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A.E. BROUWER

SOME UNITALS ON 28 POINTS AND THEIR EMBEDDINGS IN PROJECTIVE PLANES OF ORDER 9

kruislaan 413 1098 SJ amsterdam

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Some unitals on 28 points and their embeddings in projective planes of order 9

by

A.E. Brouwer

ABSTRACT

We answer three questions posed by F. Piper by exhibiting

(i) a unital that is not embeddable in a projective plane (in fact 76 non-isomorphic ones),

(ii) a unital which is embeddable, and isomorphic with its dual, but not the set of absolute points of a polarity (in fact examples exist in each of the four known projective planes of order 9),

(iii) a unital that can be embedded in two different planes.

KEY WORDS & PHRASES: *unital*

INTRODUCTION

A unital is a $2-(q^3+1,q+1,1)$ design. The classical unital with these parameters is obtained as the set of absolute points and nonabsolute lines of a unitary polarity of the projective plane $PG(2,q^2)$. Its full automorphism group is $P\Gamma U(3,q^2)$ (see O'NAN [10]). O'Nan proved that it does not contain a configuration of four blocks intersecting in six points; conversely one may ask whether any unital without "O'Nan configurations" is necessarily the classical one. This is probably true; apart from the trivial case q = 2 where the unique unital is the affine plane AG(2,3) (which indeed is free of O'Nan configurations) we have

<u>PROPOSITION</u>. A unital $2-(28,4,1)$ without O'Nan configurations is the classical one (q=3).

<u>PROOF</u>. Exhaustive computer search.

F. PIPER wrote in [11] a survey on unitals and posed several questions. To each of these questions the answer is 'no' - there exist hordes of 'ugly' unitals, counterexamples to anything you might conjecture. Even more is true: requiring that the unital be embeddable in a projective plane does not make it much nicer.

The procedure followed to obtain these results was the following:

a) construct some random unitals by hand (assuming a given small group of automorphisms);
b) do a computer search for all unitals on 28 points with an automorphism of order 7.
c) given any unital, construct all possible new ones by deleting a line so that a (3-line)-resolvable pairwise balance design results, finding a resolution of the 3-lines distinct from the given one and adding points at infinity again.

(I don't think it is feasible to search exhaustively for unitals on 28 points without any restriction, but it should be possible to obtain all these designs with a nontrivial automorphism group.)

Below we list base blocks for the designs found under a) and b) and give statistics on all designs found. We did not attempt to distinguish between designs on list c) with the same statistics.

A. Some random unitals on 28 points

1. Let $X = (\mathbb{Z}_3)^3 \cup \{\infty\}$ and take the blocks

$$\{011,021,102,202\},\ \{211,121,222,112\} \bmod(3,3,3), \text{ and}$$
$$\{\infty,000,001,002\} \ (\bmod\ 3,3,-)$$

This design is uniquely resolvable: one parallel class (replication) is obtained by taking the base blocks mod(-,-,3). In fact it has 9 spreads (forming the resolution), and 54 maximal almost-spreads (i.e., sets of six pairwise disjoint lines). Its automorphism group has order 216. It is uniquely embeddable in a projective plane PG(2,9), and this is the desarguesian plane. It is isomorphic to its dual, but not the set of absolute points of a polarity of the plane. Nine of its blocks are Baer-sublines (i.e., intersections of a projective line with a Baer-subplane). By the results of Bruck (any three points on a line are contained in a Baer-subplane and determine a unique Baer-subline) and LEFEVRE-PERCY [7] (the Buekenhout-Metz unitals in a desarguesian plane of order $q^2 > 4$ are exactly those such that for some tangent line I to the unital U all Baer-sublines that meet I intersect U in 0, 1, 2 or q+1 points) it follows that we have a Buekenhout-Metz unital (- the tangent I is the tangent at ∞).

This design seems to be the most popular one in the literature - I found it in my report [1], but HALL [4], HANANI, RAY-CHANDHURI & WILSON [5] and RAGHAVARAO [12] give isomorphic designs, remarking that it is a special case of a general construction for resolvable designs 2-(12t+4,4,1) when 4t+1 is a prime power.

The only other explicit design 2-(28,4,1) I could find in the literature (note that the entry in Takeuchi's list is incorrect) occurs in my report [1] and (an isomorphic version) in HANANI [4]. Strange enough it is not the classical unital, but the Ree unital (cf. KANTOR [6], p. 51,59, LÜNEBURG [8]). It is listed below under 3.

2. There are three designs S(2,4,28) with an automorphism of order 12 with point orbits 28 = 1 + 3 + 12 + 12. One is the classical unital, and the other two are dual - one embeddable in the translation plane and the other in the dual translation plane.

2A. The classical unital. Let $X = (\mathbb{Z}_{12} \times I_2) \cup (\{\infty\} \times \mathbb{Z}_3) \cup \{\Omega\}$ and take the blocks

(i) $\{\infty 0, \infty 1, \infty 2, \Omega\}$: 1 block

(ii) $\{0j, 4j, 8j, \Omega\}$ mod 12, j = 0,1: 8 blocks

(iii) $\{0j, 3j, 6j, 9j\}$ mod 12, j = 0,1: 6 blocks

(iv) {00,10,01,21}, {00,20,51,A1} mod 12: 24 blocks (here A = 10)

(v) $\{00, 50, 91, \infty 0\}$, $\{00, 61, 71, \infty 2\}$ mod 12: 24 blocks

This design is resolvable in 28 ways: for each tangent I at some point x of the unital U we find a resolution where the nine parallel classes are determined by the nine points of $I\backslash\{x\}$; each point y outside the unital is incident with 6 secants and 4 tangents, and the four points of intersection of the tangents with the unital form a block (namely $U \cap y^{\perp}$), so that y determines 7 pairwise disjoint blocks altogether. These are all the spreads (63 in total) and any set of 5 pairwise disjoint blocks is extendable to a spread. It is uniquely embeddable in a plane PG(2,9), and this is the desarguesian plane. It is isomorphic to its dual: $x \mapsto x^{\perp}$ defines an isomorphism. Its automorphism group is $P\Gamma U(3,3^2)$ of order 12096; it is doubly transitive on the points of U. Each of its blocks is a Baer-subline.

2B. Take the same pointset, and besides the blocks (i) - (iii) also

(iv)' {00,10,01,21}, {00,20,71,81} mod 12,

(v)' $\{00, 50, 91, \infty 0\}$, $\{00, 31, A1, \infty 2\}$ mod 12.

This design is uniquely resolvable, and uniquely embeddable in a plane PG(2,9) - it is a translation plane but not desarguesian, hence (cf. [8], p. 36) the plane Ω of [13]. Its dual is 2C. It has 15 spreads and 48 maximal almost-spreads. The automorphism group has order 48.

2C. Take the same pointset, and besides the blocks (i) - (iii) and (iv)' also

(v)" {00,50,31,∞0}, {00,41,91,∞2} mod 12.

This design is uniquely resolvable and uniquely embeddable in a plane - it is the dual translation plane Ω^d. Its dual is 2B. It has 15 spreads and 120 maximal almost-spreads. The automorphism group has order 48.

3. Let $X = I_4 \times \mathbb{Z}_7$ and take the blocks
{01,02,11,14}, {02,04,21,22}, {00,03,30,32}, {11,12,21,24},
{11,16,30,34}, {22,24,30,31}, {05,12,26,36}, {00,15,23,35},
{00,11,22,33} all mod (-17).

This design is resolvable in 10 ways, and has 45 spreads (any two resolutions having exactly one spread in common). There are no maximal almost-spreads, but an embeddable unital must possess at least 63 almost-spreads. Consequently this unital is not embeddable in a plane. Its group is PΓL(2,8) of order 1512; it is doubly transitive on the points. In fact this is the smallest member in the family of Ree-unitals.

B. Designs S(2,4,28) with an automorphism of order 7

#	s_3	s_4	s_5	s_6	s_7	r	aut	pt. orbits	bl. orbits	distr.
0	-	2016	-	-	63	28	12096	2-tra	tra	63*(40,26)
1	-	504	756	-	45	10	1512	2-tra	tra	
2	-	525	945	91	-	-	42	21+7	3*21	
3	-	924	903	63	-	-	21	21+7	3*21	
4	-	672	924	91	-	-	21	21+7	3*21	
5	-	1155	945	28	-	-	21	21+7	3*21	21*(7,1)+21*(8,1)
6	-	693	903	84	-	-	21	21+7	3*21	
7	-	910	924	56	-	-	7	4*7	9*7	7*(5,1)

There are 8 designs S(2,4,28) with an automorphism of order 7. In the above table we list their serial number, the number s_i of maximal sets of pairwise disjoint blocks of size i (i.e., s_i = # of maximal i-spreads) for $3 \le i \le 7$, the number of r resolutions, the order of the group of automorphisms, the partitioning of points and blocks into orbits, and for each block the number of 8-spreads and resolutions of the set of 32 triples on 24 points obtained by deleting the points of that block from the pointset and taking the triples arising from blocks intersecting the given block (cf. Section C). Entries (4,1) are omitted.

Of the above designs we already saw the 2-transitive ones. It turns out that none of the others is transitive. There is no cyclic S(2,4,28) - i.e., none is invariant under a 28-cycle. Only #0 is embeddable in a projective plane PG(2,9).

Below we list 9 base blocks for each of the designs, where the pointset is taken as $I_{28} = \{0,1,\ldots,27\}$ and the automorphism of order 7 acts as (0 1...6)(7 8...13)(14 15...20)(21 22...27).

#0	#1	#2	#3
0 1 7 9	0 1 7 9	0 1 7 9	0 1 7 9
0 2 12 14	0 2 14 15	0 2 12 14	0 2 14 17
0 3 18 21	0 3 21 22	0 3 21 23	0 3 21 22
0 11 26 27	0 10 18 23	0 11 15 18	0 10 11 20
0 16 17 23	0 11 17 27	0 16 17 24	0 12 23 27
0 20 22 24	0 12 16 24	0 20 22 26	0 16 18 24
7 8 21 25	7 8 22 25	7 8 20 24	7 10 18 23
7 10 15 20	7 10 17 19	7 10 21 22	7 14 20 24
7 14 18 26	14 17 21 23	7 15 17 27	7 19 21 26

#4	#5	#6	#7
0 1 7 9	0 1 7 9	0 1 7 9	0 1 7 9
0 2 14 17	0 2 14 17	0 2 14 21	0 2 14 21
0 3 21 22	0 3 21 22	0 3 16 18	0 3 16 18
0 10 11 20	0 10 11 24	0 10 11 25	0 10 11 25
0 12 23 27	0 12 16 18	0 12 23 24	0 12 23 24
0 16 18 24	0 20 23 27	0 17 22 27	0 17 22 27
7 10 18 27	7 10 17 22	7 10 16 17	7 10 16 17
7 14 21 26	7 15 16 24	7 15 18 24	7 15 19 23
7 19 20 23	7 19 23 25	7 19 23 27	7 18 24 27

C. Families of adjacent unitals

Given a S(2,4,28), that is, a B(4;28) design, we obtain a B({3,4};24) with 32 blocks of size 3 by deleting a block. In this new design the set of triples is resolvable into four 8-spreads, and by adding 4 points 'at infinity' we get a B(4;28) back again. But perhaps the set of triples was resolvable in more than one way; in this case we might obtain new unitals. I determined the families of the unitals constructed under A and B. The statistics follow:

From A.1:

#	s3	s4	s5	s6	s7	r	autom	distr.
1.0	72	1152	648	54	9	1	216	9*(16,2)
1.1	0	1608	488	94	1	0	24	1*(16,2)

From A.2C: (note that A.2B is #2.38)

#	s3	s4	s5	s6	s7	r	autom	distr.
2.0	0	1032	444	120	15	1	48	1*(16,2)+8*(22,2)+6*(12,7)
2.1	0	1113	710	82	4	0	6	3*(6,2)+3*(8,2)+1*(22,2)
2.2	0	816	716	88	7	0	48	1*(16,2)
2.3	0	1030	670	96	5	0	2	1*(10,2)+2*(9,3)+2*(8,4)+1*(12,7)
2.4	0	1044	830	44	7	0	4	2*(8,4)+1*(12,7)
2.5	0	959	866	66	2	0	2	3*(6,2)+1*(10,2)
2.6	0	1255	764	51	3	0	3	2*(8,2)+1*(18,2)
2.7	0	1258	731	54	6	0	6	3*(8,2)
2.8	0	1062	751	83	1	0	2	2*(9,3)+1*(18,3)+1*(12,7)
2.9	0	1137	730	71	4	0	2	1*(8,2)+1*(10,2)+2*(9,3)
2.10	0	1050	759	83	1	0	1	1*(7,2)+1*(8,2)+2*(8,4)
2.11	0	1006	862	57	1	0	1	1*(7,2)+2*(8,4)
2.12	0	996	928	32	3	0	4	1*(6,2)+2*(8,4)
2.13	0	961	834	73	2	0	2	3*(6,2)+2*(7,2)+1*(10,2)
2.14	0	1223	746	56	5	0	3	2*(8,2)
2.15	0	1325	630	63	8	0	6	3*(8,2)+4*(18,2)+1*(22,2)
2.16	0	1128	763	73	1	0	2	2*(7,2)+1*(12,7)
2.17	0	1117	850	49	1	0	1	1*(13,2)+1*(12,7)
2.18	0	1249	816	35	5	0	2	1*(12,7)
2.19	0	1042	861	53	1	0	2	2*(7,2)+1*(12,7)
2.20	0	1056	803	69	1	0	2	2*(8,2)+1*(16,3)+1*(12,7)
2.21	0	995	920	48	0	0	1	1*(8,3)+1*(18,3)
2.22	0	940	976	24	3	0	4	1*(6,2)
2.23	0	1053	692	91	4	0	6	3*(6,2)+1*(22,2)+2*(11,4)
2.24	0	987	902	57	0	0	1	1*(6,2)+1*(7,2)+1*(8,2)
2.25	0	1069	869	33	6	0	6	1*(18,2)
2.26	0	1201	749	57	6	0	6	1*(8,2)+1*(18,2)
2.27	0	1092	870	52	0	0	1	1*(13,2)+1*(8,3)
2.28	0	934	934	50	0	0	1	1*(8,3)+1*(16,3)
2.29	0	1033	823	67	1	0	1	1*(6,2)+1*(9,3)+1*(11,4)

(cont'd)

#	s3	s4	s5	s6	s7	r	autom	distr.
2.30	0	1046	885	55	0	0	2	1*(6,2)+2*(8,2)
2.31	0	1248	638	85	4	0	12	6*(8,2)
2.32	0	1072	831	60	1	0	2	2*(9,3)
2.33	0	1070	825	65	0	0	2	1*(6,2)+2*(9,3)
2.34	0	1101	791	67	1	0	3	3*(8,2)+1*(14,2)
2.35	0	1278	593	85	4	0	6	2*(8,2)+1*(16,2)
2.36	0	1204	677	84	3	0	3	1*(14,2)
2.37	0	1194	788	58	1	0	3	2*(8,2)
2.38	0	1440	516	48	15	1	48	9*(16,2)
2.39	0	1344	782	46	1	0	12	4*(8,2)
2.40	0	1152	692	64	7	0	48	1*(16,2)

<u>From A.2A</u>: (the classical unital):

#	s3	s4	s5	s6	s7	r	autom	distr.
3.0	0	2016	0	0	63	28	12096	63*(40,26)
3.1	0	1448	448	88	11	0	8	1*(8,2)+8*(15,2)+1*(20,2)+4*(12,3)+ +8*(14,3)+8*(16,4)+4*(28,8)+ +1*(40,26)
3.2	0	1440	320	64	31	0	192	24*(16,5)+6*(24,6)+1*(40,26)
3.3	0	1132	704	68	7	0	8	1*(8,2)+1*(20,2)+1*(24,6)
3.4	0	1279	745	51	3	0	1	1*(8,2)+4*(9,2)+1*(12,3)+1*(28,8)
3.5	0	1388	683	56	4	0	2	1*(11,3)+1*(14,3)+1*(28,8)
3.6	0	1216	716	60	5	0	2	1*(16,5)+1*(28,8)
3.7	0	1290	691	64	4	0	2	2*(9,2)+2*(10,2)+2*(11,2)+1*(14,3)+ +1*(28,8)
3.8	0	1274	739	48	5	0	2	2*(8,2)+2*(9,2)+4*(10,2)+2*(11,2)+ +2*(12,3)+2*(28,8)
3.9	0	1072	712	72	7	0	16	3*(8,2)
3.10	0	1228	723	69	1	0	1	1*(15,2)+1*(16,4)
3.11	0	1210	717	67	2	0	1	1*(8,2)+1*(9,2)+2*(10,2)+1*(11,2)+ +1*(12,2)+1*(12,3)+1*(14,3)
3.12	0	1257	716	67	1	0	1	1*(10,2)+1*(11,2)+1*(16,4)
3.13	0	1215	705	72	1	0	1	3*(9,2)+1*(10,2)+1*(12,2)+1*(16,4)
3.14	0	1120	768	32	15	0	64	5*(8,2)+2*(24,6)
3.15	0	808	760	80	7	0	16	3*(8,2)
3.16	0	1143	838	34	7	0	2	2*(8,2)+2*(12,2)
3.17	0	1202	846	42	0	0	1	2*(9,2)+1*(10,2)
3.18	0	1210	822	49	0	0	1	2*(9,2)
3.19	0	1168	854	42	1	0	1	1*(9,2)+1*(12,3)
3.20	0	1257	856	33	1	0	1	1*(9,2)+1*(12,3)
3.21	0	1192	889	33	0	0	1	1*(10,2)+1*(11,2)+1*(11,3)
3.22	1	1242	816	36	3	0	3	3*(11,2)
3.23	0	1163	798	57	1	0	1	1*(9,2)+1*(10,2)
3.24	0	1114	841	52	1	0	1	1*(9,2)+2*(10,2)
3.25	0	1030	862	52	1	0	1	2*(8,2)+1*(10,2)

(cont'd)

#	s3	s4	s5	s6	s7	r	autom	distr.
3.26	0	1121	862	48	0	0	1	1*(9,2)+1*(11,2)
3.27	0	1102	880	42	1	0	1	1*(9,2)+1*(12,2)
3.28	0	1120	916	30	1	0	1	1*(10,2)+1*(12,3)
3.29	0	1106	897	38	1	0	1	1*(10,2)+1*(11,2)+1*(12,3)
3.30	0	1088	902	37	1	0	1	1*(10,2)+1*(12,2)
3.31	0	1143	862	42	1	0	2	3*(10,2)
3.32	0	1163	863	44	0	0	2	1*(10,2)
3.33	0	1040	896	32	7	0	32	3*(8,2)
3.34	0	480	1152	0	15	0	192	15*(8,2)
3.35	0	688	992	48	7	0	32	3*(8,2)

Each of the other unitals turned out to form a family with a single member.

D. Embeddings in a projective plane

For each of the unitals mentioned their embeddings into projective planes PG(2,9) were determined. The lines of the plane are the blocks of the unital and the tangents - one for each point. The points of the plane are the points of the unital and certain 6-spreads. In fact, through an exterior point pass six secants and four tangents, so any exterior point determines a unique 6-spread in the unital (perhaps contained in a 7-spread). Therefore a necessary condition for embeddability is that the total number of 6-spread is at least 63 (even: that $s_6 + s_7 \geq 63$). But this is by no means sufficient. Of the above 86 unitals 41 satisfy $s_6 + s_7 \geq 63$ but only 10 are embeddable. Looking at the duals of embedded unitals I found an eleventh one (with s[3:7] = (0,936,656,118,1)); its family is a singleton.

#	dual	s_6	s_7	r	aut	pt. orbits	bl. orbits	plane	#Baer sublines
E.0=1.0	selfdual	54	9	1	216	27+1	54+9	Des.	9
E.1=1.1	E.2	94	1	0	24	24+4	2*24+8+6+1	dual tr.	
E.2	E.1	118	1	0	24	24+4	2*24+8+6+1	tr.	
E.3=2.0	E.4	120	15	1	48	24+3+1	2*24+8+6+1	dual tr.	
E.4=2.38	E.3	48	15	1	48	24+3+1	2*24+8+6+1	tr.	
E.5=2.23	E.6	91	4	0	6	$6^3 3^2 2 1^2$	$6^6 3^6 2^3 1^3$	Hughes	
E.6=2.35	E.5	85	4	0	6	$6^3 3^2 2 1^2$	$6^6 3^6 2^3 1^3$	Hughes	
E.7=2.36	selfdual	84	3	0	3	$3^8 1^4$	$3^{18} 1^9$	Hughes	
E.8=2.40	selfdual	64	7	0	48	24+4	2*24+8+6+1	Hughes	
E.9=3.0	selfdual	0	63	28	12096	2-tra	tra	Des.	63
E.10=3.2	selfdual	64	31	0	192	24+4	32+24+6+1	tr./dual tr.	

In fact 10 were uniquely embeddable, while the last one could be extended in exactly two ways, one yielding the translation plane and the other the dual translation plane. Note that we found several inequivalent unitals in each of the known planes of order 9. I believe that at least those in the Hughes plane are new.

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