

(2.9) $$h^{\chi} := \{X + \chi(X) \mid X \in h^{\mathbb{C}}\} \subset \mathbb{D}(G).$$

Clearly, $Df = 0$ if $f \in C^{\infty}_{H,\chi}(G)$ and $D \in h^{\chi}$. Let $\mathbb{D}(G)h^{\chi}$ be the linear span of all vw with $v \in \mathbb{D}(G)$, $w \in h^{\chi}$. Observe that, by Proposition 2.1, $\tilde{Y}_1\ldots\tilde{Y}_m \in \lambda(S_m(g))$ for $Y_1,\ldots,Y_m \in g$. The following proposition was proved in [4, Lemma X.2.5] for $\chi \equiv 1$.

<u>PROPOSITION 2.3</u>. *There are the direct sum decompositions*

$$(2.10)\qquad \lambda(S_m(\mathfrak{g})) = \lambda(S_{m-1}(\mathfrak{g}))\mathfrak{h}^{\chi} \oplus \lambda(S_m(\mathfrak{m}))$$

and

$$(2.11)\qquad \mathbb{D}(G) = \mathbb{D}(G)\mathfrak{h}^{\chi} \oplus \lambda(S(\mathfrak{m})).$$

PROOF. First we prove by complete induction with respect to m that

$$\lambda(S_m(\mathfrak{g})) \subset \lambda(S_{m-1}(\mathfrak{g}))\mathfrak{h}^{\chi} + \lambda(S_m(\mathfrak{m})).$$

This clearly holds for $m = 0$. Suppose it is true for $m < d$. Let

$$P = X_1^{d_1}\ldots X_n^{d_n}, \qquad d_1+\ldots+d_n = d.$$

If $d_{r+1}+\ldots+d_n = 0$ then $P \in S_d(\mathfrak{m})$, so $\lambda(P) \in \lambda(S_d(\mathfrak{m}))$. If $d_{r+1}+\ldots+d_n > 0$ then, by (2.4), $\lambda(P)$ is a linear combination of certain elements $\tilde{Y}_1\ldots\tilde{Y}_d$ with $Y_i \in \mathfrak{h}$ for at least one i, so

$$\lambda(P) \in \lambda(S_{d-1}(\mathfrak{g}))\mathfrak{h}^{\mathbb{C}} + \lambda(S_{d-1}(\mathfrak{g})) \subset \lambda(S_{d-1}(\mathfrak{g}))\mathfrak{h}^{\chi} + \lambda(S_{d-1}(\mathfrak{g})).$$

Now apply the induction hypothesis. This yields (2.10) and (2.11) (use Proposition 2.1) except for the directness.

To prove the directness of the sum (2.11), suppose that $P \in S(\mathfrak{m})$ and $\lambda(P) \in \mathbb{D}(G)\mathfrak{h}^{\chi}$. Then $\lambda(P)f = 0$ for all $f \in C^{\infty}_{H,\chi}(G)$, so $P = 0$ by Lemma 2.2.□

LEMMA 2.4. *Let* $D \in \mathbb{D}(G)$. *Then* $Df = 0$ *for all* $f \in C^{\infty}_{H,\chi}(G)$ *if and only if* $D \in \mathbb{D}(G)\mathfrak{h}^{\chi}$.

PROOF. Apply Proposition 2.3 and Lemma 2.2. □

Let us define

$$(2.12)\qquad \mathbb{D}_{H,\chi,\mathrm{mod}}(G) := \{D \in \mathbb{D}(G) \mid \mathrm{Ad}(h)D-D \in \mathbb{D}(G)\mathfrak{h}^{\chi} \text{ for all } h \in H\}.$$

This definition is motivated by the following lemma.

LEMMA 2.5. *Let* $D \in \mathbb{D}(G)$. *Then the following two statements are equivalent.*

(i) $D \in \mathbb{D}_{H,\chi,\mathrm{mod}}(G)$.

(ii) $f \in C^\infty_{H,\chi}(G) \Rightarrow Df \in C^\infty_{H,\chi}(G)$.

PROOF. Let $D \in \mathbb{D}(G)$. If $f \in C^\infty_{H,\chi}(G)$, $h \in H$ then

$$(*) \qquad (Df)^{R(h)} = D^{R(h)} f^{R(h)} = \chi(h^{-1}) D^{R(h)} f.$$

First assume (i). If $f \in C^\infty_{H,\chi}(G)$, $h \in H$ then $(D^{R(h)}-D)f = (\mathrm{Ad}(h)D-D)f = 0$, so combination with (*) yields $(Df)^{R(h)} = \chi(h^{-1})Df$, i.e., $Df \in C^\infty_{H,\chi}(G)$. Conversely assume (ii). If $f \in C^\infty_{H,\chi}(G)$, $h \in H$ then $(Df)^{R(h)} = \chi(h^{-1})Df$, so combination with (*) yields $(D^{R(h)}-D)f = 0$. Hence $\mathrm{Ad}(h)D - D = D^{R(h)} - D \in \mathbb{D}(G)h^\chi$ by Lemma 2.4. □

From the preceding results the following theorem is now obvious.

THEOREM 2.6.

(a) $\mathbb{D}_{H,\chi,\mathrm{mod}}(G)$ *is a subalgebra of* $\mathbb{D}(G)$.

(b) $\mathbb{D}(G)h^\chi$ *is a two-sided ideal in* $\mathbb{D}_{H,\chi,\mathrm{mod}}(G)$.

(c) *There is the direct sum decomposition*

$$(2.13) \qquad \mathbb{D}_{H,\chi,\mathrm{mod}}(G) = \mathbb{D}(G)h^\chi \oplus \lambda(S(m)) \cap \mathbb{D}_{H,\chi,\mathrm{mod}}(G).$$

(d) *Define the mappings* A *and* B *by*

$$u \overset{A}{\to} u(\mathrm{mod}\ \mathbb{D}(G)h^\chi) \overset{B}{\to} u\Big|_{C^\infty_{H,\chi}(G)} :$$

$$\lambda(S(m)) \cap \mathbb{D}_{H,\chi,\mathrm{mod}}(G) \overset{A}{\to} \mathbb{D}_{H,\chi,\mathrm{mod}}(G)/\mathbb{D}(G)h^\chi \overset{B}{\to}$$

$$\overset{B}{\to} \mathbb{D}_{H,\chi,\mathrm{mod}}\Big|_{C^\infty_{H,\chi}(G)} .$$

Then A *is a linear bijection and* B *is an algebra isomorphism onto.*

Define the mapping $\sigma\colon g \to m$ by

$$(2.14) \qquad \sigma(X+Y) := X, \qquad X \in m,\ Y \in h.$$

Consider $S(m)$ as a subalgebra of $S(g)$. Thus, if $P \in S(m)$ and $h \in H$ then $Ad(h)P \in S(g)$ and $\sigma\circ Ad(h)P \in S(m)$ are well-defined. By an application of (2.4) we see that, if $Q \in S_m(g)$ then

$$(2.15) \qquad \lambda(\sigma Q-Q) \in \lambda(S_{m-1}(g)) + \mathbb{D}(G)h^{\chi}.$$

Define the algebra

$$(2.16) \qquad I_{mod}(m) := \{P \in S(m) \mid \sigma\circ Ad(h)P = P \text{ for all } h \in H\}.$$

<u>LEMMA 2.7</u>. *Let* $P \in S(m)$ *such that* $\lambda(P) \in \mathbb{D}_{H,\chi,mod}(G)$. *Write* $P = P^m + P_{m-1}$, *where* $P^m \in S^m(m)$, $P_{m-1} \in S_{m-1}(m)$. *Then* $P^m \in I_{mod}(m)$.

<u>PROOF</u>. $\lambda(Ad(h)P-P) \in \mathbb{D}(G)h^{\chi}$ by (2.12). Hence

$$\lambda(Ad(h)P^m-P^m) \in \lambda(S_{m-1}(g)) + \mathbb{D}(G)h^{\chi}.$$

So

$$\lambda(\sigma\circ Ad(h)P^m-P^m) \in \lambda(S_{m-1}(g)) + \mathbb{D}(G)h^{\chi} \subset \lambda(S_{m-1}(m)) + \mathbb{D}(G)h^{\chi},$$

where we used (2.16) and (2.10). By directness of the decomposition (2.10):

$$\sigma \circ Ad(h)P^m - P^m \in S_{m-1}(m).$$

Hence $\sigma \circ Ad(h)P^m - P^m$, being homogeneous of degree m, is the zero polynomial. □

<u>PROPOSITION 2.8</u>. *If* $\lambda(I_{mod}(m)) \subset \mathbb{D}_{H,\chi,mod}(G)$ *then*

$$\lambda(I_{mod}(m)) = \lambda(S(m)) \cap \mathbb{D}_{H,\chi,mod}(G)$$

and the mapping

$$D \to D\big|_{C^\infty_{H,\chi}(G)} : \lambda(I_{mod}(\mathfrak{m})) \to \mathbb{D}_{H,\chi,mod}(G)\big|_{C^\infty_{H,\chi}(G)}$$

is a linear bijection.

PROOF. Use complete induction with respect to the degree of $P \in S(\mathfrak{m})$ in order to prove that $P \in I_{mod}(\mathfrak{m})$ if $\lambda(P) \in \mathbb{D}_{H,\chi,mod}(G)$ (apply Lemma 2.7). The second implication in the proposition follows from Theorem 2.6(d). □

Suppose for the moment that χ extends to a character on G and remember the mapping $f \to \tilde{f}$ defined by (2.8). For $u \in \mathbb{D}_{H,\chi,mod}(G)$ define an operator D_u acting on $C^\infty(G/H)$ by

$$(2.17) \qquad (D_u f)^\sim := u\tilde{f}, \qquad f \in C^\infty(G/H).$$

Then $\mathrm{supp}(D_u f) \subset \mathrm{supp}(f)$, hence, by Peetre's theorem (cf. for instance NARASIMHAN [7, §3.3]), D_u is a differential operator on G/H. One easily shows that $D_u \in \mathbb{D}(G/H)$, the space of G-invariant differential operators on G/H.

THEOREM 2.9. *Suppose that* χ *extends to a character on* G. *Then the mapping*

$$u\big|_{C^\infty_{H,\chi}(G)} \overset{C}{\to} D_u : \mathbb{D}_{H,\chi,mod}(G)\big|_{C^\infty_{H,\chi}(G)} \overset{C}{\to} \mathbb{D}(G/H)$$

is an algebraic isomorphism onto.

PROOF. Clearly, C is an isomorphism into. In order to prove the surjectivity let $D \in \mathbb{D}(G/H)$. Then there is a polynomial $P \in S(\mathfrak{m})$ such that

$$(Df)(g.0) = P\left(\frac{\partial}{\partial t_1},\ldots,\frac{\partial}{\partial t_r}\right) f(g \exp(t_1X_1+\ldots+t_rX_r).0)\Big|_{t_1=\ldots=t_r=0}$$

for all $f \in C^\infty(G/H)$ and for $g = e$. By the G-invariance of D this formula holds for all $g \in G$. By (2.8) and (2.2) this becomes

$$\chi(Df)^\sim = \lambda(P)(\chi\tilde{f}), \qquad \text{i.e.}$$

$$(Df)^{\sim} = (\chi^{-1}\lambda(P)\circ\chi)(\tilde{f}).$$

Clearly, $\chi^{-1}\lambda(P)\circ\chi \in \mathbb{D}(G)$ and, by Lemma 2.5, we have

$$\chi^{-1}\lambda(P)\circ\chi \in \mathbb{D}_{H,\chi,\mathrm{mod}}(G).$$

Thus, by (2.17):

$$D = D_{\chi^{-1}\lambda(P)\circ\chi}. \qquad \square$$

Suppose now that the coset space G/H is *reductive*, i.e., m can be chosen such that $\mathrm{Ad}(h)m \subset m$ for all $h \in H$. From now on assume that m is chosen in this way. Let

(2.18) $$\mathbb{D}_H(G) := \{D \in \mathbb{D}(G) \mid \mathrm{Ad}(h)D = D \text{ for all } h \in H\},$$

(2.19) $$I(m) := \{P \in S(m) \mid \mathrm{Ad}(h)P = P \text{ for all } h \in H\}.$$

Then

$$\lambda(S(m)) \cap \mathbb{D}_{H,\chi,\mathrm{mod}}(G) = \lambda(I(m)) \subset \mathbb{D}_H(G).$$

Hence (2.13) becomes

(2.20) $$\mathbb{D}_{H,\chi,\mathrm{mod}}(G) = \mathbb{D}(G)h^{\chi} \oplus \lambda(I(m)).$$

We obtain from Theorems 2.6 and 2.9:

THEOREM 2.10. *Let* G/H *be reductive. Then:*
(a) $\mathbb{D}_H(G)$ *is a subalgebra of* $\mathbb{D}(G)$.
(b) $\mathbb{D}(G)h^{\chi} \cap \mathbb{D}_H(G)$ *is a two-sided ideal in* $\mathbb{D}_H(G)$.
(c) *There is a direct sum decomposition*

(2.21) $$\mathbb{D}_H(G) = \mathbb{D}(G)h^{\chi} \cap \mathbb{D}_H(G) \oplus \lambda(I(m)).$$

(d) *Define the mappings* A, B *and* C *(*C *only if* χ *extends to a character on* G*) by*

$$u \overset{A}{\to} u(\mathrm{mod}\ \mathbb{D}(G)h^{\chi} \cap \mathbb{D}_H(G)) \overset{B}{\to} u\big|_{C^{\infty}_{H,\chi}(G)} \overset{C}{\to} D_u:$$

$$\lambda(I(m)) \overset{A}{\to} \mathbb{D}_H(G)/(\mathbb{D}(G)h^{\chi} \cap \mathbb{D}_H(G)) \overset{B}{\to} \mathbb{D}_H(G)\big|_{C^{\infty}_{H,\chi}(G)} \overset{C}{\to} \mathbb{D}(G/H).$$

Then A *is a linear bijection and* B *and* C *are algebra isomorphisms onto.*

The case $\chi \equiv 1$ of Theorem 2.10 can be found in HELGASON [4, Cor. X.2.6 and Theor. X.2.7]. See DUFLO [1] for the general case.

3. APPLICATION TO $\mathbb{D}(G/N)$ AND $\mathbb{D}(G/MN)$

Let G be a connected noncompact real semisimple Lie group. We remember some of the structure theory of G (cf. [3, Ch. VI]):

g_0 : Lie algebra of G.
g : complexification of g_0.
θ : Cartan involution of g_0, extended to automorphism of g.
$g_0 = k_0 + p_0$: Corresponding Cartan decomposition of g_0.
h_{p_0}: maximal abelian subspace of p_0, A the corresponding analytic subgroup.
h_0 : maximal abelian subalgebra of g_0 extending h_{p_0}.
$h_{k_0} := h_0 \cap k_0$, h_k its complexification.
h : complexification of h_0; this is a Cartan subalgebra of g.
Δ : set of roots of g with respect to h; the roots are real on $ih_{k_0} + h_{p_0}$.

Introduce compatible orderings on $h_{p_0}^*$ and $(ih_{k_0} + h_{p_0})^*$.

Δ^+ : set of positive roots.
P_+ : set of positive roots not vanishing on h_{p_0}.
P_- : set of positive roots vanishing on h_{p_0}.
g^{α} : root space in g of $\alpha \in \Delta$.
n : $\Sigma_{\alpha \in P_+} g^{\alpha}$.
$n_0 := n \cap g_0$
N : analytic subgroup of G corresponding to n_0.

M : centralizer of $\mathfrak{h}_{\mathfrak{p}_0}$ in G, M_0 its identity component.
$\mathfrak{m}_0$: Lie algebra of M.
$\mathfrak{m}$: complexification of $\mathfrak{m}_0$; then

$$(3.1)\qquad \mathfrak{m} = \mathfrak{h}_{\mathfrak{k}} + \sum_{\alpha\in P_-} (\mathfrak{g}^{\alpha} + \mathfrak{g}^{-\alpha}).$$

<u>PROPOSITION 3.1</u>. *The coset spaces* G/MN *and* G/N *are not reductive.*

<u>PROOF</u>. Suppose that G/MN is reductive. Then there is an $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{m}+\mathfrak{n})$-invariant subspace $\mathfrak{r}$ of $\mathfrak{g}$ complementary to $\mathfrak{m}+\mathfrak{n}$. Let $\alpha \in P_+$ and let X be a nonzero element of $\mathfrak{g}^{\alpha}$. For $H \in \mathfrak{h}$ write $H = W_H + Y_H + Z_H$ with $W_H \in \mathfrak{r}$, $Y_H \in \mathfrak{m}$, $Z_H \in \mathfrak{n}$. Then, for each $H \in \mathfrak{h}$:

$$\alpha(H)X = [W_H + Y_H + Z_h, X]$$

so

$$\alpha(H)X - [Y_H,X] - [Z_H,X] = [W_H,X] \in \mathfrak{r} \cap (\mathfrak{m}+\mathfrak{n}),$$

so

$$[Y_H,X] + [Z_H,X] = \alpha(H)X.$$

It follows from (3.1) that

$$[Y_H,X] + [Z_H,X] \in \sum_{\substack{\beta\in\Delta\\ \beta\neq\alpha}} \mathfrak{g}^{\beta}.$$

Hence $\alpha(H)X = 0$ for all $H \in \mathfrak{h}$, so $\alpha = 0$. This is a contradiction.

In the case G/N the proof is almost the same: take $\mathfrak{r}$ $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{n})$-invariant and complemantary to $\mathfrak{n}$ and $Y_H = 0$. □

HELGASON [5, p. 676] states without proof that G/MN is not in general reductive.

Let $\mathfrak{l}_0$ be the orthoplement of $\mathfrak{m}_0$ in $\mathfrak{k}_0$ with respect to the Killing form on $\mathfrak{g}_0$. In order to apply Proposition 2.8 and Theorem 2.9 to $\mathbb{D}$(G/MN) and

$\mathbb{D}(G/N)$ we take $\ell_0 + h_{p_0}$ respectively $k_0 + h_{p_0}$ as complementary subspaces of $m_0 + n_0$ respectively n_0 in g_0. Now we have:

$$(3.2) \qquad I_{mod}(\ell_0 + h_{p_0}) = S(h_{p_0}),$$

$$(3.3) \qquad I_{mod}(k_0 + h_{p_0}) = S(m_0 + h_{p_0}).$$

(3.2) is proved in HELGASON [5, Lemma 4.2] and by only slight modifications in this proof, (3.3) is obtained. It follows from Lemma 2.5 that

$$\lambda(S(h_{p_0})) \subset \mathbb{D}_{MN,1,mod}(G)$$

and

$$\lambda(S(m_0 + h_{p_0})) \subset \mathbb{D}_{N,1,mod}(G),$$

since M centralizes h_{p_0} and $m_0 + h_{p_0}$ normalizes n_0. Consider $\mathbb{D}(A)$ and $\mathbb{D}(M_0A)$ as subalgebras of $\mathbb{D}(G)$. Then $\mathbb{D}(A) \subset \mathbb{D}_{MN,1,mod}(G)$ and $\mathbb{D}(M_0A) \subset \mathbb{D}_{N,1,mod}(G)$. It follows by application of Proposition 2.8 and Theorem 2.9 that:

THEOREM 3.2. *The mapping* $u \to D_u$ *(cf.* (2.17)*) is an algebra isomorphism from* $\mathbb{D}(A)$ *onto* $\mathbb{D}(G/MN)$ *and from* $\mathbb{D}(M_0A)$ *onto* $\mathbb{D}(G/N)$. *In particular,* $\mathbb{D}(G/MN)$ *is a commutative algebra.*

The statements about $\mathbb{D}(G/MN)$ are in HELGASON [5, Theorem 4.1]. FARAUT [2, p. 393] observes that Helgason's result can be extended to the context of pseudo-Riemannian symmetric spaces.

A special case of Theorem 6.2 can be formulated in the situation that G is a connected complex semisimple Lie group. Let g be its (complex) Lie algebra and put:

u : compact real form of g.
a : maximal abelian subalgebra of u.
$h := a + ia$; this is a Cartan subalgebra of g.
Δ : set of roots of g with respect to h.

Δ^+ : set of positive roots with respect to some ordering.

$\mathfrak{g}^\alpha$: root space of $\alpha \in \Delta$.

$\mathfrak{n} := \Sigma_{\alpha\in\Delta^+} \mathfrak{g}^\alpha$, N the corresponding analytic subgroup.

$\mathfrak{g}^{\mathbb{R}} := \mathfrak{g}$ considered as real Lie algebra.

$\mathfrak{h}^{\mathbb{R}} := \mathfrak{h}$ considered as real subalgebra

Then $\mathfrak{g}^{\mathbb{R}} = \mathfrak{u} + i\mathfrak{a} + \mathfrak{n}$ is an Iwasawa decomposition for $\mathfrak{g}^{\mathbb{R}}$ (cf. [4, Theorem VI.6.3]) and $\mathfrak{a}$ is the centralizer of $i\mathfrak{a}$ in $\mathfrak{u}$. Hence we obtain from Theorem 3.2:

THEOREM 3.3. *The mapping* $P \to D_{\lambda(P)}$ *is an algebra isomorphism from* $S(\mathfrak{h}^{\mathbb{R}})$ *onto* $\mathbb{D}(G/N)$. *In particular,* $\mathbb{D}(G/N)$ *is commutative.*

This theorem was proved by HELGASON [6, Lemma 3.3] without use of Theorem 3.2.

4. APPLICATION TO ISOTROPIC SPACES

We preserve the notation and conventions of Section 2. First we prove an extension of [4, Cor. X.2.8] to the case that G/H is not necessarily reductive. In the following, A and B are as in Theorem 2.6(d).

LEMMA 4.1. *If the algebra* $I_{\text{mod}}(\mathfrak{m})$ *is generated by* $P_1,\dots,P_\ell$ *and if there are* $Q_1,\dots,Q_\ell \in S_m$ *such that* degree $(P_i - Q_i) <$ degree P_i *and* $\lambda(Q_i) \in \mathbb{D}_{H,\chi,\text{mod}}(G)$ *then the algebra*

$$\mathbb{D}_{H,\chi,\text{mod}}\Big|_{C^\infty_{H,\chi}(G)}$$

is generated by $BA\lambda(Q_1),\dots,BA\lambda(Q_\ell)$.

PROOF. We prove by complete induction with respect to m that, for each $P \in S_m(\mathfrak{m})$ with $\lambda(P) \in \mathbb{D}_{H,\chi,\text{mod}}(G)$, $BA\lambda(P)$ depends polynomially on $BA\lambda(Q_1),\dots$ $\dots,BA\lambda(Q_\ell)$. In view of Theorem 2.6 this will prove the lemma. Suppose the above property holds up to m-1. Let $P \in S_m(\mathfrak{m})$ such that $\lambda(P) \in \mathbb{D}_{H,\chi,\text{mod}}(G)$. By using Lemma 2.7 we find that $P = \Pi(P_1,\dots,P_\ell) (\text{mod } S_{m-1}(\mathfrak{m}))$ for some polynomial Π in ℓ indeterminates. Hence, $P = \Pi(Q_1,\dots,Q_\ell)(\text{mod } S_{m-1}(\mathfrak{m}))$,

$$\lambda(P) = \lambda(\Pi(Q_1,\dots,Q_\ell))(\text{mod } \lambda(S_{m-1}(\mathfrak{m}))$$

$$= \Pi(\lambda(Q_1),\dots,\lambda(Q_\ell))(\text{mod } \lambda(S_{m-1}(\mathfrak{g})),$$

$$\lambda(P) - \Pi(\lambda(Q_1),\dots,\lambda(Q_\ell)) \in \lambda(S_{m-1}(\mathfrak{g})) \cap \mathbb{D}_{H,\chi,\text{mod}}(G).$$

By Theorem 2.6 and formula (2.10) we have

$$BA\lambda(P) - \Pi(BA\lambda(Q_1),\dots,BA\lambda(Q_\ell)) = BA\lambda(P')$$

for some $P' \in S_{m-1}(\mathfrak{m})$ such that $\lambda(P') \in \mathbb{D}_{H,\chi,\text{mod}}(G)$. Now apply the induction hypothesis. □

Let τ denote the action of G on G/H. Its differential $d\tau$ yields an action of H on the tangent space $(G/H)_0$ to G/H at 0.

<u>THEOREM 4.2</u>. *Suppose there is a nondegenerate* $d\tau(H)$*-invariant bilinear form* $\langle .,. \rangle$ *on* $(G/H)_0$ *of signature* (r_1,r_2) $(r_1+r_2 = r,\ r_1 \geq r_2)$ *such that, for each* $\lambda > 0$, $d\tau(H)$ *acts transitively on* $\{X \in (G/H)_0 \mid \langle X,X \rangle = \lambda\}$ *(or on the connected components of these hyperbolas if* $r_1 = r_2 = 1$*). Let* Δ *be the Laplace-Beltrami operator on* G/H *corresponding to the* G*-invariant pseudo-Riemannian structure on* G/H *associated with* $\langle .,. \rangle$*. Then the algebra* $\mathbb{D}(G/H)$ *is generated by* Δ*, and hence commutative.*

<u>PROOF</u>. Choose a complementary subspace $\mathfrak{m}$ to $\mathfrak{h}$ in $\mathfrak{g}$. The mapping $d\pi$ identifies the H-spaces $\mathfrak{m}$ (under $\sigma \circ Ad_G(H)$) and $(G/H)_0$ (under $d\tau(H)$) with each other. Transplant the form $\langle .,. \rangle$ to $\mathfrak{m}$ and choose an othonormal basis $X_1,\dots$ $\dots,X_r$ of $\mathfrak{m}$: $\langle X_i,X_j \rangle = \varepsilon_i\delta_{ij}$, $\varepsilon_i = 1$ or -1 for $i \leq r_1$ or $> r_1$, respectively. Then the algebra $I_{\text{mod}}(\mathfrak{m})$ is generated by $\Sigma_{u=1}^{r}\ \varepsilon_i X_i^2$. It follows from the proof of Theorem 2.9 that $\Delta = D_{\lambda(P)}$ with $P \in S(\mathfrak{m})$ of degree 2 such that $\lambda(P) \in \mathbb{D}_{H,1,\text{mod}}(G)$. Thus, by Lemma 2.7, we get

$$P = c \sum_{i=1}^{r} \varepsilon_i X_i^2 (\text{mod } S_1(\mathfrak{m}))$$

with $c \neq 0$. Now apply Lemma 4.1 and Theorem 2.9. □

Theorem 4.2 extends [4, Prop. X.2.10], where the case is considered that G/H is a Riemannian symmetric space of rank 1. A pseudo-Riemannian manifold M is called *isotropic* if for each $x \in M$ and for tangent vectors $X,Y \neq 0$ at x with $\langle X,X \rangle = \langle Y,Y \rangle$ there is an isometry of M fixing x which sends X to Y. Connected isotropic spaces can be written as homogeneous spaces G/H satisfying the conditions of Theorem 4.2 with G being the full isometry group (cf. WOLF [8, Lemma 11.6.6]). It follows from Wolf's classification [8, Theorem 12.4.5] that such spaces are symmetric and reductive. However, our proof of Theorem 4.2 does not use this fact.

REFERENCES

[1] DUFLO, M., *Opérateurs différentiels invariants sur un espace symétrique*, C.R. Acad. Sci. Paris Sér. A 289 (1979), 135-137.

[2] FARAUT, J., *Distributions sphériques sur les espaces hyperboliques*, J. Math. Pures Appl. 58 (1979), 369-444.

[3] HELGASON, S., *Differential operators on homogeneous spaces*, Acta Math. 102 (1959), 239-299.

[4] HELGASON, S., *Differential geometry and symmetric spaces*, Academic Press, New York, 1962.

[5] HELGASON, S., *Duality and Radon transform for symmetric spaces*, Amer. J. Math. 85 (1963), 667-692.

[6] HELGASON, S., *Invariant differential operators and eigenspace representations*, pp. 236-286 in: *Representation theory of Lie groups*, (M. Atiyah e.a.), Cambridge University Press, Cambridge, 1979.

[7] NARASIMHAN, R., *Analysis on real and complex manifolds*, Masson, Paris, 1968.

[8] WOLF, J.A., *Spaces of constant curvature*, McGraw-Hill, New York, 1967.