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A CHARACTERIZATION OF THE CLASSICAL UNITALS

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A characterization of the classical unitals *)

by

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ABSTRACT

A characterization of the classical unitals is given in terms of certain geometrical properties.

KEY WORDS & PHRASES: *Unital, inversive plane, generalized quadrangle, Minkowski plane*

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*) This report will be submitted for publication elsewhere.

1. INTRODUCTION

A *unital* or *unitary block design* is a $2-(q^3+1,q+1,1)$ design, i.e. an incidence structure of q^3+1 points, $q^2(q^2-q+1)$ lines, such that each line contains $q+1$ points and any two distinct points are on a unique line. If q is a prime power, the absolute points and non-absolute lines of a unitary polarity of $PG(2,q^2)$ form a unital (see [2]). These unitals are called *classical*.

In [6], O'NAN showed that a classical unital satisfies the following condition.

(I) No four distinct lines intersect in six distinct points (see Figure 1).

No:

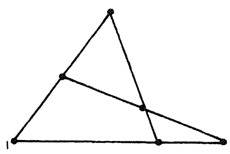

Fig. 1

In [5], PIPER conjectured that this property characterizes the classical unitals. Here we shall give a characterization for even q under the assumption that also the following condition holds.

(II) If L is a line, x a point not on L, M a line through x meeting L and and $y \neq x$ a point on M, then there exists a line $L' \neq M$ through y intersecting all lines through x which meet L.

To achieve this result we shall give another characterization for all q under the additional assumption that a third condition holds. To formulate this condition we need some notation. If x and y are distinct points, then we denote by xy the line through x and y. Given a point x, two lines L and L' missing x are called *x-parallel* (notation $L \parallel_x L'$) if and only if they intersect the same lines through x. Clearly, $\parallel_x$ is an equivalence relation on the set of lines missing x, and by (I) and (II), each equivalence class consists of q disjoint lines. Our third condition now reads as follows.

(III) Given a point x, three distinct lines M_1, M_2, M_3 through x and points y_i, z_i on M_i $(i = 1,2,3)$ such that $(y_1y_2) \|_x (z_1z_2)$ and $(y_1y_3) \|_x (z_1z_3)$, then also $(y_2y_3) \|_x (z_2z_3)$.

Clearly, the presence of unitary transvections in PΓU(3,q) implies that the classical unitals satisfy conditions (II) and (III). In Section 2 we shall study unitals satisfying (I) and (II). Section 3 is devoted to the proof that unitals satisfying all three conditions are classical. Finally, in Section 4, we shall show that for even q, (III) is a consequence of (I) and (II).

2. UNITALS SATISFYING (I) AND (II)

Throughout this section $\mathcal{U}$ is a unital on $q^3 + 1$ points with point set X and line set $\mathcal{L}$ satisfying (I) and (II) above. If $x \in X$, then we denote by $\mathcal{L}^x$ the set of lines incident with x, and $\mathcal{L}_x$ will be the set of lines missing x. Furthermore, $\mathcal{C}_x$ will stand for the set of $\|_x$-equivalence classes on $\mathcal{L}_x$. From [1] it is clear that we want to show that the incidence structure which has $\mathcal{L}^x$ as the set of points, $\mathcal{C}_x$ as the set of blocks and LIC $(L \in \mathcal{L}^x, C \in \mathcal{C}_x)$ iff L meets one (hence all) lines of C, is the residual of an inversive plane of order q. We denote this incidence structure by $\mathcal{I}^*(x) = (\mathcal{L}^x, \mathcal{C}_x)$. Clearly, $\mathcal{I}^*(x)$ is a $2-(q^2,q+1,q)$ design.

<u>LEMMA 1</u>. *If* $x \in X$ *and* $L, L' \in \mathcal{L}_x$ *such that* L *and* L' *both meet three distinct lines* $M_1, M_2, M_3 \in \mathcal{L}^x$, *then* $L \|_x L'$, *i.e. three distinct points of* $\mathcal{I}^*(x)$ *are in at most one block of* $\mathcal{I}^*(x)$.

<u>PROOF</u>. Let $y \in M_1 \cap L'$ and let L" be the line through y which is x-parallel to L, then L' ≠ L" contradicts (I). □

If M and M' are two distinct lines through a point x, then an easy counting argument shows that there are q −2 lines $N_1,\ldots,N_{q-2}$ through x such that no line of $\mathcal{L}_x$ meets M, M' and N_i, $i = 1,\ldots,q-2$. Put $C^*(M,M') := \{M_1M'\} \cup \{N_1,N_2,\ldots,N_{q-2}\}$. We have to show that the $C^*(M,M')$ correspond to circles which will make $\mathcal{I}^*(x)$ into an inversive plane. We have to show that $N,N' \in C^*(M,M') \Rightarrow C^*(M,M') = C^*(N,N')$. Clearly, $C^*(M,M') = C^*(M',M)$ and so it suffices to show that $M'' \in C^*(M,M') \Rightarrow C^*(M,M') = C^*(M,M'')$. This is the contents of the next lemma.

<u>LEMMA 2</u>. *Fix a line* $M \in L$ *and two distinct points* x *and* y *in* M. *For* $M', M'' \in L^x\backslash\{M\}$ *write* $M' \sim M''$ *iff no line of* $L^y\backslash\{M\}$, *intersects both* M' *and* M'' *or* $M' = M''$. *Then* $\sim$ *is an equivalence relation on* $L^x\backslash\{M\}$.

<u>PROOF</u>. For $u,v \in M$ let $A^*(u,v)$ be the incidence structure with $L^u\backslash\{M\}$ as points, $L^v\backslash\{M\}$ as lines, and incidence defined by PIB iff P and B meet ($P \in L^u\backslash\{M\}$, $B \in L^v\backslash\{M\}$). If u, v, w are distinct points of M, then clearly the mapping $\tilde{\tau}^u_{v,w}: A^*(u,v) \to A^*(u,w)$ defined by

$$\tilde{\tau}^u_{v,w}(P) := P, \qquad P \in L^u\backslash\{M\},$$

$$\tilde{\tau}^u_{v,w}(B) := \text{u-parallel of B through w}, \qquad B \in L^v\backslash\{M\},$$

is an isomorphism of $A^*(u,v)$ onto $A^*(u,w)$. Now fix $x,y \in X$, $x \neq y$. If $q > 2$ and u, v are distinct points in M, $u,v \neq x,y$, then

$$\tilde{\delta}^{x,y}_{u,v} := \tilde{\tau}^y_{v,x}\, \tilde{\tau}^v_{u,y}\, \tilde{\tau}^u_{x,v}\, \tilde{\tau}^x_{y,u},$$

is an automorphism of $A^*(x,y)$.

Now we claim that

(i) For all $u,v \in M\backslash\{x,y\}$, $u \neq v$ and for all $P \in L^x\backslash\{M\}$, $\tilde{\delta}^{x,y}_{u,v}(P) \neq P$ and $\tilde{\delta}^{x,y}_{u,v}(P) \sim P$.

(ii) For all $u,v,v' \in M\backslash\{x,y\}$, $u \neq v \neq v' \neq u$ and for all $P \in L^x\backslash\{M\}$, $\tilde{\delta}^{x,y}_{u,v}(P) \neq \tilde{\delta}^{x,y}_{u,v'}(P)$ and $\tilde{\delta}^{x,y}_{u,v}(P) \sim \tilde{\delta}^{x,y}_{u,v'}(P)$.

To prove these claims, write $u\,P\,v$ for the u-parallel of P incident with v. Then $\tilde{\delta}^{x,y}_{u,v}(P) = y(u\,P\,v)x$. Suppose $y(u\,P\,v)x \sim P$ or $y(u\,P\,v)x = P$. Then there is a line L incident with x intersecting P and $y(u\,P\,v)x$. Then L intersects $u\,P\,v$ in a point a, say. Since au intersects P, we now have an O'Nan configuration on the lines M, P, L and au, contradicting (I).

Suppose $y(u\,P\,v)x \sim y(u\,P\,v')x$ or $y(u\,P\,v)x = y(u\,P\,v')x$. Let L be the line through y intersecting $y(u\,P\,v)x$ and $y(u\,P\,v')x$. Then L intersects $u\,P\,v$ and $u\,P\,v'$ in points a and a', say. Since au intersects $u\,P\,v'$, we have an O'Nan configuration on M, L, $u\,P\,v'$ and au, again in contradiction with (I).

For a given $P \in L^x \backslash \{M\}$, there are $q-2$ $Q \in L^x \backslash \{M\}$, $Q \neq P$ such that $Q \sim P$. Fixing u we can make $q-2$ choices for $v \in M \backslash \{x,y,u\}$. Thus, each $Q \in L^x \backslash \{M\}$, $Q \neq P$ can be written as $Q = \tilde{\delta}^{x,y}_{u,v}(P)$. If $Q = \tilde{\delta}^{x,y}_{u,v}(P) \sim P$ and $Q' = \tilde{\delta}^{x,y}_{u,v'}(P) \sim P$, then $Q \sim Q'$ by (ii). □

Lemma 2 and its proof have a number of important corollaries.

COROLLARY 3. *Let $x \in X$ and let ∞_x be a new symbol. Put*

$$C^x := \{c^*(M,M') \cup \{\infty_x\} \mid M,M' \in L^x,\ M \neq M'\}.$$

Then $I(x) := (L^x \cup \{\infty_x\},\ C^x \cup C_x)$ is an inversive plane of order q with point set $L^x \cup \{\infty_x\}$ and block set $C^x \cup C_x$ and incidence defined in the obvious way.

PROOF. See the discussion preceding Lemma 2. □

COROLLARY 4. *For $x,y \in X$, $x \neq y$, the incidence structure $A^*(x,y)$ of Lemma 2 is isomorphic to the derived design $I(x)^{xy}$ with ∞_x and the lines through ∞_x removed. The affine plane $I(x)^{xy}$ admits a dilatation group of order $q-1$ with centre ∞_x.*

PROOF. The automorphisms $\tilde{\delta}^{x,y}_{u,v}$ of $A^*(x,y)$ induce $q-2$ distinct nonidentity dilatations with centre ∞_x on $I(x)^{xy}$. Since $I(x)^{xy}$ has order q, these are the non-identity elements of the dilatation group with centre ∞_x of order $q-1$.□

COROLLARY 5. *Let $L \in L$ and let $x_1,x_2,\ldots,x_{q+1}$ be the points on L. It is possible to partition the set of lines which meet L into classes A_{ij}, $1 \le i,j \le q+1$, such that for all i and j*

(i) $|A_{ij}| = q-1$

(ii) $M \in A_{ij} \Rightarrow x_i \in M$,

(iii) every point $x \in X\backslash L$ is on exactly one line of $\bigcup_k A_{kj}$,

(iv) no line which meets L in a point $\neq x_i$, meets two lines of A_{ij},

(v) for all k, i', $1 \le k,i' \le q+1$, $k \neq i,i'$ and for all $M \in A_{ij}$, the x_k-parallel of M through $x_{i'}$ is in $A_{i'j}$,

(vi) if $1 \le i' \le q+1$, $i' \neq i$ and $M \in A_{ij}$, $M' \in A_{i',j}$ then there exists a unique $k \in \{1,\ldots,q+1\}$ such that M and M' are x_k-parallel.

<u>PROOF</u>. Consider $\mathcal{I}(x_1)$. Number the circles of $\mathcal{I}(x_1)$ through the two points ∞_{x_1} and L of $\mathcal{I}(x_1)$ from 1 upto q+1. Apart from ∞_{x_1} and L each such circle contains (q-1) lines through x_1. These will be the sets $A_{1,j}$, $j = 1,\dots,q+1$. For $i > 1$ and $1 \le j \le q+1$, let $A_{i,j}$ consist of the (q-1) lines through x_i which in $\mathcal{I}(x_1)$ correspond to the (q-1) circles (not through ∞_{x_1}) in the pencil with carrier L and which contains circle j through ∞_{x_1} and L. Now (i) and (ii) are trivially satisfied. For (iii), note that the q+1 lines xx_i, $i = 1,\dots,q+1$ are in A_{ij}'s with distinct j since the circles in a pencil with carrier L partition the set of points $\neq$ L of $\mathcal{I}(x_1)$. To prove the other cases, observe that our subdivision of the set of lines meeting L into the classes A_{ij} would have remained the same if we had started by considering $\mathcal{I}(x_i)$, $i > 1$ instead of $\mathcal{I}(x_1)$. Thus, to prove (iv), it suffices to show that no line $M \in \mathcal{L}^{x_1}\setminus\{L\}$ can intersect two distinct lines $N_1, N_2 \in A_{ij}$ with $i > 1$. This follows at once, since N_1 and N_2 correspond to tangent circles in $\mathcal{I}(x_1)$. Also (v) is clear if we take $k = 1$ for then M and the x_k-parallel of M through x_i, represent the same circle in $\mathcal{I}(x_1)$. Finally (vi) follows from (i), (v) and the easily shown fact two lines which meet L cannot be x_k- and x_ℓ-parallel for distinct k and ℓ. $\square$

Following PIPER [5], we are now able to associate with each line L of $\mathcal{U}$ an incidence structure $GQ(L)$ as follows. The points of $GQ(L)$ are the points $x \in X\setminus L$ and the sets A_{ij}, $1 \le i,j \le q+1$. The lines of $GQ(L)$ are the lines M of $\mathcal{U}$ meeting L, and 2(q+1) new lines, $A_1, A_2, \dots, A_{q+1}$, $B_1, B_2, \dots, B_{q+1}$. Incidence in $GQ(L)$ is defined as displayed in the following table.

	line of type M	line of type A_k or B_ℓ
point of type $x \in X\setminus L$	$x \in M$	never
point of type A_{ij}	$M \in A_{ij}$	$i=k$ or $j=\ell$

Incidence in $GQ(L)$

THEOREM 6. *Let* $U = (X, L)$ *be a unital with* q+1 *points on a line satisfying* (I) *and* (II). *Then for each line* $L \in L$, $GQ(L)$ *is a generalized quadrangle with* q+1 *points on a line and* q+1 *lines through a point. Moreover, any two nonintersecting lines* m_1 *and* m_2 *of* $GQ(L)$ *form a regular pair (in the sense of* [7]*) provided* m_1 *and* m_2 *do not correspond to lines* M_1 *and* M_2 *of* U *such that* $M_1 \in A_{ij}$ *and* $M_2 \in A_{k\ell}$ *with* $i \neq k$ *and* $j \neq \ell$. *In particular, the lines* $A_1,\ldots,A_{q+1},B_1,\ldots,B_{q+1}$ *of* $GQ(L)$ *are regular.*

PROOF. Straightforward verification. □

We shall see in Section 4 that if all lines of $GQ(L)$ are regular, then U is classical.

3. UNITALS SATISFYING (I), (II) AND (III)

Let $U = (X, L)$ be a unital satisfying (I), (II) and (III). Using (III) it is easy to see that for any three distinct points x, y, z on a line L there is a unique automorphism $\tau^x_{y,z}$ of U fixing x and all lines through x and mapping y onto z: if $u \notin L$ then $\tau^x_{y,z}(u)$ is defined to be the point of intersection of xu and the x-parallel of yu through z, if $v \in L\setminus\{x\}$, fix a point $u \notin L$ and define $\tau^x_{y,z}(v)$ to be the point of intersection of L and the x-parallel of uv through $\tau^x_{y,z}(u)$.

THEOREM 7. *Let* $U = (X, L)$ *be a unital with* q+1 *points on a line satisfying* (I), (II) *and* (III), *and let* G *be the automorphism group of* U *generated by the* $\tau^x_{y,z}$. *Then* U *is classical,* G *is isomorphic to* $PSU(3,q^2)$ *and acts on* U *in the usual way.*

PROOF. Clearly G is transitive on X. We claim that G acts 2-transitively on X if $q > 2$ (the case $q = 2$ is left to the reader). To prove this, note that the mappings $\tilde{\tau}^x_{y,z}$ of Lemma 2 are induced by the automorphisms $\tau^x_{y,z}$ of U. Hence, also the mappings $\tilde{\delta}^{x,y}_{u,v}$ of Lemma 2 are induced by automorphisms $\delta^{x,y}_{u,v} \in G$ of U. Since incidence in the inversive plane $I(x)$ is determined by incidence in U, $\delta^{x,y}_{u,v}$ induces an automorphism of $I(x)$. By Corollary 4, this is a dilatation of $I(x)^{xy}$ with centre ∞_x. Therefore, it can also be viewed as a dilatation of $I(x)^{\infty_x}$ with centre xy. Thus in the affine plane $I(x)^{\infty_x}$, each point is the centre of a dilatation. Hence $\mathcal{I}(x)^{\infty x}$ is a translation plane and the group generated by the dilatations contains the full translation group of $\mathcal{I}(x)^{\infty x}$ ([2, p.187]). Let T(x) be the normal subgroup of G_x consisting of elements which induce (possibly identity) translations of $\mathcal{I}(x)^{\infty x}$. Then T(x) acts regularly on the points of $I(x)^{\infty x}$, i.e. on $\mathcal{L}^x$, and for each line $L \in \mathcal{L}^x$, $T(x)_L$ acts regularly on $L\setminus\{x\}$. Thus T(x) is a normal subgroup of G_x acting regularly on $X\setminus\{x\}$, and G is 2-transitive. Applying [4] we get that G has a normal subgroup M such that $M \le G \le \mathrm{Aut}\, M$ and M acts on X as one of the following groups in its usual 2-transitive representation: a sharply 2-transitive group, $PSL(2,q^3)$, $Sz(q^{3/2})$, $PSU(3,q^2)$, or a group of Ree type. Since $q^3 + 1 = (q+1)(q^2-q+1)$ is not a prime power for $q > 2$, the first alternative will not occur. If $H \le G$ and x, y, z are three distinct points of X, then the H_{xy}-orbit of z is contained in xy, so has length $\le q-1$. This excludes $M = PSL(2,q^3)$ and $M = Sz(q^{3/2})$. Moreover, this argument shows that if $M = PSU(3,q^2)$ then $\mathcal{U}$ is classical, for M_{xy} has a unique orbit of length q-1 on $X\setminus\{x,y\}$, all other orbits have length $(q^2-1)/(q+1,3)$ ([6, p. 499]). Now the $\tau^x_{y,z}$ can be identified with the unitary transvections and it follows that $G \simeq PSU(3,q^2)$. Thus we are left with the case that M is a group of Ree type. Since $q = 3^{2a+1}$, G contains an involution δ fixing at least two points $x,y \in X$ (Corollary 4). By [4], Lemma 3.3(v) and (ix), $\delta \in M$ and δ fixes q+1 points. Since δ is a dilatation on $\mathcal{I}(x)^{\infty x}$ these must be the q+1 points of xy and so $\mathcal{U}$ is nothing but the Ree unital associated with M. Now, for $L \in \mathcal{L}$, $\langle\delta\rangle \times PSL(2,q) \simeq M_L \trianglelefteq G_L$ and so $\langle\tau^x_{y,z} \mid x,y,z \in L\rangle \le \mathrm{Aut}(PSL(2,q)) = P\Gamma L(2,q)$, which shows that at least one, and hence all, $\tau^x_{y,z} \in M$, i.e. $G = M$ of order $(q^3+1)q^3(q-1)$. Now for a 3-Sylow group T(x) of G, $T(x)/T(x)_L$ (x on L) is the elementary abelian translation group of $\mathcal{I}(x)^{\infty x}$. Hence, for the derived group $T(x)^{(1)}$ of T(x) we find $|T(x)^{(1)}| \le |T(x)_L| = q$, contradicting Lemma 3.3(iii) of [4]. □

4. MORE CHARACTERIZATIONS

Let $\mathcal{U} = (X,\mathcal{L})$ be a unital satisfying (I) and (II). Consider the following two conditions.

(III') Given a point x and three distinct lines M_1, M_2, M_3 through x and points y_i, z_i on M_i (i = 1,2,3) such that $(y_1y_2) \parallel_x (z_1z_2)$, $(y_1y_3) \parallel_x (z_1z_3)$

and one of the lines (y_iy_j) or (z_iz_j) meets all three of M_1, M_2 and M_3, then $(y_2y_3)\|_x(z_2z_3)$.

(IV) Given a point x and two distinct lines M_1 and M_2 through x and points y_1, y_3, z_1, z_3 on M_1, y_2, y_4, z_2, z_4 on M_2 such that $(y_1y_2)\|_x(z_1z_2)$, $(y_1y_4)\|_x(z_1z_4)$ and $(y_2y_3)\|_x(z_2z_3)$, then also $(y_3y_4)\|_x(z_3z_4)$.

Clearly, (III) implies (III') and (IV). The converse is also true.

LEMMA 8. *Let $\mathcal{U} = (X,\mathcal{L})$ be a unital satisfying* (I), (II), (III') and (IV), *then also* (III) *holds.*

PROOF. Let x, M_i, y_i, z_i, i = 1,2,3 be as in (III). Suppose that M_1, M_2 and M_3 determine a circle in $\mathcal{I}(x)$ not containing ∞_x, i.e. suppose there is a line through y_1 intersecting M_2 in u_2 and M_3 in u_3, say. Let v_2 (v_3) be the point of intersection of the x-parallel of y_1u_2 through z_1 and $M_2(M_3)$. Using (III') we find that $(u_2y_2)\|_x(v_2z_2)$ and $(u_3y_3)\|_x(v_3z_3)$. Hence by (IV), $(y_2y_3)\|_x(z_2z_3)$ and (III) is shown to hold in this case. The remaining case is where M_1, M_2 and M_3 are on a circle of $\mathcal{I}(x)$ containing ∞_x, i.e. no line of $\mathcal{L}_x$ meets all three of M_1, M_2 and M_3. Since the two circles of $\mathcal{I}(x)$ corresponding to y_1y_2 and y_1y_3 cannot be tangent (for otherwise $y_1y_2 = y_1y_3$ and there is a line intersecting M_1, M_2 and M_3), there is a line M_4 through x which meets y_1y_2 in y_4 and z_1z_2 in z_4, say, and which also meets y_1y_3 and z_1z_3. Now looking at M_1, M_3 and M_4 are applying (III') we see that $(y_3y_4)\|_x(z_3z_4)$. Since M_2, M_3 and M_4 are not on a circle of $\mathcal{I}(x)$ (for otherwise this would be the circle determined by M_1, M_2 and M_3), we can apply the previous case and conclude that $(y_2y_3)\|_x(z_2z_3)$. □

The reason for considering (III') and (IV) is that in both cases there is a line M_i which is intersected by all lines mentioned in the condition. Thus, both (III') and (IV) have a (no doubt ackward) equivalent formulation $(\overline{\text{III}}')$ respectively $(\overline{\text{IV}})$ into terms of $\mathcal{GQ}(M_i)$. Since the classical unital satisfies (III') and (IV), the classical generalized quadrangle Q(4,q) on the points and lines of a hyperquadric in PG(4,q) must satisfy $(\overline{\text{III}}')$ and $(\overline{\text{IV}})$. So, conversely, if a unital U satisfying (I) and (II) has the property that $\mathcal{GQ}(L)$ is isomorphic to Q(4,q) for each line L of $\mathcal{U}$, then $\mathcal{U}$ is classical.

THEOREM 9. *Let $\mathcal{U}$ be a unital with* q+1 *points on a line satisfying* (I) *and* (II). *If for each line* L *of* $\mathcal{U}$, $\mathcal{GQ}(L) \simeq Q(4,q)$, *i.e. if every line of* $\mathcal{GQ}(L)$ *is regular, then* $\mathcal{U}$ *is classical.*

We are now in a position to prove that for even q, (I) and (II) suffice to characterize $\mathcal{U}$.

THEOREM 10. *Let* $\mathcal{U} = (X,\mathcal{L})$ *be a unital with* q+1 *points on a line satisfying* (I) *and* (II). *If* q *is even, then* $\mathcal{U}$ *is classical.*

PROOF. Let L be a line of $\mathcal{U}$ and let A_{ij}, $1 \le i,j \le q+1$ and A_i, B_i, $i = 1,\ldots$ $\ldots,q+1$ be defined as before. For each $x \in X\backslash L$ put

$$C(x) := \{A_{ij} \mid \exists \text{ line } M \in A_{ij} \text{ incident with } x\}.$$

By Corollary 5, C(x) has exactly one point on each of the lines A_i and B_i, $i = 1,\ldots,q+1$. We claim that if $x,y \in X\backslash L$, $x \ne y$, then $|C(x) \cap C(y)| \le 2$. First suppose xy is a line meeting L, $xy \in A_{ij}$, say. then by Corollary 5(iv), $C(x) \cap C(y) = \{A_{ij}\}$. Now consider the case where xy is a line of $\mathcal{U}$ not meeting L. Suppose x_1, x_2, x_3 are distinct points of L such that $xx_1, yx_1 \in A_{1,1}$, $xx_2, yx_2 \in A_{2,2}$ and $xx_3, yx_3 \in A_{3,3}$. In $\mathcal{I}(x)$, L, yx, yx_2, yx_3 correspond to circles with the following properties: yx_1, yx_2, yx_3 all go through the point xy of $\mathcal{I}(x)$ and are tangent to L in respectively xx_1, xx_2 and xx_3. Since q is even, there is a point $\ne$ xy of $\mathcal{I}(x)$ which is also on the circles yx_1, yx_2, yx_3, i.e. there is a line $M \ne xy$ through x intersecting yx_i, $i = 1,2,3$. By Lemma 1, $L \parallel_y M$ and so xy intersects L, a contradiction. We have shown that each triple A_{i_1,j_1}, A_{i_2,j_2}, A_{i_3,j_3} with $|\{i_1,i_2,i_3\}| = |\{j_1,j_2,j_3\}| = 3$ is covered at most once by a C(x). Since there are $q^3 - q$ C(x), each such triple is covered exactly once. Thus, with the A_{ij} as points, the A_i and B_i as lines and the C(x) as circles, we have obtained a Minkowski plane $\mathcal{M}(L)$ of even order q. By [3], $\mathcal{M}(L)$ is isomorphic to the geometry of points, lines and plane sections of a quadric of index two in PG(3,q). Since $\mathcal{GQ}(L)$ is determined by $\mathcal{M}(L)$ (the points of $\mathcal{GQ}(L)$ correspond to the points and circles of $\mathcal{M}(L)$, the lines of $\mathcal{GQ}(L)$ correspond to the lines and pencils of $\mathcal{M}(L)$, etc.) $\mathcal{GQ}(L)$ is isomorphic to Q(4,q) and so $\mathcal{U}$ is classical. □

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